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Stochastic Processes
Solutions to the 9th Problem sheet

Solution to problem 1

a) By definition of the Brownian motion we have

$$(B_s, B_n - B_s, B_t - B_n)^\top \sim \mathcal{N}(0, \Sigma) \text{ with } \Sigma = \begin{pmatrix} s & 0 & 0 \\ 0 & n-s & 0 \\ 0 & 0 & t-n \end{pmatrix}$$

Now with

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

we have $(B_s, B_n, B_t) = A \cdot (B_s, B_n - B_s, B_t - B_n)^\top$ and therefore $(B_s, B_n, B_t) \sim \mathcal{N}(0, A \cdot \Sigma \cdot A^\top)$. Thus the common density is given by

$$f(x_1, x_2, x_3) = \frac{1}{\sqrt{s}} e^{-\frac{1}{2}\frac{x_1^2}{s}} \frac{1}{\sqrt{n-s}} e^{-\frac{1}{2}\frac{(x_2-x_1)^2}{n-s}} \frac{1}{\sqrt{t-n}} e^{-\frac{1}{2}\frac{(x_3-x_2)^2}{t-n}}.$$

b) The conditional density is given by

$$f(x_2|x_1, x_3) = \frac{f(x_1, x_2, x_3)}{f(x_1, x_3)}$$

and like in a) we get

$$f(x_1, x_3) = \frac{1}{\sqrt{s}} e^{-\frac{1}{2}\frac{x_1^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{1}{2}\frac{(x_3-x_1)^2}{t-s}}.$$

Then since $n - s = \frac{t-s}{2} = t - n$ we have

$$f(x_2|x_1, x_3) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x_2 - \frac{x_1 + x_3}{2})^2\right)$$

with $\sigma := \frac{\sqrt{t-s}}{2}$.

Solution to problem 2

First we prove the hint. Let $\Omega := \mathbb{R}^T$, $\mathcal{B}^T := \sigma(\pi_t \,|\, t \in T)$, $\pi_t(x) := x(t)$, $x \in \mathbb{R}^T$ and

$$\mathcal{B}_*^T := \left\{A \subset \mathbb{R}^T \,|\, \exists \,\text{abz. } T(A) \subset T \,\forall x \in \mathbb{R}^T \,\forall y \in A \,:\, (x(t) = y(t) \,\forall t \in T(A) \Rightarrow x \in A)\right\}.$$

Then $\mathcal{B}_*^T$ is a σ-algebra, which contains $\mathcal{B}^T$:

(i) $\mathbb{R}^T \in \mathcal{B}_*^T$ trivial (with an arbitrary countable set $T(A) \subset T$).

(ii) Let $A \in \mathcal{B}_*^T$, i.e. there is a countable set $T(A) \subset T$, such that for all $x \in \mathbb{R}^T$, $y \in A : (x(t) = y(t)$ for all $t \in T(A) \Rightarrow x \in A)$. Now choose $T(A^c) = T(A)$. Then for $x \in \mathbb{R}^T$ and $y \in A^c : (x(t) = y(t)$ for all $t \in T(A) = T(A^c) \Rightarrow x \in A^c)$. Else we would have $x \in A$ and therefore $y \in A$ by the assumption $A \in \mathcal{B}_*^T$, which is a contradiction.

(iii) Let $(A_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{B}_*^T$. Thus there is a sequence of countable sets $T(A_i) \subset T$, s.t. for all $x \in \mathbb{R}^T$, $y \in A_i : (x(t) = y(t)$ for all $t \in T(A_i) \Rightarrow x \in A_i)$. Define $T(\bigcup_{i\in\mathbb{N}} A_i) := \bigcup_{i\in\mathbb{N}} T(A_i)$. then for $x \in \mathbb{R}^T$, $y \in \bigcup_{i\in\mathbb{N}} A_i$: if $x(t) = y(t)$ for all $t \in T(A)$, then there is $i \in \mathbb{N}$ with $x \in A_i$, thus $x \in \bigcup_{i\in\mathbb{N}} A_i$.

Now consider the generating system $\mathcal{E} := \{\pi_t^{-1}(B) \,|\, B \in \mathcal{B}, t \in T\}$ of $\mathcal{B}^T$. Choose $T(\pi_t^{-1}(B)) = \{t\}$, then $\pi_t^{-1}(B) \in \mathcal{B}_*^T$ for all $B \in \mathcal{B}$: If $x \in \mathbb{R}^T$, $y \in \pi_t^{-1}(B)$ and $x(t) = y(t)$, then obviously also $x \in \pi_t^{-1}(B)$ holds. therefore we have $\mathcal{B}^T = \sigma(\mathcal{E}) \subset \mathcal{B}_*^T$.
Now we want to show $\mathcal{C}(T) \notin \mathcal{B}_*^T$, assume $\mathcal{C}(T) \in \mathcal{B}_*^T$. Then there is a countable set $T(\mathcal{C}(T)) \subset T$, s.t. for all $x \in \mathbb{R}^T$, $y \in \mathcal{C}(T) : (x(t) = y(t)$ for all $t \in T(\mathcal{C}(T)) \Rightarrow x \in \mathcal{C}(T))$. Now consider $y \equiv 0 \in \mathcal{C}(T)$. Let $t_0 \in T \setminus T(\mathcal{C}(T))$ (since $T(\mathcal{C}(T))$ is countable, there is such a t_0). Define

$$x(t) := \begin{cases} 0, & t \neq t_0, \\ 1, & t = t_0. \end{cases}$$

x obviously is not continuous, i.e. $x \notin \mathcal{C}(T)$, but for all $t \in T(\mathcal{C}(T))$ we have $x(t) = y(t)$. Therefore sich a set cannot exist.

Solution to problem 3

i) $P(X_0 = 0) = P(B_0^{(1)} = 0) = 1$, since $B^{(1)}$ is a Brownian motion.

ii) On $\mathbb{R}_+ \setminus \mathcal{N}_0$, $(X_t)_{t\geq 0}$ obviously is continuous P-a.s., for $n \in \mathcal{N}_0$ we have:

$$\lim_{t\downarrow n} X_t = \sum_{i=1}^{n} B_1^{(i)} = X_n \text{ a.s.,}$$

since $B_t^{(n+1)}$ is a.s. continuous in 0 and $P(B_0^{(n+1)} = 0) = 1$;

$$\lim_{t\uparrow n} X_t = \sum_{i=1}^{n} B_1^{(i)} = X_n \text{ a.s.,}$$

since $B_t^{(n)}$ is a.s. continuous in 1 and $P(B_0^{(n+1)} = 0) = 1$.

iii) Let $t > s \geq 0$. We have

$$\begin{aligned} X_t - X_s &= \Big(\sum_{i=1}^{\lfloor t\rfloor} B_1^{(i)} + B_{t-\lfloor t\rfloor}^{(\lfloor t\rfloor+1)}\Big) - \Big(\sum_{i=1}^{\lfloor s\rfloor} B_1^{(i)} + B_{s-\lfloor s\rfloor}^{(\lfloor s\rfloor+1)}\Big) \\ &= \Big(B_{t-\lfloor t\rfloor}^{(\lfloor t\rfloor+1)} - B_0^{(\lfloor t\rfloor+1)}\Big) + \Big(\sum_{i=\lfloor s\rfloor+2}^{\lfloor t\rfloor} B_1^{(i)}\Big) + \Big(B_1^{(\lfloor s\rfloor+1)} - B_{s-\lfloor s\rfloor}^{(\lfloor s\rfloor+1)}\Big) \end{aligned}$$

independent of

$$\mathcal{F}_s = \sigma(X_u \,|\, u \leq s) = \sigma(B_t^{(i)}, 1 \leq i \leq \lfloor s\rfloor, 0 \leq t \leq 1,\; B_t^{(\lfloor s\rfloor+1)}, 0 \leq t \leq s - \lfloor s\rfloor),$$

since the $B^{(i)}$, $i \geq \lfloor s\rfloor + 2$, are independent of $\mathcal{F}_s$ because the $B^{(i)}$ are family of independent processes, $B_1^{(\lfloor s\rfloor+1)} - B_{s-\lfloor s\rfloor}^{(\lfloor s\rfloor+1)}$ is independent of $\mathcal{F}_s$ because of the independence of $B^{(\lfloor s\rfloor+1)}$ of every $B^{(i)}$, $i = 1, ..., \lfloor s \rfloor$ and independence of $B_1^{(\lfloor s \rfloor+1)} - B_{s-\lfloor s \rfloor}^{(\lfloor s \rfloor+1)}$ of $\sigma(B_t^{(\lfloor s \rfloor+1)}, 0 \leq t \leq s - \lfloor s \rfloor)$, since $B^{(\lfloor s \rfloor+1)}$ is a Brownian motion.

Since we now have a representation of $X_t - X_s$ as a sum of i.i.d. normally distributed random variables, $X_t - X_s$ is normally distributed and the parameters are:

$$\begin{aligned} E(X_t - X_s) &= 0, \\ Var(X_t - X_s) &= t - \lfloor t \rfloor + \sum_{i=\lfloor s \rfloor+2}^{\lfloor t \rfloor} 1 + (1 - (s - \lfloor s \rfloor)) \\ &= t - \lfloor t \rfloor + (\lfloor t \rfloor - (\lfloor s \rfloor + 1)) + (1 - (s - \lfloor s \rfloor)) \\ &= t - s. \end{aligned}$$

Solution to problem 4

1. $\tilde{B}_t := -B_t$, $t \geq 0$ is obviously measurable. $P(\tilde{B}_0 = 0) = P(B_0 = 0) = 1$. P-a.s. continuity of the paths of $(B_t)_{t \geq 0}$ implies P-a.s. continuity of the paths of $(\tilde{B}_t)_{t \geq 0}$. Finally for $t > s \geq 0$: $\tilde{B}_t - \tilde{B}_s = -(B_t - B_s)$ is independent of $\mathcal{F}_s$ and $\mathcal{N}(0, t-s)$-distributed.

2. $\tilde{B}_t := B_{a+t} - B_a$, $t, a \geq 0$. This process is obviously measurable w.r.t. the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ with $\tilde{\mathcal{F}}_t := \mathcal{F}_{t+a}$. We have $P(\tilde{B}_0 = 0) = P(B_a = B_a) = 1$. Obviously like $(B_t)_{t \geq 0}$ also $(B_{a+t} - B_a)_{t \geq 0}$ has a.s. continuous paths. For $t > s \geq 0$: $\tilde{B}_t - \tilde{B}_s = B_{a+t} - B_{a+s}$ is independent of $\mathcal{F}_{a+s} = \tilde{\mathcal{F}}_s$ and $\mathcal{N}(0, (a+t) - (a+s)) = \mathcal{N}(0, t-s)$-distributed.

3. $\tilde{B}_t := cB_{t/c^2}$, $t \geq 0$, $c \neq 0$. This process is obviously measurable w.r.t. the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ with $\tilde{\mathcal{F}}_t := \mathcal{F}_{t/c^2}$. We have $P(\tilde{B}_0 = 0) = P(cB_0 = 0) = 1$. Obviously like $(B_t)_{t \geq 0}$ also $(cB_{t/c^2})_{t \geq 0}$ has a.s. continuous paths. For $t > s \geq 0$: $\tilde{B}_t - \tilde{B}_s = c(B_{t/c^2} - B_{s/c^2})$ is independent of $\mathcal{F}_{s/c^2} = \tilde{\mathcal{F}}_s$ and $\mathcal{N}(0, c^2(t-s)/c^2) = \mathcal{N}(0, t-s)$-distributed.