Prof. Dr. N. Bäuerle · Dipl.-Math. oec. D. Joos

Stochastic Processes
Solutions to the 4th Problem sheet

Solution to problem 1

(a) S is finite, (X_n) is irreducible. By Remark 3.11 ii) (X_n) is positive recurrent. Since P is doubly stochastic, $\pi = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|}\right)$ is the stationary distribution.

(b) WLOG $S = \mathbb{N}$. Assume (X_n) is positive recurrent. By theorem 3.10 there is a stationary distribution π. Since $\sum_{i\in\mathbb{N}} \pi_i = 1$, we have $\lim_{i\to\infty} \pi_i = 0$ and thus $\pi_{i_0} = \max\{\pi_i \,;\, i \in S\} > 0$ is well defined. $\pi P^n = \pi$ implies $\pi_{i_0} = \sum_{j\in S} \pi_j p_{ji_0}^{(n)}$ und therefore $1 = \sum_{j\in S} \frac{\pi_j}{\pi_{i_0}} p_{ji_0}^{(n)}$ for all $n \in \mathbb{N}$.

Assume there is some $j_0 \in S$ satisfying $\pi_{j_0} < \pi_{i_0}$. Since (X_n) is irreducible, i_0 and j_0 communicate, i.e. $\exists\, n \in \mathbb{N}$: $p_{j_0 i_0}^{(n)} > 0$. Then:

$$1 = \sum_{j\in S} \frac{\pi_j}{\pi_{i_0}} p_{ji_0}^{(n)} = \sum_{j\neq j_0} \frac{\pi_j}{\pi_{i_0}} p_{ji_0}^{(n)} + \frac{\pi_{j_0}}{\pi_{i_0}} p_{j_0 i_0}^{(n)} < \sum_{j\in S} p_{ji_0}^{(n)} = 1\,.$$

Thus $\pi_j = \pi_{i_0}$ for all $j \in S$. With $\pi_{i_0} > 0$ this is a contradiction:

$$1 = \sum_{j\in S} \pi_j = \sum_{j\in S} \pi_{i_0} = |S|\pi_{i_0} = \infty\,.$$

Therefore the Markov chain has to be transient or null recurrent.

Solution to problem 2

i) Let X be transient. Let $\tau_0 := \inf\{n \geq 1 \,|\, X_n = i_0\}$ be the return time to i_0. The first-step method implies that

$$h(i) := P_i(\tau_0 = \infty), \quad i \in S,$$

satisfies the given equation. Assume $h \equiv 0$, then the first-step method implies $f_{i_0 i_0}^* = 1$, which is a contradiction to X transient. Thus $h \neq 0$ and obviously h is bounded.

ii) Now assume there is a function h with the desired properties. Let

$$\tilde{h}(i) = \begin{cases} h(i)\,, i \neq i_0 \\ 0 \quad\,, i = i_0. \end{cases}$$

and $\alpha := \sum_{j\in S} p_{i_0 j} \tilde{h}(j) \geq 0$ since h is non-negative. Then $\tilde{h}$ is subharmonic and bounded by assumption. If X were recurrent, $\tilde{h}$ would be constant. $\tilde{h}(i_0) = 0$ implies that this constant must be 0, which is a contradiction. Thus X is transient.

Solution to problem 3

Problem 2 implies that X is transient, if and only if there is a bounded, non–constant function $h : \mathbb{N}_0 \to \mathbb{R}$, s.t.

$$\underbrace{(p_{i,i-1} + p_{ii} + p_{i,i+1})}_{=1} \cdot h(i) = h(i) = p_{i,i+1}h(i+1) + p_{ii}h(i) + p_{i,i-1}h(i-1), \quad i \in \mathbb{N},$$

holds (WLOG we set $h(i_0) := 0$). This is equivalent to

$$p_{i,i+1} \cdot (h(i+1) - h(i)) = p_{i,i-1} \cdot (h(i) - h(i-1)), \quad i \in \mathbb{N}.$$

After i recursive steps we get

$$h(i+1) = h(i) + (h(1) - h(0)) \cdot \prod_{k=1}^{i} \frac{p_{k,k-1}}{p_{k,k+1}}, \quad i \in \mathbb{N},$$

thus

$$h(i+1) = h(1) + (h(1) - h(0)) \cdot \sum_{j=1}^{i} \prod_{k=1}^{j} \frac{p_{k,k-1}}{p_{k,k+1}}, \quad i \in \mathbb{N},$$

The solution h is bounded and non-constant, if and only if condition (1) holds. In the special case $p_{01} = p$, $p_{i,i+1} = p$, $p_{i,i-1} = 1 - p =: q$ $(i \in \mathbb{N})$ this means

$$\sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{q}{p} < \infty,$$

which holds iff $q < p$ (cf example 3.14).

Solution to problem 4

a) A state is determined by the M (distinct) sites $i_1, \dots, i_M$ which are in the cache. Thus the state space is given by

$$S = \{(i_1, \dots, i_M) : i_1, \dots,\ i_M \in \{1, \dots, N\},\ i_1 < i_2 < \dots < i_M\}$$

For $N = 2$, $M = 4$ this is

$$S = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

The probability to stay in a state (i, j) is exactly the probability that one of the pages i or j gets visited next, thus

$$p_{(i,j),(i,j)} = \alpha_i + \alpha_j$$

For $k \notin \{i, j\}$ both the probability to go from state (i, j) to $(\min\{i, k\}, \max\{i, k\})$ and the probability to go from state (i, j) to $(\min\{j, k\}, \max\{j, k\})$ are $\frac{1}{2}\alpha_k$, since i and j are deleted with probability $\frac{1}{2}$ respectively.

Since only one page is deleted, all other transition probabilities are 0, i.e. for $k, l \notin \{i, j\}$

$$p_{(i,j),(k,l)} = 0.$$

Thus with $\alpha = \frac{1}{25}(12, 6, 4, 3)$ the transition matrix is given by

$$P = \frac{1}{50} \begin{pmatrix} 36 & 4 & 3 & 4 & 3 & 0 \\ 6 & 32 & 3 & 6 & 0 & 3 \\ 6 & 4 & 30 & 0 & 6 & 4 \\ 12 & 12 & 0 & 20 & 3 & 3 \\ 12 & 0 & 12 & 4 & 18 & 4 \\ 0 & 12 & 12 & 6 & 6 & 14 \end{pmatrix}$$

b) Solutions to $\nu \cdot P = \nu$ are given by multiples of $(12, 8, 6, 4, 3, 2)$ (since we know that invariant measures are unique up top multiples we just have to find one eigenvector of $P^\top$ with eigenvalue 1; a CAS (e.g. Maple) provides us the above result). Thus the stationary distribution is given by

$$\pi = (\frac{12}{35}, \frac{8}{35}, \frac{6}{35}, \frac{4}{35}, \frac{3}{35}, \frac{2}{35})$$

We can write this in a closed form dependent on the state as

$$\pi(i,j) = \frac{\frac{1}{ij}}{\sum_{(k,l)\in S} \frac{1}{kl}}$$

Therefore our ansatz for general M and N is

$$\pi(s) = \frac{\mu(s)}{\sum_{t\in S} \mu(t)} \quad \text{with} \quad \mu(s) = \mu(i_1, \dots, i_M) = \frac{1}{\prod_{j=1}^M i_j} \qquad (*)$$

Now we have to check the balance equations. Let $s = (i_1, \dots, i_M) \in S$. For $t \in S \setminus \{s\}$ the transition probability p_{ts} is positive if and only if t and s differ in exactly one component. We denote by $\mathcal{M}_s$ the set $\{1, \dots, N\} \setminus \{i_1, \dots i_M\}$ and by $s^{(jk)}$ the state which is created from s by deleting site j and putting site k into the cache instead ($k \in \mathcal{M}_s$). Then we have

$$\sum_{t\in S} p_{ts}\pi(t) = \sum_{k\in\mathcal{M}_s} \sum_{j=1}^M p_{s^{(i_j k)}s}\pi(s^{(i_j k)}) + p_{ss}\pi(s) = \sum_{k\in\mathcal{M}_s} \sum_{j=1}^M \frac{1}{M}\alpha_{i_j}\pi(s^{(i_j k)}) + \sum_{j=1}^M \alpha_{i_j}\pi(s)$$

With $(*)$ we get

$$\alpha_{i_j}\pi(s^{(i_j k)}) = \frac{\frac{1}{i_j}}{\sum_{l=1}^N \frac{1}{l}} \frac{\frac{1}{k\prod_{l=1,l\neq j}^M i_l}}{\sum_{t\in S}\mu(t)} = \frac{\frac{1}{k}}{\sum_{l=1}^N \frac{1}{l}} \frac{\frac{1}{\prod_{j=1}^M i_j}}{\sum_{t\in S}\mu(t)} = \alpha_k\pi(s)$$

$$\Longrightarrow \sum_{t\in S} p_{ts}\pi(t) = \sum_{k\in\mathcal{M}_s} \sum_{j=1}^M \frac{1}{M}\alpha_k\pi(s) + \sum_{j=1}^M \alpha_{i_j}\pi(s) = \pi(s)$$

Since $s \in S$ was arbitrary, the balance equations hold and π is the stationary distribution.

Now we calculate the expected hit ratio using the stationary distribution:

$$r = \sum_{s\in S} P(X_1 = s | X_0 = s) \cdot \pi(s) = \sum_{(i,j)\in S} (\alpha_i + \alpha_j)\pi(i,j) = \frac{103}{175} \approx 58.857\%$$