Stochastic Processes
Solutions to the 7th Problem sheet

Solution to problem 1

Let $Y_n = X_n$ for all $n \in \mathbb{N}$. This is equivalent to $p(1) = P$, i.e. $p_{00}(1) = p_{11}(1) = \alpha$.

In example 8.8 we derived that for a Markov chain $Y = (Y_t)_{t\geq 0}$ with state space $S = \{0,1\}$ and generator

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}, \quad \lambda, \mu > 0.$$

the transition probabilities are

$$p_{00}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu},$$

$$p_{11}(t) = \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}.$$

Since

$$p_{00}(1) = \alpha = p_{11}(1),$$

we have

$$\alpha = \frac{1}{2}(p_{00}(1) + p_{11}(1)) = \frac{1}{2}\left(1 + e^{-(\lambda+\mu)}\right) > \frac{1}{2}.$$

Now let $\alpha > \frac{1}{2}$, then with $\lambda := \mu := -\frac{1}{2}\log(2\alpha - 1)$ we have $p_{00}(1) = p_{11}(1) = \alpha$ (and therefore the transition matrices are the same, i.e. $Y_n = X_n$ for all $n \in \mathbb{N}$).

Alternative proof: By

$$\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)} + \frac{\mu}{\lambda+\mu} = \alpha,$$

$$\frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)} + \frac{\lambda}{\lambda+\mu} = \alpha$$

we have

$$e^{-(\lambda+\mu)} + 1 = 2\alpha,$$

thus $\lambda + \mu = -\log(2\alpha - 1)$. Inserting, substracting and applying $2\alpha - 1 < 1$ proves the assertion and on top we have uniqueness of λ and μ.

Solution to problem 2

Define the matrices A and D by

$$A = \begin{pmatrix} \lambda & 1 \\ -\mu & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -(\lambda+\mu) & 0 \\ 0 & 0 \end{pmatrix}$$

then $Q = ADA^{-1}$ and

$$P(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n = A \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \right) A^{-1} = A \begin{pmatrix} \exp(-t(\lambda+\mu)) & 0 \\ 0 & 1 \end{pmatrix} A^{-1}$$

$$= \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda\exp(-t(\lambda+\mu))+\mu & -\lambda\exp(-t(\lambda+\mu))+\lambda \\ -\mu\exp(-t(\lambda+\mu))+\mu & \mu\exp(-t(\lambda+\mu))+\lambda \end{pmatrix}$$

Solution to problem 3

The Kolmogorov forward differential equations are

$$p'_{ii}(t) = -\lambda p_{ii}(t), \quad p'_{ij}(t) = -\lambda p_{ij}(t) + \lambda p_{ij-1}(t)$$

The solution is

$$p_{ij}(t) = \begin{cases} 0 & , \quad j < i \\ e^{-\lambda t} & , \quad j = i \\ \lambda e^{-\lambda t} \int_0^t e^{\lambda s} p_{ij-1}(s)ds, & j > i \end{cases}$$

Now we prove inductively for $j \geq i$

$$p_{ij}(t) = e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}$$

i) $j = i$ (✓)

ii) $j \to j+1$:

$$p_{ij+1}(t) = \lambda e^{-\lambda t}\int_0^t e^{\lambda s} p_{ij}(s)ds = \lambda e^{-\lambda t}\int_0^t e^{\lambda s}e^{-\lambda s}\frac{(\lambda s)^{j-i}}{(j-i)!}ds = e^{-\lambda t}\frac{(\lambda t)^{j+1-i}}{(j+1-i)!}$$

Therefore $P_0(X_t = i) = e^{-\lambda t}\frac{(\lambda t)^j}{j!}$.

Solution to problem 4

a)

$$P(X > t+s | X > t) = \frac{P(X > t+s, X > t)}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)}$$

$$= \frac{e^{-\lambda_1(t+s)}}{e^{-\lambda_1 s}} = e^{-\lambda_1 t} = P(X > t)$$

b)

$$P(X \wedge Y > x) = P(X > x, Y > x) = P(X > x)P(Y > x) = e^{-\lambda_1 x}e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}$$

c)

$$P(X > Y) = \int_0^\infty \int_v^\infty \lambda_1 e^{-\lambda_1 u}\lambda_2 e^{-\lambda_2 v} du dv = \int_0^\infty \lambda_2 e^{-\lambda_2 v} dv \int_v^\infty \lambda_1 e^{-\lambda_1 u} du$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 v}e^{-\lambda_1 v} dv = \frac{\lambda_2}{\lambda_1+\lambda_2}$$