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Stochastic Processes
Problem sheet 11

Problem 1

Suppose $X, X_1, X_2, \ldots$ are random variables in $L^2(\Omega, \mathcal{F}, P)$.
Prove or disprove the following implications:

1. $X_n \xrightarrow{L^2} X \quad \Rightarrow \quad X_n \xrightarrow{P} X$
2. $X_n \xrightarrow{L^2} X \quad \Rightarrow \quad X_n \to X \quad$ P-a.s.
3. $X_n \to X \quad$ P-a.s. $\quad \Rightarrow \quad X_n \xrightarrow{L^2} X$

Problem 2

By Theorem 12.10, for a sequence $\mathcal{Z}_n$ of partitions of $[0,t]$ with vanishing length the quadratic variation $V_{\mathcal{Z}_n}$ of a Brownian motion satisfies $V_{\mathcal{Z}_n} \xrightarrow{L^2} t$. Show that with the partition

$$t_{n,j} := \frac{j}{2^n}\, t, \quad n \in \mathbb{N},\ j = 0, ..., k_n := 2^n,$$

we even have $V_{\mathcal{Z}_n} \xrightarrow{P-a.s.} t$.

Hint:

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty \text{ for all } \epsilon > 0 \quad \Rightarrow \quad X_n \xrightarrow{P-a.s.} X.$$

Problem 3

Show that P-almost-every path of a Brownian motion is not monotone on any interval $[a,b] \subset \mathbb{R}_+$ with $a < b$.

Problem 4

Let $N, Y_1, Y_2, ...$ be independent random variables, N Poisson-distributed with parameter λ and Y_k with distribution μ for all $k \in \mathbb{N}$. By $\text{CompP}(\lambda, \mu)$ we denote the distribution of the random sum $\sum_{k=1}^{N} Y_k$, a *compound Poisson distribution* with intensity parameter $\lambda > 0$ and summand distribution μ.

Now let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ and $(Y_k)_{k\in\mathbb{N}}$ an i.i.d.-sequence of random variables with distribution μ, independent of N. Show that $X = (X_t)_{t\geq 0}$ defined by

$$X_t := \sum_{k=1}^{N_t} Y_k \text{ für alle } t \geq 0$$

is a Markov process with transition probability

$$P_t(x, A) = \text{CompP}(\lambda t, \mu)(A - x)$$

and determine its generator.

This sheet will be discussed in the problem sessions on the 30th of June.