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# Stochastic Processes
## Problem sheet 7

**Problem 1**
Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a discrete-time Markov chain with state space $S = \{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}, \quad 0 < \alpha < 1.$$

Show that there is a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$, in which $X$ can be embedded in terms of

$$X_n = Y_n \quad \text{for all } n \in \mathbb{N}_0,$$

if and only if $\alpha > 1/2$. In this case, determine the generator $Q$ of $Y$.

**Problem 2**
Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov chain with state space $S = \{1, 2\}$ and generator

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}, \quad \lambda, \mu > 0.$$

Determine the transition probabilities by using the formula $P(t) = \exp(tQ)$.
(Hint: First decompose $Q$ into $ADA^{-1}$ with a diagonal matrix $D$.)

**Problem 3**
Let $(N_t)$ be a Poisson process with parameter $\lambda$. Thus the generator of $N$ is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & \dots \\ \vdots & & \ddots & & \dots \end{pmatrix}$$

Solve the Kolmogorov forward equation in order to determine $P(N_t = i)$ for all $i \in \mathbb{N}_0, t \geq 0$.
Note that the Kolmogorov forward equation holds, since we have $\sup_{i\in S} q_i < \infty$.

**Problem 4**
Let $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$ be independent with $\lambda_1, \lambda_2 > 0$. Prove the following statements:

a) *Forgetfulness property*: For $s, t > 0$

$$P(X > t + s | X > s) = P(X > t) = e^{-\lambda_1 t}.$$

b)

$$X \wedge Y \sim Exp(\lambda_1 + \lambda_2)$$

c)

$$P(X > Y) = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Remark: the opposite direction in a) also holds, i.e. if $X$ is a random variable with values in $\mathbb{R}_+$ and for all $s, t \geq 0$ $P(X > t + s \mid X > t) = P(X > s)$, then $X \sim Exp(\lambda)$ with some $\lambda > 0$.

**This sheet will be discussed in the problem sessions on the 2nd of June.**