Stochastic Processes
Solutions to the 3rd Problem sheet

Solution to problem 1
We want to calculate the probabilities f^*_{i0} for $i = 0, \ldots, 2B$.
In general, we have

$$f^*_{ij} = P_i(\exists k \geq 1 : X_k = j) = \sum_{l \in S} P_i(\exists k \geq 1 : X_k = j, X_1 = l)$$

$$= P_i(\exists k \geq 1 : X_k = j, X_1 = j) + \sum_{l \in S \setminus \{j\}} P_i(\exists k \geq 1 : X_k = j, X_1 = l)$$

$$= p_{ij} + \sum_{l \in S \setminus \{j\}} P(\exists k \geq 2 : X_k = j, X_1 = l | X_0 = i)$$

$$= p_{ij} + \sum_{l \in S \setminus \{j\}} P(X_1 = l | X_0 = i) \cdot P(\exists k \geq 2 : X_k = j | X_1 = l)$$

Thus

$$f^*_{ij} = p_{ij} + \sum_{l \in S \setminus \{j\}} p_{il} f^*_{lj}, \quad (*)$$

(Note that if j is recurrent this implies $f^*_{(j)} = P f^*_{(j)}$ with $f^*_{(j)} = (f^*_{ij})_{i \in S}$.)

Here the transition matrix is given by $P = (p_{ij})$ with $p_{00} = p_{2B,2B} = 1$ and

$$p_{ij} = \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \end{cases} \text{ if } 0 < i < 2B.$$

Therefore we have $f^*_{00} = 1$, $f^*_{2B,0} = 0$ and by $(*)$

$$f^*_{i0} = (1-p) f^*_{i-1,0} + p f^*_{i+1,0}, \quad 1 \leq i \leq 2B-1$$

We conclude

$$f^*_{i+1,0} - f^*_{i,0} = \frac{1-p}{p}(f^*_{i,0} - f^*_{i-1,0}), \quad 1 \leq i \leq 2B-1$$

and inductively obtain

$$f^*_{i+1,0} - f^*_{i,0} = \left(\frac{1-p}{p}\right)^i (f^*_{1,0} - f^*_{0,0}).$$

Obviously this formula also holds for $i = 0$, thus altogether for $i = 0, \ldots, 2B-1$.
With $\rho := \frac{1-p}{p}$ we get for $i = 0, \ldots, 2B$:

$$f^*_{i0} = f^*_{00} + \sum_{k=0}^{i-1} \underbrace{(f^*_{k+1,i} - f^*_{k,i})}_{= \rho^k (f^*_{1,0} - f^*_{0,0})} = 1 - (1 - f^*_{10}) \frac{1-\rho^i}{1-\rho}$$

Inserting $i = 2B$ we get

$$0 = 1 - (1 - f_{10}^*)\frac{1-\rho^{2B}}{1-\rho} \Leftrightarrow 1 - f_{10}^* = \frac{1-\rho}{1-\rho^{2B}}.$$

and therefore

$$f_{i0}^* = 1 - \frac{1-\rho^i}{1-\rho^{2B}} = \frac{\rho^i - \rho^{2B}}{1-\rho^{2B}}$$

Solution to problem 2

a) Obviously, an admissible Markov chain is also irreducible. The state space $S = S_N$ is finite, thus by remark 3.5a there is a stationary distribution.
Since the shuffling procedure is symmetric, the transition matrix satisfies $P = P^\top$, i.e. $\sum_{i\in S} p_{ij} = 1$ for all $j \in S$. Such a matrix is called *doubly stochastic*. Thus for any $c \in \mathbb{R}$ $\nu \equiv c$ is an invariant measure. Since S is finite, we can normalize and the invariant measure is given by $\pi(i) = \frac{1}{|S|} = \frac{1}{N!}$ for all $i \in S$.

b) (i)
- symmetric
  WLOG $\varpi \neq \tau$. Assume there are no two postitions i and j s.t. ϖ becomes τ by interchanging i and j. Then by construction the one-step probabilities are 0:

  $$p_{\tau\varpi} = P(X_1 = \varpi | X_0 = \tau) = 0 = P(X_1 = \tau | X_0 = \varpi) = p_{\varpi\tau}$$

  In the case where there are postitions i and j s.t. ϖ becomes τ by interchanging i and j, we have

  $$p_{\tau\varpi} = P(X_1 = \varpi | X_0 = \tau) = \alpha_i\alpha_j + \alpha_j\alpha_i = P(X_1 = \tau | X_0 = \varpi) = p_{\varpi\tau}$$

- admissible
  Any permutation of length N can be expressed as the composition of finitely many transpositions. In particular for $\varpi_1, \varpi_2 \in S_N$ we have

  $$\exists k \in \mathbb{N}\ \exists i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \ldots, N\}: \quad \varpi_2 \circ \varpi_1^{-1} = (i_k\ j_k) \circ \ldots \circ (i_1\ j_1)$$

  where $(i\ j)$ denotes the transposition between i and j. Thus

  $$p_{\varpi_1\varpi_2}^{(k)} \geq \underbrace{p_{\varpi_1,(i_1\ j_1)\circ\varpi_1}}_{=2\alpha_{i_1}\alpha_{j_1}>0} \cdot \ldots \cdot \underbrace{p_{(i_{k-1}\ j_{k-1})\circ\ldots\circ(i_1\ j_1)\circ\varpi_1,\varpi_2}}_{=2\alpha_{i_k}\alpha_{j_k}>0} > 0$$

  (i.e. admissibility follows from the fact that every transpostion occurs with positive probability)

(ii)
- symmetric
  If $p_{\varpi\tau} > 0$, then there exist $i \in \{1, \ldots, N\}$ and $j \in \{0, \ldots, N-1\}$ s.t. we obtain τ from ϖ by taking the i-th card and putting it under the j-th card of the rest of the deck. But then ϖ is obtained from τ by taking the $(j+1)$-st card and putting it under the $(i-1)$-st card of the rest of the deck. Thus $p_{\varpi\tau} = \frac{1}{N^2} = p_{\tau\varpi}$. The rest follows with the same argument as in (i).
- admissible
  We can create any transposition by 2 steps of taking a card and putting it back. Thus for any $\varpi \in S$ and any transposition $(i\ j)$ we have $p_{\varpi,(i\ j)\circ\varpi} \geq \frac{1}{N^4} > 0$. As in (i), admissibility follows.

Solution to problem 3

a) We have $X_{n+1} = X_n + Z_{n+1}$ with $P(Z_{n+1} = 1|X_n = i) = \frac{i}{N} = 1 - P(Z_{n+1} = -1|X_n = i)$. Thus

$$p_{i,i+1} = \frac{N-i}{N}, \quad i = 0, \ldots, N-1$$

$$p_{i,i-1} = \frac{i}{N}, \quad i = 1, \ldots, N$$

and $p_{kj} = 0$ else.

b) By a) $\pi P = \pi$ is equal to

$$\pi(i) = \pi(i-1)\left(1 - \frac{i-1}{N}\right) + \pi(i+1)\frac{i+1}{N}, \quad i = 1, \ldots, N-1,$$

$$\pi(0) = \pi(1)\frac{1}{N}, \quad \pi(N) = \pi(N-1)\frac{1}{N}$$

One obtains successively

$$\pi(i) = \pi(0)\binom{N}{i}$$

and $1 = \sum_{i=0}^{N} \pi(i) = \sum_{i=0}^{N} \pi(0)\binom{N}{i} = \pi(0)2^N$ implies $\pi(0) = \frac{1}{2^N}$, thus

$$\pi(i) = \frac{1}{2^N}\binom{N}{i}.$$

Solution to problem 4

The state space is $S = \{1, 2, 3, 4, 5\}$ and the transition matrix is given by

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

With $(*)$ from problem 1 we get

$$f_{15}^* = \frac{1}{2}f_{25}^* + \frac{1}{2}f_{45}^* = \frac{1}{4}f_{15}^* + \frac{1}{4}\underbrace{f_{35}^*}_{=0} + \frac{1}{6}f_{15}^* + \frac{1}{6}\underbrace{f_{35}^*}_{=0} + \frac{1}{6}\underbrace{f_{55}^*}_{=1} = \frac{5}{12}f_{15}^* + \frac{1}{6}$$

This holds if and only if

$$f_{15}^* = \frac{2}{7}$$