Karlsruhe Institute of Technology
Institute for Stochastics
Prof. Dr. N. Bäuerle · Dipl.-Math. oec. D. Joos

Stochastic Processes
Problem sheet 5

Problem 1
Let $(N_t^1)_{t\geq 0}$ and $(N_t^2)_{t\geq 0}$ be two independent Poisson processes with parameters $\lambda_1, \lambda_2 > 0$, respectively. Show that $(N_t)_{t\geq 0}$ defined by $N_t := N_t^1 + N_t^2$ is a Poisson process and determine its parameter.

Problem 2
Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$ and $(X_i)_{i\in\mathbb{N}}$ an iid sequence of random variables, independent of N, with $P(X_1 = 1) = p = 1 - P(X_1 = 0)$. Show that $(N_t^1)_{t\geq 0}$ and $(N_t^2)_{t\geq 0}$ defined by $N_t^1 := \sum_{l=1}^{N_t} X_l$ and $N_t^2 = N_t - N_t^1$ are independent Poisson processes and determine their parameters.

Problem 3
Let $\lambda > 0$ and $(U_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d $\mathcal{U}(0,1)$–distributed random variables. Show that for $t > 0$ the random variable

$$N_t := \min\left\{n \in \mathbb{N} \,\middle|\, \prod_{k=1}^{n} U_k < \mathrm{e}^{-\lambda t}\right\} - 1$$

is Poisson distributed with parameter λ.

Problem 4
Let X be a random walk on $\mathbb{Z}$, i.e.

$$P(X_{n+1} = i+1 | X_n = i) = p = 1 - P(X_{n+1} = i-1 | X_n = i), \quad X_0 = 0$$

with $p \in (0,1)$.

a) Find a harmonic function for P. Differentiate the cases $p = \frac{1}{2}$ (X recurrent) and $p \neq \frac{1}{2}$ (X transient).

b) Use part a) to prove that for $p = \frac{1}{2}$ the process X is a martingale and for $p \neq \frac{1}{2}$ the process Y defined by $Y_n := \left(\frac{1-p}{p}\right)^{X_n}$, $n \in \mathbb{N}_0$ is a martingale.

This sheet will be discussed in the problem sessions on the 19th of May.

Please note that the English tutorial has been moved to Wednesday 11:30-13:00, Plank-Hörsaal (40.32)