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Stochastic Processes
Solutions to the 5th Problem sheet

Solution to problem 1

(A1) ✓

(A2) Let $n \in \mathbb{N}$, $0 \le t_0 \le t_1 \le \ldots \le t_n$, then

$$N_{t_0} = N^1_{t_0} + N^2_{t_0},$$
$$N_{t_i} - N_{t_{i-1}} = (N^1_{t_i} - N^1_{t_{i-1}}) + (N^2_{t_i} - N^2_{t_{i-1}}), \quad i = 1, \ldots, n$$

are independent by independence of N^1 and N^2 and (A2) for N^1 and N^2.

(A3) Let $s, t > 0$, $n \in \mathbb{N}_0$, then

$$P(N_{t+s} - N_s = n) = P((N^1_{t+s} - N^1_s) + (N^2_{t+s} - N^2_s) = n) = \sum_{i=1}^{n} P(N^1_{t+s} - N^1_s = i) \cdot P(N^2_{t+s} - N^2_s = n - i)$$

$$= \sum_{i=1}^{n} P(N^1_t - N^1_0 = i) \cdot P(N^2_t - N^2_0 = n - i) = P((N^1_t - N^1_0) + (N^2_t - N^2_0) = n) = P(N_t - N_0 = n)$$

(A4)

$$P(N_h \ge 2) = P(N^1_h \ge 2 \vee N^2_h \ge 2) + P(N^1_h = N^2_h = 1)$$
$$\le \underbrace{P(N^1_h \ge 2)}_{=o(h)} + \underbrace{P(N^2_h \ge 2)}_{=o(h)} + \underbrace{P(N^1_h = 1) \cdot P(N^2_h = 1)}_{=e^{-2\lambda h}(\lambda h)^2} = o(h)$$

thus N is a Poisson process. Since $N_t = N^1_t + N^2_t \sim Po(\lambda_1 t + \lambda_2 t)$, the parameter is $\lambda_1 + \lambda_2$.

Solution to problem 2

i) N^1 and N^2 are Poisson processes

First we check (A1)-(A4) for N^1:

(A1) ✓

(A2) Let $n \in \mathbb{N}$, $0 \le t_0 \le t_1 \le \ldots \le t_n$, then

$$N^1_{t_0} = \sum_{l=1}^{N_{t_0}} X_i,$$

$$N^1_{t_i} - N^1_{t_{i-1}} = \sum_{l=N_{t_{i-1}}}^{N_{t_i}} X_l, \quad i = 1, \ldots, n$$

are independent by independence of $X_1, X_2, \ldots$

(A3) Let $s, t > 0$, $n \in \mathbb{N}_0$, then

$$\begin{aligned}
& P(N^1_{t+s} - N^1_s = n) \\
= \quad & P(\textstyle\sum_{l=N_s+1}^{N_{t+s}} X_l = n) = \sum_{j \geq i} \underbrace{P(\displaystyle\sum_{l=i+1}^{j} X_l = n, N_s = i, N_{t+s} = j)}_{=P(\sum_{l=i+1}^{j} X_l = n) \cdot P(N_s = i, N_{t+s} = j)} \\
= \quad & \textstyle\sum_{j \geq i} P(\sum_{l=1}^{j-i} X_l = n) \cdot \underbrace{P(N_s = i, N_{t+s} - N_s = j - i)}_{=P(N_s = i) \cdot P(N_{t+s} - N_s = j - i)} \\
\overset{k:=j-i}{=} \quad & \textstyle\sum_{i,k \geq 0} P(\sum_{l=1}^{k} X_l = n) \cdot P(N_s = i) \cdot P(N_t - N_0 = k) \\
= \quad & \textstyle\sum_{k \geq 0} P(\sum_{l=1}^{k} X_l = n) \cdot P(N_t - N_0 = k) = P(\sum_{l=N_0+1}^{N_t} X_l = n) = P(N^1_t = n)
\end{aligned}$$

(A4)

$$P(N^1_h \geq 2) \leq P(N_h \geq 2) = o(h) \quad \Longrightarrow \quad P(N^1_h \geq 2) = o(h)$$

Since $N^2_t = N_t - N^1_t = \sum_{l=1}^{N_t} \underbrace{(1 - X_l)}_{=:Y_l}$ with $P(Y_l = 1) = 1 - p = 1 - P(Y_l = 0)$, the proof obviously also holds for N^2.

ii) the parameters

By i) we know that $N^1_t \sim Po(\tilde{\lambda})$ with some parameter λ. We can use the formula we derived in the proof of (A3):

$$P(N^1_t = n) = \sum_{k \geq 0} P(\sum_{l=1}^{k} X_l = n) \cdot P(N_t = k)$$

With $\sum_{l=1}^{k} X_l \sim Bin(k, p)$ and $N_t \sim Po(\lambda)$ we get (note that for the first probability to be positive $n \leq k$ has to hold):

$$P(N^1_t = n) = \sum_{k \geq n} \binom{k}{n} p^n (1-p)^{k-n} \cdot e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \frac{p^n}{n!} \sum_{k \geq n} \frac{1}{(k-n)!} (1-p)^{k-n} (\lambda t)^k = e^{-\lambda p t} \frac{(\lambda p t)^n}{n!}$$

Thus the parameter of N^1 is λp and again by symmetry the parameter of N^2 is $\lambda(1-p)$.

iii) N^1 and N^2 are independent

Define for $n, l \in \mathbb{N}$, $0 = t_0 \leq t_1 \leq \ldots \leq t_n < \infty$, $0 = s_0 \leq s_1 \leq \ldots \leq s_l < \infty$, $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathbb{N}$

$$A := A(t_1, \ldots, t_n; i_1, \ldots, i_n) := \{f \in D_0 : f(t_j) - f(t_{j-1}) = i_j,\ j = 1, \ldots, n\}$$

$$B := A(s_1, \ldots, s_l; j_1, \ldots, j_l)$$

WLOG we can assume that $n = l$ and $t_1 = s_1, \ldots, t_n = s_n$:
Else consider $\tilde{t}_1 \leq \ldots \leq \tilde{t}_m$ with $\{\tilde{t}_1, \ldots, \tilde{t}_m\} = \{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_l\}$, then by case differentiation over the possible states rewrite A and B as the disjoint union of sets that have the form $A(\tilde{t}_1, \ldots, \tilde{t}_m, u_1, \ldots, u_m)$.

Thus $B = A(t_1, \ldots, t_n; j_1, \ldots, j_n)$ and we have

$$P(N^1 \in A, N^2 \in B)$$

$$= P(N^1_{t_n} - N^1_{t_{n-1}} = i_n, \ldots, N^1_{t_2} - N^1_{t_1} = i_2, N^1_{t_1} = i_1, N^2_{t_n} - N^2_{t_{n-1}} = j_n, \ldots, N^2_{t_2} - N^2_{t_1} = j_2, N^2_{t_1} = j_1)$$

$$= P(N_{t_n} - N_{t_{n-1}} = i_n + j_n, \dots, N_{t_2} - N_{t_1} = i_2 + j_2, N_{t_1} = i_1 + j_1, \sum_{l=1}^{i_1+j_1} X_l = i_1, \dots, \sum_{l=\sum_{k=1}^{n-1}(i_k+j_k)+1}^{\sum_{k=1}^{n}(i_k+j_k)} X_l = i_n)$$

all ocurring random variables are independent of each other (follows by (A2) for N, $(X_i)_{i\in\mathbb{N}}$ i.i.d.-sequence and independence of $(X_i)_{i\in\mathbb{N}}$ and N) and we know their distributions:

$$P(N^1 \in A, N^2 \in B)$$

$$= \prod_{k=1}^{n} \left(e^{-\lambda(t_k - t_{k-1})} \frac{(\lambda(t_k - t_{k-1}))^{i_k + j_k}}{(i_k + j_k)!} \right) \cdot \prod_{k=1}^{n} \left(\binom{i_k + j_k}{i_k} p^{i_k} (1-p)^{j_k} \right)$$

$$= \prod_{k=1}^{n} \left(e^{-\lambda p(t_k - t_{k-1})} \frac{(\lambda p(t_k - t_{k-1}))^{i_k}}{i_k!} \right) \cdot \prod_{k=1}^{n} \left(e^{-\lambda(1-p)(t_k - t_{k-1})} \frac{(\lambda(1-p)(t_k - t_{k-1}))^{j_k}}{j_k!} \right)$$

$$= P(N^1 \in A) \cdot P(N^2 \in B)$$

Thus we have shown independence on a π-system of $\mathcal{B}(D_0)$, which is sufficient.

Solution to problem 3

We have

$$N_t = \max\Big\{ n \in \mathbb{N} \,\big|\, \prod_{k=1}^{n} U_k \geq \mathrm{e}^{-\lambda t} \Big\}.$$

Thus for $n \in \mathbb{N}_0$

$$\begin{aligned} P(N_t = n) &= P\Big(\prod_{k=1}^{n} U_k \geq \mathrm{e}^{-\lambda t} > \prod_{k=1}^{n+1} U_k \Big) \\ &= P\Big(-\frac{1}{\lambda} \ln\Big(\prod_{k=1}^{n} U_k \Big) \leq t < -\frac{1}{\lambda} \ln\Big(\prod_{k=1}^{n+1} U_k \Big) \Big) \\ &= P\Big(\sum_{k=1}^{n} \big(-\frac{1}{\lambda} \ln U_k \big) \leq t < \sum_{k=1}^{n+1} \big(-\frac{1}{\lambda} \ln U_k \big) \Big) \\ &= P\Big(\sum_{k=1}^{n} \tau_k \leq t < \sum_{k=1}^{n+1} \tau_k \Big) \end{aligned}$$

with $\tau_k := -\frac{1}{\lambda} \ln U_k \sim \mathrm{Exp}(\lambda)$, $k \in \mathbb{N}$. Now consider a Poisson process $(X_t)_{t\geq 0}$, with interarrival times τ_k. Then $P\Big(\sum_{k=1}^{n} \tau_k \leq t < \sum_{k=1}^{n+1} \tau_k \Big)$ equals $P(X_t = n)$.
Thus X_t and N_t are identically distributed, which proves the assumption.

Solution to problem 4

We have

$$\begin{aligned} Ph = h \quad &\Longleftrightarrow \quad \forall j \in \mathbb{Z} \quad h(j) = p \cdot h(j+1) + (1-p) \cdot h(j-1) \\ &\Longleftrightarrow \quad \forall j \in \mathbb{Z} \quad h(j+1) - h(j) = \left(\frac{1-p}{p} \right) \cdot (h(j) - h(j-1)) \end{aligned}$$

thus for $p = \frac{1}{2}$ the identity function $h(j) = j$ is harmonic and for $p \neq \frac{1}{2}$ $h(j) := \left(\frac{1-p}{p} \right)^j$ is harmonic.

By Theorem 5.2 $h(X_n) = Y_n$ is a martingale, since X_n can take only finitely many values, namely $X_0 - n, \dots, X_0 + n$.