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Stochastic Processes
Solutions to the 13th Problem sheet

Solution to problem 1
Since B_s is $\mathcal{N}(0,s)$-distributed, we have

$$F_{|B_s|}(x) = P(|B_s| \leq x) = P(-x \leq B_s \leq x) = \int_{-x}^{x} \frac{1}{\sqrt{s}}\varphi(\frac{y}{\sqrt{s}})dy$$

Differentiating w.r.t. x we get the density

$$f_{|B_s|}(x) = \frac{1}{\sqrt{s}}\varphi(\frac{x}{\sqrt{s}}) - (-\frac{1}{\sqrt{s}}\varphi(-\frac{x}{\sqrt{s}})) = 2\frac{1}{\sqrt{s}}\varphi(\frac{x}{\sqrt{s}}).$$

Solution to problem 2
We have

$$\rho(s,t) = P(\exists u \in (s,t) : B_u = 0) = P(B_s > 0, \exists u \in (s,t) : B_u = 0) + P(B_s < 0, \exists u \in (s,t) : B_u = 0)$$
$$= P(B_s > 0, \min_{s\leq u\leq t} B_u < 0) + P(B_s < 0, \max_{s\leq u\leq t} B_u > 0).$$

Since $B_s \stackrel{d}{=} -B_s$ and $\min B_u = -\max -B_u \stackrel{d}{=} -\max B_u$, both probabilities are the same and we get

$$\rho(s,t) = 2P(B_s > 0, \min_{s\leq u\leq t} B_u < 0) = 2E(1_{(0,\infty)}(B_s)1_{(-\infty,0)}(\min_{s\leq u\leq t} B_u)).$$

Now we decompose the second indicator funtion w.r.t. the value of B_s:

$$\rho(s,t) = 2E(1_{(0,\infty)}(B_s)\underbrace{E[1_{(-\infty,-B_s)}(\min_{s\leq u\leq t}(B_u - B_s))|\mathcal{F}_s]}_{=:\psi(B_s)})$$

Let $(\tilde{B}_u)_{u\geq 0}$ with $\tilde{B}_u := B_{u+s} - B_s$ denote the post-s-process, then

$$\psi(x) = E1_{(-\infty,x)}(\min_{0\leq u\leq t-s} \tilde{B}_u) = P(\max_{0\leq u\leq t-s} \tilde{B}_u > x) = \int_x^\infty \frac{2}{\sqrt{t-s}}\varphi(\frac{v}{\sqrt{t-s}})dv = 2\int_{x/\sqrt{t-s}}^\infty \varphi(y)dy$$

where φ denotes the density of a $\mathcal{N}(0,1)$-distributed random variable. Thus

$$\rho(s,t) = 2E(1_{(0,\infty)}(B_s)\psi(B_s)) = 2\int_0^\infty \psi(x)P^{B_s}(dx) = 2\int_0^\infty \psi(x)\frac{1}{\sqrt{s}}\varphi(\frac{x}{\sqrt{s}})dx$$
$$= 2\int_0^\infty \psi(z\sqrt{s})\varphi(z)dz = 2\int_0^\infty 2\int_{z\sqrt{s}/\sqrt{t-s}}^\infty \varphi(y)dy\varphi(z)dz.$$

This means that

$$\rho(s,t) = 4P(Y > \frac{\sqrt{s}}{\sqrt{t-s}}Z > 0) = 2P(\frac{Y}{Z} > \frac{\sqrt{s}}{\sqrt{t-s}})$$

for two independent $\mathcal{N}(0,1)$-distributed random variables Y and Z, the last equation follows by symmetry. Now $\frac{Y}{Z}$ has a Cauchy-distribution, therefore we get

$$\rho(s,t) = 2(1-(\frac{1}{\pi}\arctan(\frac{\sqrt{s}}{\sqrt{t-s}})+\frac{1}{2})) = \frac{2}{\pi}(\frac{\pi}{2}-\arctan(\frac{\sqrt{s}}{\sqrt{t-s}})) = \frac{2}{\pi}\arccos(\sqrt{\frac{s}{t}})$$

since $\arccos(x) = \frac{\pi}{2} - \arctan(\frac{x}{\sqrt{1-x^2}})$.

Remark: The equation

$$E(1_{(0,\infty)}(B_s)1_{(-\infty,0)}(\min_{s\le u\le t} B_u)) = E(1_{(0,\infty)}(B_s)E[1_{(-\infty,-B_s)}(\min_{s\le u\le t}(B_u - B_s))|\mathcal{F}_s]$$

can be easily proved with the formula

$$E(XY) = E(E[XY|\mathcal{F}]) = E(XE[Y|\mathcal{F}])$$

for all $\mathcal{F}$-measurable random variables X (in this case $\mathcal{F} = \mathcal{F}_s$ and $X = 1_{(0,\infty)}(B_s)$).

The interpretation is as follows: We decompose a random variable Y w.r.t. the value of another random variable X. In the discrete case this can be proved more intuitively:

$$E(XY) = \sum_{x,y} xyP(X=x, Y=y) = \sum_{x,y} xyP(Y=y|X=x)P(X=x)$$

$$= \sum_{x} P(X=x)\left(x\sum_{y} yP(Y=y|X=x)\right) = \sum_{x} P(X=x)\left(xE[Y|X=x]\right) = E(XE[Y|X])$$

with $x \in \{x \in X(\Omega) : P(X=x) > 0\}$ and $y \in Y(\Omega)$.