Karlsruhe Institute of Technology

Institute for Stochastics

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Stochastic Processes

Problem sheet 9

Problem 1

Let $(B_t)_{t\geq 0}$ be a Brownian motion, $0 \leq s < t < \infty$ and $n = \frac{s+t}{2}$

a) Determine a common density of (B_s, B_n, B_t).

b) Determine the conditional density of B_n, given B_s and B_t.

Problem 2

Let $T \in \{[0,1], \mathbb{R}_+\}$, $\mathcal{C}(T)$ be the set of all continuous functions on T and $\mathcal{B}^T$ the σ-algebra on $\mathbb{R}^T$ generated by the projections. Prove:

$$\mathcal{C}(T) \notin \mathcal{B}^T \ .$$

Hint: First show that for any $A \in \mathcal{B}^T$ there is a countable set $T(A) \subset T$ with

$$\forall x \in \mathbb{R}^T,\ y \in A \quad x(t) = y(t) \quad \text{for all } t \in T(A) \Rightarrow x \in A \ .$$

Problem 3

Let $(B_t^{(i)})_{t\in T}$, $i \in \mathbb{N}$, be a family of independent Brownian motions with time set $T = [0,1]$. Show that the process $(X_t)_{t\geq 0}$ defined by

$$X_t := \sum_{i=1}^{\lfloor t \rfloor} B_1^{(i)} + B_{t-\lfloor t \rfloor}^{(\lfloor t \rfloor + 1)}$$

is a Brownian motion with time set $[0, \infty)$.

Problem 4 (Theorem 9.7)

Let $a \geq 0$, $c \neq 0$ and $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the following processes $(\tilde{B}_t)_{t\geq 0}$ are Brownian motions too:

$$\text{(i) } \tilde{B}_t := -B_t,\ \forall t \geq 0, \quad \text{(ii) } \tilde{B}_t := B_{a+t} - B_a,\ \forall t \geq 0, \quad \text{(iii) } \tilde{B}_t := cB_{t/c^2},\ \forall t \geq 0\ .$$

This sheet will be discussed in the problem sessions on the 16th of June.

Exam dates:
Wednesday, July 28th
Wednesday, September 15th

Registration is possible from the 10th of June until the 16th of July at Tatjana Dominic's office (room 5A-22). The exam will be oral.