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Stochastic Processes
Solutions to the 10th Problem sheet

Solution to problem 1

i) The system is normalized:
$||g_{01}|| = 1$ is clear, let $(n,k) \in S$ with $n \geq 1$:

$$||g_{nk}|| = \int_0^1 2^{n-1}(1_{[(k-1)2^{-n},k2^{-n})}(t) + 1_{[k2^{-n},(k+1)2^{-n})}(t))dt = 2 \cdot 2^{-n} \cdot 2^{n-1} = 1$$

ii) The system is orthogonal:
Let $(n,k),(m,l) \in S$, $(n,k) \neq (m,l)$:

1. $m = n$
Then $l \neq k$ and since both numbers are odd by definition of S, we even have $|l-k| \geq 2$. Thus $[(k-1)2^{-n},(k+1)2^{-n}) \cap [(l-1)2^{-n},(l+1)2^{-n}) = \emptyset$, i.e. $g_{nk} \cdot g_{ml} \equiv 0$ and therefore $\langle g_{nk}, g_{ml}\rangle = 0$.
2. $m \neq n$
WLOG assume $m > n$ and $[(k-1)2^{-n},(k+1)2^{-n}) \cap [(l-1)2^{-m},(l+1)2^{-m}) \neq \emptyset$ (otherwise we're done as in part 1).
Define $j := \lfloor \frac{l-1}{2^{m-n}} \rfloor$. Since l is odd, we have $\frac{l-1}{2} \in \mathbb{N}$ and therefore $j \geq \frac{l-1}{2^{m-n}} - \left(1 - \frac{1}{2^{m-n-1}}\right)$ (for $p,q \in \mathbb{N}: \lfloor \frac{p}{q} \rfloor \geq \frac{p}{q} - (1-\frac{1}{q})$).
Rearranging this equation yields $\frac{l+1}{2^{m-n}} \leq j+1$ and therefore $\frac{l-1}{2^{m-n}}, \frac{l+1}{2^{m-n}} \in [j, j+1]$. Thus $[(l-1)2^{-m},(l+1)2^{-m}) \subset [(k-1)2^{-n},k2^{-n}]$ <u>or</u> $[(l-1)2^{-m},(l+1)2^{-m}) \subset [k2^{-n},(k+1)2^{-n}]$. This means that g_{nk} is constant (either 1 or -1) almost everywhere on $\{g_{ml} \neq 0\}$ and $g_{ml} \perp g_{nk}$.

Solution to problem 2

(a) Consider the function $F : [0,1] \to \mathbb{R}$, $t \mapsto \int_0^t f(s)\,ds$. Since $f \perp g_{01} \equiv 1$, we have that $F(0) = F(1) = 0$. Furthermore $f \perp g_{11}$ implies $\int_0^{1/2} f(x)\,dx = \int_{1/2}^1 f(x)\,dx$, thus by what we have shown above $F(1/2) = 0$. then with $f \perp g_{12}, g_{32}$ we deduce $F(1/4) = F(3/4) = 0$ and inductively $F(t) = 0$ in all points $t = k2^{-n}$, $(n,k) \in S$. Now F is obviously continuous, thus $F \equiv 0$ and therefore $f = F' = 0$ almost everywhere.

(b) For all $(i,j) \in S$:

$$\begin{aligned} & \Big\langle f - \sum_{(n,k)\in S} \langle f, g_{nk}\rangle g_{nk}, g_{ij} \Big\rangle \\ = \; & \langle f, g_{ij}\rangle - \sum_{(n,k)\in S} \langle f, g_{nk}\rangle \underbrace{\langle g_{nk}, g_{ij}\rangle}_{\delta_{\{nk=ij\}}} \\ = \; & \langle f, g_{ij}\rangle - \langle f, g_{ij}\rangle \; = \; 0, \end{aligned}$$

thus together with part a) the equation we had to prove.

Solution to problem 3

(a) Continuity of the paths (a.s.) is obvious. thus we have to check the martingale property. Let $t > s \geq 0$, $\alpha > 0$.

$$\begin{aligned}
E[X_t \,|\, \mathcal{F}_s] &= E\big[\exp\big(\alpha B_t - \alpha^2 \tfrac{t}{2}\big) \,|\, \mathcal{F}_s\big] \\
&= \exp\big(-\alpha^2 \tfrac{t}{2}\big) \cdot E\big[\exp\big(\alpha(B_t - B_s)\big) \exp\big(\alpha B_s\big) \,|\, \mathcal{F}_s\big] \\
&= \exp\big(-\alpha^2 \tfrac{t}{2}\big) \cdot E \exp\big(\alpha(B_t - B_s)\big) \cdot \exp\big(\alpha B_s\big) \\
&= \exp\big(-\alpha^2 \tfrac{t}{2}\big) \cdot \varphi_{\mathcal{N}(0,t-s)}(-i\alpha) \cdot \exp\big(\alpha B_s\big) \\
&= \exp\big(-\alpha^2 \tfrac{t}{2}\big) \cdot \exp\big(-\tfrac{1}{2}(t-s)(-i\alpha)^2\big) \cdot \exp\big(\alpha B_s\big) \\
&= \exp\big(\alpha B_s + \tfrac{1}{2}\alpha^2(t-s) - \alpha^2 \tfrac{t}{2}\big) \\
&= \exp\big(\alpha B_s - \alpha^2 \tfrac{s}{2}\big) \;=\; X_s.
\end{aligned}$$

(b) For all $\alpha > 0$:

$$\begin{aligned}
P(M_t \geq z) &= P\big(\sup_{0 \leq s \leq t} B_s \geq z\big) \;=\; P\big(\sup_{0 \leq s \leq t} \exp\big(\alpha B_s - \frac{\alpha^2 s}{2} + \frac{\alpha^2 s}{2}\big) \geq \exp(\alpha z)\big) \\
&\leq P\big(\sup_{0 \leq s \leq t} X_s \cdot \exp\big(\frac{\alpha^2 t}{2}\big) \geq \exp(\alpha z)\big) \\
&= P\big(\sup_{0 \leq s \leq t} X_s \geq \exp\big(\alpha z - \frac{\alpha^2 t}{2}\big)\big) \\
&\leq \exp\big(\frac{\alpha^2 t}{2} - \alpha z\big) \cdot E\exp\big(\alpha B_t - \frac{\alpha^2 t}{2}\big)\big), \quad \text{hint with } p = 1, \\
&= \exp\big(\frac{\alpha^2 t}{2} - \alpha z\big) \cdot E\exp\big(\alpha B_0 - \frac{\alpha^2 0}{2}\big)\big), \quad \text{martingale,}
\end{aligned}$$

now since α was arbitrary, we can minimize w.r.t. α, (minimum occurs in $\alpha = z/t (> 0)$) and get

$$P(M_t \geq z) \leq \exp\big(-\frac{z^2}{2t}\big).$$

(c) Let $T_a := \inf\{t > 0 : B_t = a\}$. Then:

$$tP(T_a > t) = tP(M_t < a) = t(1 - P(M_t \geq a)) \geq t \cdot \Big(1 - \exp\big(-\frac{a^2}{2t}\big)\Big).$$

Thus

$$\begin{aligned}
\liminf_{t\to\infty} tP(T_a > t) &\geq \lim_{t\to\infty} t \cdot \Big(1 - \exp\big(-\frac{a^2}{2t}\big)\Big) \\
&= \lim_{s\to 0} \frac{1 - \exp(-\alpha^2 s/2)}{s} \;=\; \frac{\alpha^2}{2} \;>\; 0.
\end{aligned}$$

In particular, there is $t_0 > 0$, s.t. for all $t \geq t_0$

$$P(T_a > t) \geq \frac{\alpha^2}{4t}.$$

Thus

$$ET_a = \int_0^\infty P(T_a > t)\,dt \geq \underbrace{\int_0^{t_0} P(T_a > t)\,dt}_{\geq 0} + \frac{\alpha^2}{4} \underbrace{\int_{t_0}^\infty \frac{1}{t}\,dt}_{=\infty} = \infty.$$

Therefore $P(T_a < \infty) = 1$, but $ET_a = \infty$.

Solution to problem 4
With the hint we get

$$\begin{aligned}\limsup_{t\to\infty} t^{-\gamma} B_t &= \limsup_{t\to\infty}\ t^{\frac{1}{2}-\gamma} \sqrt{\log\log t} \frac{B_t}{\sqrt{t \log\log t}} \\ &\leq \sqrt{2} \cdot \limsup_{t\to\infty} \frac{\sqrt{\log\log t}}{t^{\gamma - \frac{1}{2}}} = 0,\end{aligned}$$

since $\gamma - 1/2 > 0$. Analoguously we deduce $\liminf_{t\to\infty} t^{-\gamma} B_t \geq 0$. This shows the assertion for $\gamma > 1/2$.