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# Stochastic Processes
## Solutions to the 8th Problem sheet

### Solution to problem 1

a) The Markov chain $(Y_n)_{n\in\mathbb{N}}$ is irreducible. For the state $0 \in S$ we have $P_0(T_0 = 1) = 0$ and $P_0(T_0 = n) = p^{n-2}(1-p)$ for $n \geq 2$. Then with $p < 1$ for $T_0 = \inf\{n \geq 1\,; Y_n = 0\}$ it follows

$$E_0(T_0) = \sum_{n=2}^{\infty} n\, P_0(T_0 = n) = (1-p)\sum_{n=0}^{\infty} np^n < \infty\,,$$

i.e. $(Y_n)$ is positive recurrent.

Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables with parameter 1, independent of $(Y_n)$. Then obviously $\frac{Z_n}{q_i} \sim \exp(q_i)$. Thus for the time-continuous Markov chain $(X_t)_{t\geq 0}$ associated to $(Y_n)$ and $\tau_0 = \inf\{t \geq 0\,; X_t = 0 \text{ and } X_{t^-} \neq 0\}$ we have

$$E_0(\tau_0) = E_0\left(\sum_{n=0}^{T_0-1} \frac{Z_n}{q_{Y_n}}\right) = E_0\left(\sum_{n=0}^{T_0-1} \frac{Z_n}{q_n}\right) = E_0\left(\sum_{n=0}^{\infty} \frac{Z_n}{q_n} 1_{\{n<T_0\}}\right)$$

$$\overset{(Z_n)\text{and}(Y_n)\text{independent}}{=} \sum_{n=0}^{\infty} \frac{1}{q_n} P(n < T_0) = \frac{1}{q_0} + \sum_{n=1}^{\infty} \frac{1}{q_n} p^{n-1}\,.$$

Now choose $q_n = p^{n-1}$, then $E_0(\tau_0) = \infty$. Therefore in this case $(Y_n)$ is positive recurrent and $(X_t)$ is null recurrent.

b) Define the generator $Q$ by

$$q_{0j} = \begin{cases} -1, & j = 0 \\ \frac{6}{(\pi j)^2}, & j \geq 1 \end{cases}$$

and for $i \geq 1$:

$$q_{ij} = \begin{cases} -(i+1)^2, & j = i \\ (i+1)^2, & j = i-1 \\ 0, & otherwise \end{cases}$$

The waiting time until the first jump has expectation 1. Then, if the next state is $n$, the waiting time for the return to 0 ($n$ jumps) has expectation $\sum_{i=1}^{n} \frac{1}{i^2} \leq \frac{\pi^2}{6}$. By $\frac{\pi^2}{6} + 1 < \infty$, 0 is positive recurrent.

Embedded Markov chain:

$$p_{0j} = \begin{cases} 0, & j = 0 \\ \frac{6}{(\pi j)^2}, & j \geq 1 \end{cases}$$

and for $i \geq 1$:

$$p_{ij} = \begin{cases} 0, & j \geq i \\ 1, & j = i-1 \end{cases}$$

Again we need exactly $j$ steps to return to 0 after a jump to the state $j$, thus $m_0 = E_0(T_0) = 1 + \sum_{j=1}^{\infty} \frac{6}{(\pi j)^2} \cdot j = 1 + \frac{6}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j} = \infty$.
Therefore 0 is null recurrent.

**Solution to problem 2**
We have a $(M|M|c)$-queueing model.

(a) $(X_t)$ is a BDP with parameters $\lambda_i \equiv \lambda$ and $\mu_i = \mu \cdot \min\{c, i\}$.

(b) Solving $\pi Q = 0$, we get

$$\pi_k = \begin{cases} \frac{\eta^k}{k!}\pi_0 & , \quad 0 \le k \le c \\ \frac{\eta^c}{c!}\rho^{k-c}\pi_0 & , \quad k \ge c \end{cases}$$

with $\eta := \frac{\lambda}{\mu}$ and $\rho := \frac{\lambda}{\mu c}$. If $X$ is positive recurrent, by $\sum_{k=0}^{\infty} \pi_k = \pi_0 \left( \sum_{k=0}^{c-1} \frac{\eta^k}{k!} + \frac{\eta^c}{c!} \sum_{k=0}^{\infty} \rho^k \right) \stackrel{!}{=} 1$ we get a stationary distribution $\pi$ defined by:

$$\pi_k = \begin{cases} \frac{1}{a}\frac{\eta^k}{k!} & , \quad 0 \le k \le c \\ \frac{1}{a}\frac{\eta^c}{c!}\rho^{k-c} & , \quad k \ge c \end{cases}$$

with $a = \sum_{k=0}^{c-1} \frac{\eta^k}{k!} + \frac{\eta^c}{c!}\frac{1}{1-\rho}$. Thus we have

$$E(X_t) < \infty \text{ für alle } t \ge 0 \Longleftrightarrow \sum_{k=c}^{\infty} k\rho^{k-c} < \infty \Longleftrightarrow \rho < 1\,.$$

(c) For general $\lambda, \mu > 0$ the intensity matrix of $(X_t)$ on $S = \{0, 1, \ldots, 6\}$ is:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & 0 & 0 \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & 0 & 0 & 0 \\ 0 & 0 & 3\mu & -(3\mu+\lambda) & \lambda & 0 & 0 \\ 0 & 0 & 0 & 4\mu & -(4\mu+\lambda) & \lambda & 0 \\ 0 & 0 & 0 & 0 & 4\mu & -(4\mu+\frac{1}{2}\lambda) & \frac{1}{2}\lambda \\ 0 & 0 & 0 & 0 & 0 & 4\mu & -4\mu \end{pmatrix}$$

Solving $\pi Q = 0$ under the condition $\sum_{i=0}^{6} \pi_i = 1$, $\pi_i \ge 0$ with $\lambda = 6$ and $\mu = 3$ yields

$$\pi = \frac{1}{89}(12, 24, 24, 16, 8, 4, 1)\,.$$

**Solution to problem 3**
By Problem 2 $(X_t)$ is a BDP with parameters $\lambda_i \equiv \lambda$ and $\mu_i = \mu \cdot \min\{c, i\}$ By example 8.16 $(X_t)$ is positive recurrent, if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdot \ldots \cdot \lambda_n}{\mu_1 \cdot \ldots \cdot \mu_{n+1}} = \sum_{n=1}^{c-1} \frac{\lambda^{n+1}}{(n+1)!\,\mu^{n+1}} + \sum_{n=c}^{\infty} \frac{\lambda^{n+1}}{c!\,c^{n-c}\mu^{n+1}} < \infty \Leftrightarrow \frac{c^{c+1}}{c!} \sum_{n=c}^{\infty} \left(\frac{\lambda}{c\mu}\right)^{n+1} < \infty\,,$$

i.e. $(X_t)$ is positive recurrent, if $\rho := \frac{\lambda}{c\mu} < 1$. The expected number of waiting customers doesn't become infinitely large, if $\rho < 1$. For $\lambda = 6$ and $\mu = 3$, this means $c > 2$. Thus there have to be at least 3 counters.

**Solution to problem 4**
Let $i = (i_1, \ldots, i_N) \in \{0, 1, \ldots, M\}^N$. Then

$$q_{ij} = \begin{cases} \mu_k\,\delta(i_k) & , \quad j = i - e_k + e_{k+1} \quad (k = 1, \ldots, N-1) \\ \mu_N\,\delta(i_N) & , \quad j = i - e_N + e_1 \end{cases}$$

Global balance:

$$\pi(j) \sum_{k=1}^{N} \mu_k \, \delta(j_k) = \pi(j + e_N - e_1) \, \mu_N \, \delta(j_1) + \sum_{k=2}^{N} \pi(j + e_{k-1} - e_k) \, \mu_{k-1} \, \delta(j_k) \quad \forall j \in S$$

Local balance:

$$\begin{aligned} &\text{(i)} \quad \pi(j) \, \mu_1 = \pi(j + e_N - e_1) \, \mu_N \quad (j_1 > 0) \\ &\text{(ii)} \quad \pi(j) \, \mu_k = \pi(j + e_{k-1} - e_k) \, \mu_{k-1} \quad (j_k > 0, k = 2, \ldots, N) \end{aligned}$$

(ii) implies $\pi(j) = \pi(j + e_{k-1} - e_k) \, \frac{\mu_{k-1}}{\mu_k} = \ldots = \pi(j + j_k e_{k-1} - j_k e_k) \, \left(\frac{\mu_{k-1}}{\mu_k}\right)^{j_k}$ for $k = 2, \ldots, N$. Thus we have

$$\begin{aligned} \pi(j) &= \pi\big((M, 0, \ldots, 0)\big) \left(\frac{\mu_{N-1}}{\mu_N}\right)^{j_N} \left(\frac{\mu_{N-2}}{\mu_{N-1}}\right)^{j_N + j_{N-1}} \cdot \ldots \cdot \left(\frac{\mu_1}{\mu_2}\right)^{j_N + \ldots + j_2} \\ &= \pi\big((M, 0, \ldots, 0)\big) \left(\frac{1}{\mu_N}\right)^{j_N} \left(\frac{1}{\mu_{N-1}}\right)^{j_{N-1}} \cdot \ldots \cdot \left(\frac{1}{\mu_2}\right)^{j_2} \mu_1^{j_N + \ldots + j_2} \\ &= \pi\big((M, 0, \ldots, 0)\big) \, \mu_1^M \prod_{k=1}^{N} \left(\frac{1}{\mu_k}\right)^{j_k} . \end{aligned}$$

The condition $\sum_{j \in S} \pi(j) = 1$ implies

$$\pi\big((M, 0, \ldots, 0)\big) \, \mu_1^M \sum_{j \in S} \prod_{k=1}^{N} \left(\frac{1}{\mu_k}\right)^{j_k} = \pi\big((M, 0, \ldots, 0)\big) \, \mu_1^M \prod_{k=1}^{N} \sum_{j_k=0}^{M} \left(\frac{1}{\mu_k}\right)^{j_k} \stackrel{!}{=} 1$$

and therefore

$$\pi\big((M, \ldots, 0)\big) = \left( \mu_1^M \prod_{k=1}^{N} \frac{1 - \left(\frac{1}{\mu_k}\right)^{M+1}}{1 - \frac{1}{\mu_k}} \right)^{-1} .$$