Institute for Stochastics
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Stochastic Processes
Problem sheet 12

Problem 1
Find a martingale (in discrete time), that is not a Markov process.

Problem 2
Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and $a, b \in \mathbb{R}$ with $a < 0 < b$;

$$\tau := \inf\{t \geq 0 \,:\, B_t \notin (a,b)\}$$

is the point in time, in which B leaves the interval (a,b).

a) Show

$$P(B_\tau = a) = \frac{b}{b-a}, \quad P(B_\tau = b) = \frac{-a}{b-a}.$$

b) Show, that

$$(M_t)_{t\geq 0}, \quad M_t := (B_t - a)(b - B_t) + t,$$

is a martingale with continuous paths. Use this result to prove

$$E\tau = -ab.$$

Hint: You can use without proof that $EX_\tau = EX_0$ for a martingale $(X_t, \mathcal{F}_t)_{t\geq 0}$ with continuous paths and a bounded stopping time τ.

Problem 3
Let $B = (B_t)_{t\geq 0}$ be a Brownian motion $(\Omega, \mathcal{A}, P)$; for sake of simplicity assume that all paths are continuous.

a) Show, that B as a function $[0,\infty) \times \Omega \to \mathbb{R}$, $(t,\omega) \mapsto B_t(\omega)$, is $(\mathcal{B}_{[0,\infty)} \otimes \mathcal{A}, \mathcal{B})$-measurable.

b) Show, that the random set $L_a(\omega) := \{t \geq 0 \,:\, B_t(\omega) = a\}$ is a Lebesgue-null set with P-probability 1 for any $a \in \mathbb{R}$.

Problem 4
Determine the generator of a Markov chain $(X_t)_{t\geq 0}$ with intensity matrix $Q = (q_{ij})$.

This sheet will be discussed in the problem sessions on the 7th of July.