Karlsruhe Institute of Technology

Institute for Stochastics

Prof. Dr. N. Bäuerle · Dipl.-Math. oec. D. Joos

Stochastic Processes
Solutions to the 11th Problem sheet

Solution to problem 1

1. By Markov's inequality with $g(x) := x^2$, for all $\epsilon > 0$

$$P(|X_n - X| > \epsilon) \leq \frac{E|X_n - X|^2}{\epsilon^2}$$

which proves the implication.

2. The implication does not hold, counterexample:
Let $\Omega := [0,1]$, $\mathcal{F} := \mathcal{B}_{[0,1]}$, $P := \lambda_{[0,1]}$. For $n \in \mathbb{N}\ \exists_1 k \in \mathbb{N} \exists_1 j \in \mathbb{N}, 0 \leq j < 2^k : n = 2^k + j$. Define $X_n := 1_{A_n}$ with $A_n := \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)$. Then $EX_n^2 = 2^{-k}$ and since with $n \to \infty$ we also have $k \to \infty$:

$$X_n \overset{L^2}{\to} 0$$

But since $P(\lim_{n\to\infty} X_n = 0) = 0$ we obviously do not have $X_n \to 0$ P-a.s..

3. The implication does not hold, counterexample:
On the same probability space as in 2, define $X_n := n \cdot 1_{A_n}$ with $A_n := [0, \frac{1}{n})$. Then $P(\lim_{n\to\infty} X_n = 0) = P((0,1]) = 1$ and therefore

$$X_n \to 0 \quad \text{P-a.s.}$$

But $EX_n^2 = n^2 \cdot \frac{1}{n} = n$, thus we do not have $X_n \overset{L^2}{\to} 0$.

Solution to problem 2

As in the proof of Theorem 12.10

$$V_{\mathcal{Z}_n} := \sum_{j=1}^{k_n} \left(B_{t_{n,j}} - B_{t_{n,j-1}}\right)^2,$$

and we have

$$E(V_{\mathcal{Z}_n} - t) = 0$$

and

$$Var(V_{\mathcal{Z}_n} - t) = 2 \sum_{j=1}^{k_n} \left(t_{n,j} - t_{n,j-1}\right)^2.$$

Inserting the partition we get

$$Var(V_{\mathcal{Z}_n} - t) = 2 \sum_{j=1}^{2^n} (2^{-n} t)^2 = t^2 2^{1-n}.$$

The Chebyshev inequality now yields

$$\sum_{n=1}^{\infty} P(|V_{\mathcal{Z}_n} - t| > \epsilon) \leq \sum_{n=1}^{\infty} \frac{1}{\epsilon^2} t^2 2^{1-n} = 2\frac{t^2}{\epsilon^2} < \infty \text{ for all } \epsilon > 0.$$

Together with the hint this proves the assertion.

Solution to problem 3
Let $[a,b] \subset \mathbb{R}_+$ with $a < b$. For all $n \in \mathbb{N}$ we have

$$A^+ := \{\omega \in \Omega : t \mapsto B_t(\omega) \text{ is monotonely increasing on } [a,b]\} \subset A_n^+$$

with

$$A_n^+ := \Big\{\omega \in \Omega : B\Big(a + \frac{i(b-a)}{n}, \omega\Big) - B\Big(a + \frac{(i-1)(b-a)}{n}, \omega\Big) \geq 0,\ i = 1, ..., n\Big\}.$$

Since we have normal distributions with expectation 0, we can deduce $P(A_n^+) = 2^{-n}$ for all $n \in \mathbb{N}$ and therefore $P(A^+) = 0$. Analoguously we get $P(A^-) = 0$ for $A^- := \{\omega \in \Omega : t \mapsto B_t(\omega)$ is monotonely decreasing on $[a,b]\}$. That means $P(B$ monotone on $[a,b]) = 0$.
Now for every $a < b$ we can choose $p, q \in \mathbb{Q}$ with $p < q$ and $[p,q] \subset [a,b]$. Thus

$$P(\exists [a,b] \subset \mathbb{R}_+ : B \text{ monotone on } [a,b]) \leq$$

$$P(\exists [p,q] \subset \mathbb{R}_+, p, q \in \mathbb{Q}_+ : B \text{ monotone on } [p,q]) \leq$$

$$\sum_{p,q \in \mathbb{Q}_+} \underbrace{P(B \text{ monotone on } [p,q])}_{=0} = 0, \quad \text{(countably many)}$$

Solution to problem 4
Lemma *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F}$ a sub-σ-algebra of $\mathcal{A}$, $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ measurable spaces, $X : \Omega \to \Omega_1$ and $Y : \Omega \to \Omega_2$ $(\mathcal{A}, \mathcal{A}_1)$-/ $(\mathcal{F}, \mathcal{A}_2)$-measurable mappings respectively, $\phi : \Omega_1 \times \Omega_2 \to \mathbb{R}$ a $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B})$-measurable mapign satisfying $E|\phi(X,Y)| < \infty$.*
Then, if X and $\mathcal{F}$ are independent, $\psi(Y)$ defined by

$$\psi : \Omega_2 \to \mathbb{R}, \quad \omega_2 \mapsto E\phi(X, \omega_2)$$

is a version of the conditional expectation of $\phi(X,Y)$ under $\mathcal{F}$.

We have to show that

$$\mathbb{P}[X_{t+s} \in A | \mathcal{F}_t] = P_s(X_t, A)$$

written in terms of conditional expectation this is

$$E[1_A(X_{t+s}) | \mathcal{F}_t] = P_s(X_t, A).$$

By definition this means that for all $F \in \mathcal{F}_t$

$$\int_F 1_A(X_{t+s})\, d\mathbb{P} = \int_F P_s(X_t, A)\, d\mathbb{P}.$$

Now taking into account that we can write

$$P_s(X_t, A) = \int 1_A(y) P_s(X_t, dy)$$

we have to show that

$$\int 1_A(y) P_s(X_t, dy).$$

is a version of

$$E[1_A(X_t + (X_{t+s} - X_t))|\mathcal{F}_t].$$

Obviously this process is $\mathcal{F}_t$-measurable. Furthermore X_t and $X_{t+s} - X_t$ are independent. Applying the Lenmma stated above, $\Psi(X_t)$ with $\Psi(x) := E1_A(x + X_{t+s} - X_t)$ is a version of $E[1_A(X_t + (X_{t+s} - X_t))|\mathcal{F}_t]$. Since $X_{t+s} - X_t$ and X_s have the same distribution, we have that

$$\begin{aligned}\Psi(x) &= E1_A(x + X_s) \\ &= \int 1_A(x+y) \, \mathrm{CompP}(\lambda s, \mu)(dy) \\ &= \int 1_A(y) P_s(x, dy).\end{aligned}$$

In order to determine the generator, let f be a bounded function. We have to calculate

$$\lim_{t \to 0} \frac{1}{t}\big(P_t f - f\big)(x)$$

for all $x \in \mathbb{R}$. We have

$$P_t f(0) = \int f(y) P_t(0, dy) = Ef\big(\sum_{k=1}^{N_t} Y_k\big) = Ef(X_t),$$

since $P_t(0, A) = \mathrm{CompP}(\lambda t, \mu)(A)$. Furthermore

$$Ef(X_t) = Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t=0\}} + Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t=1\}} + Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t>1\}}$$

We have

$$Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t=0\}} = Ef(0) 1_{\{N_t=0\}} = f(0) P(N_t = 0) = f(0) e^{-\lambda t},$$

$$Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t=1\}} = Ef(Y_1) 1_{\{N_t=1\}} = Ef(Y_1) P(N_t = 1) = Ef(Y_1) \lambda t e^{-\lambda t}$$

and

$$\big|Ef\big(\sum_{k=1}^{N_t} Y_k\big) 1_{\{N_t>1\}}\big| \le \|f\|_\infty \cdot P(N_t > 1) = \mathcal{O}(t^2).$$

Therefore

$$\frac{1}{t}\big(P_t f - f\big)(0) = \frac{1}{t} f(0)\big(e^{-\lambda t} - 1\big) + Ef(Y_1)\, \frac{\lambda t e^{-\lambda t}}{t} + \mathcal{O}(t),$$

thus

$$\lim_{t \to 0} \frac{1}{t}\big(P_t f - f\big)(0) = -\lambda f(0) + \lambda E f(Y_1) = \lambda \int \big(f(y) - f(0)\big)\, \mu(dy).$$

Now we have

$$P_t f(x) = P_t f_x(0),$$

where $f_x(y) := f(y + x)$, i.e. in general

$$Gf(x) = \lambda \int \big(f(y+x) - f(x)\big)\, \mu(dy).$$