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Stochastic Processes
Solutions to the 12th Problem sheet

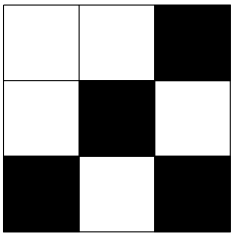
Solution to problem 1

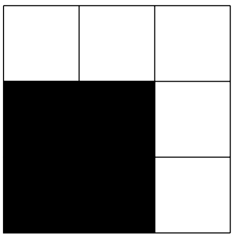
We use example 5.7 with $N = 3$ and $L = 2$. Since the probability to choose a panel of a certain colour only depends on the number of panels of that colour and not on the location of those panels, with the same argument as in the lecture we prove that $(Y_n)_{n\in\mathbb{N}_0}$ is a martingale w.r.t to the filtration $(\sigma(Y_0, \dots, Y_n))$.

Now assume that Y_n is a Markov proces w.r.t. $(\sigma(Y_0, \dots, Y_n))$. Then there is $p = p_{45} \in [0, 1]$ with

$$p = P(Y_{n+1} = 5|Y_n = 4).$$

Now consider the following two cases:

1. 

2. 

In the first example, the probability to go from 4 to 5 black panels is

$$p = \frac{1}{9} \cdot \frac{1}{4} \cdot (2 + 2 + 2 + 3 + 3) = \frac{1}{3}.$$

In the second expample we get

$$p = \frac{1}{9} \cdot \frac{1}{4} \cdot (2 + 2 + 0 + 2 + 2) = \frac{2}{9},$$

which is a contradiction, therefore Y cannot be a Markov chain.

Solution to problem 2

(a) Theorem 12.1 implies $P(\tau < \infty) = 1$. Since B is a martingale with continuous paths, the hint implies $EB_{\tau\wedge n} = EB_0 = 0$ for all $n \in \mathbb{N}$. Furthermore, $B_{\tau\wedge n} \to B_\tau$ a.s. for $n \to \infty$ and $\max\{-a, b\}$ is an integrable majorant to $B_{\tau\wedge n}$, thus:

$$aP(B_\tau = a) + bP(B_\tau = b) = EB_\tau = \lim_{n\to\infty} EB_{\tau\wedge n} = 0.$$

With $P(B_\tau = a) + P(B_\tau = b) = 1$ the assertion follows.

(b) Continuous paths are obvious, thus we have to show the martingale property: For all $s, t \geq 0$ we have:

$$\begin{aligned} E[M_{t+s} - M_t|\mathcal{F}_t] &= E[-B_{t+s}^2 + B_t^2 + (a+b)B_{t+s} - (a+b)B_t + s|\mathcal{F}_t] \\ &= -E[(B_{t+s}^2 - (t+s)) - (B_t^2 - t)|\mathcal{F}_t] + (a+b)E[B_{t+s} - B_t|\mathcal{F}_t] \\ &= 0, \end{aligned}$$

since $(B_t)_{t\geq 0}$ and $(B_t^2 - t)_{t\geq 0}$ are martingales. To prove that $(B_t^2 - t)_{t\geq 0}$ is in fact a martingale, let $s, t \geq 0$:

$$\begin{aligned} E[B_{t+s}^2 - (t+s) - B_t^2 + t|\mathcal{F}_t] &= E[B_{t+s}^2 - B_t^2|\mathcal{F}_t] - s \\ &= E[(B_{t+s} - B_t)(B_{t+s} - B_t + 2B_t)|\mathcal{F}_t] - s \\ &= \underbrace{E(B_{t+s} - B_t)^2}_{=s} + 2B_t \underbrace{E(B_{t+s} - B_t)}_{=0} - s \\ &= 0. \end{aligned}$$

With the hint we now get:

$$EM_{\tau\wedge n} = EM_0 = -ab,$$

on the other hand

$$EM_{\tau\wedge n} = E(B_{\tau\wedge n} - a)(b - B_{\tau\wedge n}) + E\tau \wedge n.$$

Now for the first term on the right hand side use the dominated convergence theorem and for the second one the monotone convergence theorem in order to get:

$$-ab = E\tau + \underbrace{E(B_\tau - a)(b - B_\tau)}_{=0}.$$

Solution to problem 3

a) For any $(\mathcal{A}, \mathcal{B})$-measurable function Y and all $0 \leq t_1 < t_2$ mappings of the form

$$X : (t, \omega) \mapsto Y(\omega) \cdot 1_{[t_1,t_2)}(t)$$

are $(\mathcal{B}_{[0,\infty)} \otimes \mathcal{A}, \mathcal{B})$-measurable, since

$$\{X \geq a\} = \begin{cases} [t_1, t_2) \times \{y \geq a\}, & a > 0, \\ [t_1, t_2) \times \{y \geq a\} \cup [t_1, t_2)^c \times \Omega, & a \leq 0, \end{cases}$$

and thus $\{X \geq a\} \in \mathcal{B}_{[0,\infty)} \otimes \mathcal{A}$ for all $a \in \mathbb{R}$.

Now we have

$$B_t(\omega) = \lim_{n\to\infty} \sum_{k=0}^{n2^n} 1_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}(t) B_{\frac{k}{2^n}}(\omega)$$

because of the continuity, thus B as limit of measurable functions is itself measurable.

b) By part a) we have $L'_a := \{(t, \omega) \,|\, B_t(\omega) = a\} \in \mathcal{B}_{[0,\infty)} \otimes \mathcal{A}$. Thus $L_a(\omega)$ is λ^1-mesasurable and with Fubini II we have:

$$E\lambda^1(L_a) = E \int_0^\infty 1_{\{B_t=a\}}\, dt = \int_0^\infty E 1_{\{B_t=a\}}\, dt = \int_0^\infty P(B_t = a)\, dt = 0,$$

thus, since $\lambda^1(L_a)$ is non-negative, $\lambda^1(L_a) = 0$.

Solution to problem 4

Let f be bounded, $i \in S$ and $t > 0$, then

$$\frac{1}{t}(P_t f - f)(i) = \sum_{j \in S} \frac{1}{t}(f(j) - f(i))p_{ij}(t) = \sum_{j \in S} \frac{1}{t}(f(j) - f(i))(p_{ij}(t) - \delta_{ij}(t)).$$

Since f is bounded, we have

$$\lim_{t \to 0} \frac{1}{t}(P_t f - f)(i) = \sum_{j \in S} \lim_{t \to 0}(\frac{1}{t}(f(j) - f(i))(p_{ij}(t) - \delta_{ij}(t))) = \sum_{j \in S}(f(j) - f(i))q_{ij}$$