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Stochastic Processes
Solutions to the 2nd Problem sheet

Solution to problem 1

a) By induction we get:

$$A^{2n} = \begin{pmatrix} 1-p & 0 & p \\ 0 & 1 & 0 \\ 1-p & 0 & p \end{pmatrix}$$

and

$$A^{2n+1} = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}$$

b) For $0 < p < 1$ there is only one class. Since the state space is finite, all states are recurrent.
For $p = 0$, the class $K(1) = \{1, 2\}$ is recurrent and the class $K(3) = \{3\}$ is transient.
For $p = 1$, the class $K(2) = \{2, 3\}$ is recurrent and the class $K(3) = \{3\}$ is transient.

Solution to problem 2

1st case $p_1 = 1$

the TM $P = (p_{ik})_{i,k}$ is given by $p_{ik} = \delta_{ik}$. Every state $k \in \mathbb{N}$ is recurrent:

$$f^*_{kk} = \sum_{n=1}^{\infty} f^{(n)}_{kk} = \sum_{n=1}^{\infty} P(X_n = k, X_j \neq k \text{ für } 1 \leq j < n \,|\, X_0 = k) = P(X_1 = k \,|\, X_0 = k) = 1\,.$$

2nd case: $p_1 < 1$

With $p_{0k} = \delta_{0k}$ we get that the state 0 is recurrent ($f^*_{00} = 1$, cf. 1st case). The other states $k \geq 1$ are transient:

(a) Let $p_0 = 0$. Thus any individual has at least 1 descendant and

$$P(X_n = k, X_i \neq k \text{ für } 1 \leq i < n \,|\, X_0 = k) = 0$$

for all $n \geq 2$. Thus

$$f^*_{kk} = f^{(1)}_{kk} = P(X_1 = k \,|\, X_0 = k) = P(X_{n+1,j} = 1 \text{ für } j = 1, \ldots, k) = p_1^k < 1\,.$$

(b) Let $p_0 = 1$. Then $p_{k0} = 1$ and thus $f^*_{kk} = 0$.

(c) Let $0 < p_0 < 1$. Then

$$\begin{aligned} f^*_{kk} &= P(X_n = k \text{ for some } n \geq 0 \,|\, X_0 = k) \\ &\leq P(X_1 \neq 0 \,|\, X_0 = k) = 1 - P(X_1 = 0 \,|\, X_0 = k) = 1 - p_0^k < 1\,. \end{aligned}$$

Solution to problem 3
Obviously (N_k) is a Markov chain:
for $n_1, \dots, n_{k+1} \in \mathbb{N}$ we have

$$P(N_{k+1} = n_{k+1} \,|\, N_1 = n_1, \dots, N_k = n_k) = p_{n_k n_{k+1}} = e^{-n_k} \frac{(n_k)^{n_{k+1}}}{n_{k+1}!}$$

independent of $n_1, \dots, n_{k-1}$. For all $n \geq 1$ we have

$$\begin{aligned} f^*_{n0} = P(N_k = 0 \text{ for some } k \geq 2 \,|\, N_1 = n) &= \sum_{m=0}^{\infty} P(N_k = 0 \text{ for some } k \geq 2, N_2 = m \,|\, N_1 = n) \\ &= \sum_{m=1}^{\infty} P(N_k = 0 \text{ for some } k \geq 3, N_2 = m \,|\, N_1 = n) + P(N_2 = 0 \,|\, N_1 = n) \\ &= \sum_{m=1}^{\infty} p_{nm} f^*_{m0} + p_{n0} = \sum_{m=1}^{\infty} e^{-n} \frac{n^m}{m!} f^*_{m0} + e^{-n} \overset{f^*_{00}=1}{=} e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} f^*_{m0} \,. \end{aligned}$$

This system of equation holds, if $f^*_{n0} = 1$ for all $n \geq 1$. If this is the only solution, we are done, because $f^*_{n0} = 1 \; \forall n \in \mathbb{N}$ implies, that 0 occurs in (N_k) with probability 1.
For $n \geq 1$ we have

$$f^*_{nn} = 1 - P(N_k \neq n \; \forall k \geq 2 \,|\, N_1 = n) \leq 1 - P(N_2 = 0 \,|\, N_1 = n) = 1 - e^{-n} < 1 \,,$$

i.e. n is transient. By 2.6 we have $p^{(k)}_{mn} \xrightarrow{k \to \infty} 0$ for any states $n, m \geq 1$. For $Q = (p_{mn})_{m,n \geq 1}$ we conclude

$$(I - Q) \sum_{j=0}^{n} Q^j = \sum_{j=0}^{n} Q^j - Q^{j+1} = Q^0 - Q^{n+1} = I - Q^{n+1} \xrightarrow{n \to \infty} I \,,$$

i.e. $\sum_{k=0}^{\infty} Q^k$ exists and is the inverse of $(I - Q)^{-1}$. Now let $\mathbf{f}^* = (f^*_{n0})_{n \geq 1}$. Then the above system of equations is equivalent to

$$\mathbf{f}^* - Q\mathbf{f}^* = (I - Q)\mathbf{f}^* = (e^{-n})_{n \geq 1} \,.$$

Because $(I - Q)^{-1}$ exists, $\mathbf{f}^*$ is uniquely determined and we are done.

Solution to problem 4
We can assume $i \neq j$, because the case $i = j$ is obvious. Now let $n_0 := \inf\{n \in \mathbb{N}; p^{(n)}_{ji} > 0\}$ and $T_j := \inf\{n \in \mathbb{N}; X_n = j\}$. For $n > n_0$ we have

$$\begin{aligned} P_j(T_j \leq n, X_{n_0} = i) &= \sum_{k=1}^{n} P_j(T_j = k, X_{n_0} = i) \\ &= \sum_{k=1}^{n_0 - 1} f^{(k)}_{jj} p^{(n_0 - k)}_{ji} + \sum_{k=n_0+1}^{n} P_j(T_j \geq n_0, X_{n_0} = i) f^{(k - n_0)}_{ij} \,. \end{aligned}$$

By definition $p^{(n_0 - k)}_{ji} = 0$ for $k = 1, \dots, n_0 - 1$, i.e. the first sum on the right hand side is 0. In the second sum we use the inequality $P_j(T_j \geq n_0, X_{n_0} = i) \leq p^{(n_0)}_{ji}$ and deduce

$$P_j(T_j \leq n, X_{n_0} = i) \leq \sum_{k=n_0+1}^{n} p^{(n_0)}_{ji} f^{(k - n_0)}_{ij} \leq p^{(n_0)}_{ji} f^*_{ij} \,.$$

Now we let $n \to \infty$ and take into account that $\lim_{n \to \infty} P_j(T_j \leq n) = \lim_{n \to \infty} \sum_{k=1}^{n} f^{(k)}_{jj} = f^*_{jj} = 1$. Thus we have

$$p^{(n_0)}_{ji} = P_j(X_{n_0} = i) = \lim_{n \to \infty} P_j(T_j \leq n, X_{n_0} = i) \leq p^{(n_0)}_{ji} f^*_{ij}$$

and therefore with $p^{(n_0)}_{ji} > 0$ the assertion follows.