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Stochastic Processes

Solutions to the 1st Problem sheet

Solution to problem 1

WLOG let $S = \mathbb{N}$. Define the function $g : S \times [0,1] \to S$ by

$$g(i,y) = \sum_{k \in S} k \; 1_{(\sum_{l=1}^{k-1} p_{il}, \sum_{l=1}^{k} p_{il}]}(y)\,.$$

Let (Y_n) be a sequence of independently $\mathcal{U}(0,1)$-distributed random variables and $X_n := g(X_{n-1}, Y_n)$. Then for $i_{n-1}, i_n \in S$ we have

$$\begin{aligned} P(X_n = i_n \,|\, X_{n-1} = i_{n-1}) &= P\big(g(X_{n-1}, Y_n) = i_n \,|\, X_{n-1} = i_{n-1}\big) = P\big(g(i_{n-1}, Y_n) = i_n\big) \\ &= P\big(Y_n \in (\sum_{l=1}^{i_n - 1} p_{i_{n-1}l}, \sum_{l=1}^{i_n} p_{i_{n-1}l}]\big) = p_{i_{n-1}i_n}\,, \end{aligned}$$

because the interval $(\sum_{l=1}^{i_n-1} p_{i_{n-1}l}, \sum_{l=1}^{i_n} p_{i_{n-1}l}]$ is of length $p_{i_{n-1}i_n}$. Thus (X_n) defined by $X_n = g(X_{n-1}, Y_n)$ obviously is a Markov chain with transition matrix P.

Solution to problem 2

Let $X_{n,j}$ be the number of individuals in the nth Generation who directly descend from the jth member of generation $n-1$. Thus for all $n \in \mathbb{N}$ $(X_{n,j})_j$ is a (finite) sequence of i.i.d. random variables with distribution $(p_k)_{k \in \mathbb{N}}$. The number of individuals in the nth generation is given by

$$X_n = \sum_{j=1}^{X_{n-1}} X_{n,j} \quad (n \in \mathbb{N}).$$

Then we have

$$P(X_n = i_n \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = P(\sum_{j=1}^{i_{n-1}} X_{n,j} = i_n) = P(X_n = i_n \,|\, X_{n-1} = i_{n-1})\,.$$

thus (X_n) is a Markov chain with transition matrix $P = (p_{ik})_{i,k \in \mathbb{N}}$, where p_{ik} is defined by

$$p_{ik} = \begin{cases} P(\sum_{j=1}^{i} X_{n+1,j} = k) & , \quad i > 0 \\ \delta_0(k) & , \quad i = 0 \end{cases}\,.$$

Solution to problem 3

In general the assertion is false. Counterexample:

> Let $S = \{1, 2\}$ and (X_n) be an S-valued Markov chain with initial distribution $p = (\frac{1}{2}, \frac{1}{2})$ and transition matrix
> $$P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$
> Set $j = 1, A_0 = \{1\}$ and $A_1 = \{1, 2\}$, then we have:
> $$\begin{aligned} P(X_2 = j \,|\, X_0 \in A_0, X_1 \in A_1) &= \frac{P(X_2 = 1, X_0 = 1, X_1 \in \{1,2\})}{P(X_0 = 1, X_1 \in \{1,2\})} \\ &= \frac{1}{P(X_0 = 1)} \sum_{i=1}^{2} P(X_2 = 1 \,|\, X_1 = i)\, P(X_1 = i \,|\, X_0 = 1)\, P(X_0 = 1) = 1\,, \\ P(X_2 = j \,|\, X_1 \in A_1) &= \frac{P(X_2 = 1, X_1 \in \{1,2\})}{P(X_1 \in \{1,2\})} \\ &= \sum_{i=1}^{2} \sum_{k=1}^{2} P(X_2 = 1, X_1 = k, X_0 = i) = \frac{1}{2} + 0 + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}\,. \end{aligned}$$

Look out: If $A_n = \{\, i\}$ for $i \in S$, the assertion is true!

Solution to problem 4

P is an upper triangular matrix, thus P^n is an upper triangular matrix for all $n \in \mathbb{N}$. For given states $i \neq j$ we have $p_{ij}^{(m)} \cdot p_{ji}^{(n)} = 0$ for all $n, m \in \mathbb{N}$ and thus $i \nleftrightarrow j$. We conclude $K(i) = \{i\}$ for $i = 1, \ldots, 8$, i.e. any state forms an own communicating class.

Furthermore we have

$$f_{17}^* = \sum_{n=0}^{\infty} f_{17}^{(n)} = \sum_{n=0}^{\infty} P(X_n = 7, X_k \neq 7 \text{ für } 1 \leq k < n \,|\, X_0 = 1) = (0.5)^3 \cdot 0.6 \cdot 0.8 \cdot 0.01 = 0.0006\,.$$

That means a field worker ascends from level 1 to a lifetime employment with probability 0.06%.