Karlsruhe Institute of Technology

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Stochastic Processes
Problem sheet 8

Problem 1

Let $S = \mathbb{N}_0$.

a) Let $(Y_n)_{n\in\mathbb{N}}$ be a discrete-time Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1-p & 0 & p & 0 & \dots \\ 1-p & 0 & 0 & p & \\ \vdots & \vdots & & & \ddots \end{pmatrix},$$

where $0 < p < 1$. Show that $(Y_n)_{n\in\mathbb{N}}$ is positive recurrent.
Construct a null recurrent continuous-time Markov chain $(X_t)_{t\geq 0}$, which has $(Y_n)_{n\in\mathbb{N}}$ as an embedded Markov chain by adequate choice of $(q_i)_{i\in S}$.

b) Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with generator Q defined by

$$q_{0j} = \begin{cases} -1, & j = 0 \\ \frac{6}{(\pi j)^2}, & j \geq 1 \end{cases}$$

and for $i \geq 1$:

$$q_{ij} = \begin{cases} -(i+1)^2, & j = i \\ (i+1)^2, & j = i-1 \\ 0, & else \end{cases}$$

Show that $(X_t)_{t\geq 0}$ is positive recurrent and that its embedded Markov chain is null recurrent.

Problem 2

In a waiting room with c counters let the customers arrive according to a Poisson process with parameter $\lambda = 6$. The services at the counters are assumed to be independent and exponentially distributed with parameter $\mu = 3$. Let X_t denote the total number of customers waiting at time t.

(a) Determine the generator Q of the queueing process $(X_t)_{t\geq 0}$.

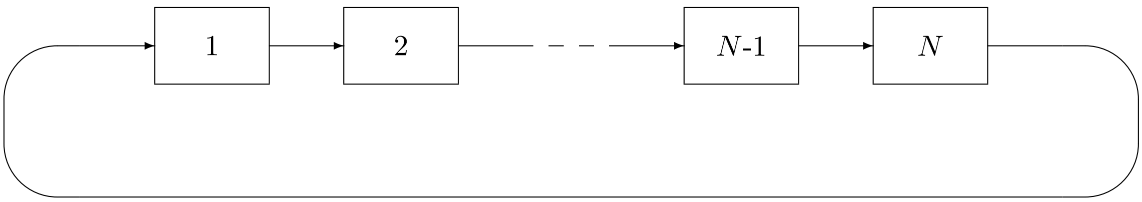
(b) Determine a stationary distribution π by solving the equation $\pi Q = 0$. In which case does $E_\pi X_t < \infty$ hold?

(c) Assume $c = 4$ and the following rule for the behaviour of an arriving customer: If there are up to 4 waiting customers, he stays in the waiting room. If there are 5 customers, he stays with probability $p = 0.5$ and if there are 6 or more, he leaves. Determine a stationary distribution for $(X_t)_{t\geq 0}$.

Problem 3

Consider the situation from Problem 2a). Which condition on c has to hold such that the number of waiting customers doesn't become infinitely large?

Problem 4
Let N stations with one server each be arranged as follows:



Let the serving rate at station i be $\mu_i > 0$. If a customer has finished service at station i, he directly proceeds to station $i+1$, if $i < N$. For $i = N$ the customer goes back to the queue at station 1. The overall number of customers in the system is M.
Determine the stationary distribution of the Markov chain $X_t = \big(X_1(t), \ldots, X_N(t)\big)$, where $X_i(t)$ denotes the number of customers at station i at time t.

This sheet will be discussed in the problem sessions on the 9th of June.