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Stochastic Processes
Solutions to the 6th Problem sheet

Solution to problem 1
We have for $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots t_n$, $t, h > 0$, $i_k \in S$ $(k = 0, \dots, n)$ with $P(X_{t_k} = i_k, 0 \leq k \leq n) > 0$:

$$P(X_{t_n+h} = i_{n+1} | X_{t_k} = i_k, 0 \leq k \leq n) = P(X_{t_n+h} = i_{n+1} | X_{t_n} = i_n) = P(X_{t+h} = i_{n+1} | X_t = i_n)$$

since $X_{t_n+h} = (-1)^{N_{t_n+h}} = (-1)^{N_{t_n}} \cdot (-1)^{N_{t_n+h}-N_{t_n}} = X_{t_n} \cdot (-1)^{N_{t_n+h}-N_{t_n}}$ and $N_{t_n+h} - N_{t_n}$ is independent of N_{t_k} $(k = 0, \dots, n)$ and independent of t_n.
Moreover,

$$P(X_{t+s} = 1 | X_s = -1) = P(\underbrace{N_{t+s} - N_s}_{\sim Po(\lambda t)} \text{ is odd}) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2k+1}}{(2k+1)!} = \frac{1}{2}(1 - e^{-2\lambda t}),$$

that is, $p_{-1,+1}(t) = \frac{1}{2}(1 - e^{-2\lambda t})$. The other transition probabilities can be calculated analogously and we have

$$P(t) = \frac{1}{2} \begin{pmatrix} 1 + e^{-2\lambda t} & 1 - e^{-2\lambda t} \\ 1 - e^{-2\lambda t} & 1 + e^{-2\lambda t} \end{pmatrix}$$

Solution to problem 2
$\forall i, j \in S$, $t \geq 0$, $k > 0$:

$$|p_{ij}(t+h) + p_{ij}(t)| = |\sum_{k \neq i} p_{ik}(h) p_{kj}(t) + p_{ij}(t) \underbrace{(p_{ii}(h) - 1)}_{= -\sum_{k \neq i} p_{ik}(h)}| \leq 1 - p_{ii}(h)$$

since $|p_{kj}(t) - p_{ij}(t)| \leq 1$.

Solution to problem 3

1. P_h is aperiodic: We have to show $d(i) := \gcd\{n \in \mathbb{N} : p_{ii}^{(n)}(h) > 0\} = 1$. By $\lim_{t \to 0} p_{ii}(t) = 1$ there is some $t_0 > 0$, s.t. for all $t \leq t_0$ $p_{ii}(t) > \frac{1}{2}$. Now for $s \geq 0$ there is $n \in \mathbb{N}_0$ and $t \leq t_0$ with $s = nt_0 + t$, thus $p_{ii}(s) \geq p_{ii}(t_0)^n p_{ii}(t) > 0$ for all $s \geq 0$. In particular for fixed $h > 0$
$$p_{ii}^{(n)}(h) \geq p_{ii}(h)^n > 0.$$
2. P_h is irreducible: Let $h > 0$ and $i, j \in S$. Since $\{p(t) : t \geq 0\}$ is irreducible, there is some $t \geq 0$ with $p_{ij}(t) > 0$. Now let $n \in \mathbb{N}$ with $nh > t$ and $s := nh - t$. By part 1 we have $p_{ii}(s) > 0$, thus
$$p_{ij}^{(n)}(h) = p_{ij}(nh) = p_{ij}(s+t) \geq p_{ii}(s) p_{ij}(t) > 0.$$

Solution to problem 4

Let $i, j \in S$, $\epsilon > 0$. By Problem 2, $t \mapsto p_{ij}(t)$ is uniformly continuous, i.e.

$$\exists \delta > 0 \; \forall h \in [0, \delta) \; \forall t \in [0, \infty) \; : \; |p_{ij}(t+h) - p_{ij}(t)| \leq \frac{\epsilon}{2}.$$

Problem 3 implies, that the discrete-time Markov chain $p(\delta)$ is aperiodic and irreducible, thus we can apply Theorem 4.2. Therefore we have:

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}(\delta) = \lim_{n \to \infty} p_{ij}(n\delta).$$

thus

$$\exists n_0 \in \mathbb{N} \; \forall n \geq n_0 \; : \; |p_{ij}(n\delta) - \pi_j| \leq \frac{\epsilon}{2}.$$

Now let $t \geq n_0 \delta$, $t = n\delta + h$ with $0 \leq h < \delta$, $n \geq n_0$. Then

$$|p_{ij}(t) - \pi_j| \leq |p_{ij}(t) - p_{ij}(n\delta)| + |p_{ij}(n\delta) - \pi_j| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

thus

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j.$$