Karlsruhe Institute of Technology
Institute for Stochastics
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Stochastic Processes
Problem sheet 10

Problem 1
Let S and $(g_{nk})_{(n,k)\in S}$ be defined as in the proof of theorem 11.1: For $n \in \mathbb{N}_0$ let

$$S_n := \{(n,k) \mid 1 \le k \le 2^n, k \text{ odd}\}$$

and $S := \bigcup_{n=0}^{\infty} S_n$. Now for $(n,k) \in S$ define

$$g_{01}(t) := 1$$
$$g_{nk}(t) := 2^{\frac{(n-1)}{2}} \left(1_{[(k-1)2^{-n},\, k2^{-n})}(t) - 1_{[k2^{-n},\, (k+1)2^{-n})}(t)\right) \quad \text{for } n \neq 0,\ 0 \le t \le 1.$$

Show that the system $\{g_{nk}(t) : (n,k) \in S\}$ is orthonormal.

Problem 2
Let $\{g_{nk}(t) : (n,k) \in S\}$ be defined as in Problem 1.

a) Let $f \in L^2([0,1])$ be a function satifsying

$$\langle f, g_{nk} \rangle = 0 \ \text{ for all } (n,k) \in S.$$

Show, that $f = 0$ almost everywhere.

b) Use part a) to prove the formula

$$f = \sum_{(n,k)\in S} \langle f, g_{nk} \rangle g_{nk} \ \text{ für alle } f \in L^2([0,1]).$$

Hint for part a): Evaluate of the function $F : [0,1] \to \mathbb{R}$, $t \mapsto \int_0^t f(s)\,ds$, in the points $k2^{-n}$, $(n,k) \in S$.

Problem 3
Let $(B_t)_{t\ge0}$ be a (one-dimensional) Brownian motion.

a) Show that for all $\alpha > 0$ the process $(X_t)_{t\ge0}$ defined by

$$X_t \ := \ \exp\left(\alpha B_t - \alpha^2 t/2\right)$$

is a martingale with continuous paths.

b) For $t \ge 0$ let $M_t := \sup_{0\le s\le t} B_s$. Show:

$$P(M_t \ge z) \ \le \ \exp\left(-z^2/(2t)\right) \qquad \text{for all } z, t \ge 0.$$

c) Let $T_a := \inf\{t > 0 : B_t = a\}$; Show that for all $a \neq 0$

$$\liminf_{t\to\infty} \ t\, P(T_a > t) \ > \ 0$$

and therefore $ET_a = \infty$.

Hint for part b): Let $T = [0, t_0]$ and $(X_t, \mathcal{F}_t)_{t \in T}$ be a martingale with continuous paths. You can use without proof that for all $p \geq 1$ and $c > 0$ the following inequality holds:

$$P\Big(\sup_{t \in T} |X_t| \geq c\Big) \;\leq\; \frac{1}{c^p}\, E|X_{t_0}|^p.$$

Problem 4

Let $(B_t)_{t \geq 0}$ be a Brownian motion. Show that for all $\gamma > 1/2$

$$\lim_{t \to \infty} t^{-\gamma} B_t \;=\; 0 \;\text{ P–a.s. .}$$

Hint: Law of the iterated logarithm:

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{t \log \log t}} = \sqrt{2}, \quad \liminf_{t \to \infty} \frac{B_t}{\sqrt{t \log \log t}} = -\sqrt{2} \;\text{ P–a.s. .}$$

This sheet will be discussed in the problem sessions on the 23rd of June.

Exam dates:
Wednesday, July 28th
Wednesday, September 15th

Registration is possible from the 10th of June until the 16th of July at Tatjana Dominic's office (room 5A-22). The exam will be oral (English or German).