COMPLETE MARKETS DO NOT ALLOW FREE CASH FLOW STREAMS

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Abstract. In this short note we prove a conjecture posed in [7]: Dynamic mean-variance problems in arbitrage-free, complete financial markets do not allow free cash flows. Moreover, we show by investigating a benchmark problem that this effect is due to the performance criterion and not due to the time inconsistency of the strategy.

Key words: Mean-Variance Problem, Time-Inconsistency, Market Completeness, Benchmark Problem

1. Introduction

Time inconsistency of the optimal dynamic mean-variance portfolio strategy is a topic which has widely been discussed recently (see among others [1, 11, 7]). It refers to the fact that the optimal portfolio strategy does not satisfy the principle of optimality and the initial capital is always part of the optimal portfolio strategy. In [7] the authors show that an investor behaves irrationally under this strategy by constructing a portfolio which yields the same mean and variance but allows with positive probability to withdraw some money out of the market. Such a strategy is constructed in a discrete T-period, incomplete financial market. The authors also show that such a construction is not possible if the market is complete for the first $T-1$ periods by using the specific construction of a complete discrete market. At the end, the authors conjecture that the appearance of free cash flows is related to market completeness.

In this short note we show for a very general financial market and with very simple ideas that this conjecture is true. For this purpose we stay within the framework of self-financing strategies and assume that instead of withdrawing the money out of the market we reinvest it in the bond and consume it only at the final time point T. Moreover, by investigating a benchmark problem we also show that the appearance of free cash flows is not related to the time inconsistency but only to the optimization criterion which is not consistent with the first order stochastic dominance.

The outline of the paper is as follows: In Section 2 we introduce our general financial market. In section 3 we prove that arbitrage-free, complete markets do not allow free cash flows and in Section 4 we finally show that the same is true for optimal portfolio strategies in benchmark problems which are time consistent.

2. The Model

We consider a very generic frictionless financial market on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ up to time horizon $T > 0$. The market consists of d stocks with price processes (S_t^k), $k = 1, \dots, d$ and one bond with price process (S_t^0). We make no specific assumption about the price processes of the stocks. As far as the bond is concerned we assume that the price process is deterministic, non-decreasing in time and strictly positive. Further we assume that trading in the market is well-defined and we denote by $(\pi^0, \pi) = (\pi_t^0, \pi_t) = (\pi_t^0, \pi_t^1, \dots, \pi_t^d)$ a self-financing trading strategy where we have in mind that π_t^k is the amount of money invested in the k-th asset at time t. Note that due to the self-financing condition it is enough to specify the investment process π in the stocks. We assume that π is adapted to the filtration $(\mathcal{F}_t)$ generated by (S_t^k)

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for $k = 1, \ldots, d$. The corresponding wealth process is denoted by (X_t^π). Trading may be in continuous or discrete time. By $\mathcal{A} := \{\pi : \pi \text{ is self-financing, } X_T^\pi \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ we denote all *admissible strategies*. We make now the following crucial assumptions:

(A1) The mapping $\pi \mapsto X_t^\pi$ is linear, i.e. $X_t^{\alpha_1\pi_1+\alpha_2\pi_2} = \alpha_1 X_t^{\pi_1} + \alpha_2 X_t^{\pi_2}$ for all $t \in [0,T]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

(A2) The financial market is free of arbitrage, i.e. there are no trading strategies $\pi \in \mathcal{A}$ s.t. $X_0^\pi = 0$, $X_T^\pi \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(X_T^\pi > 0) > 0$.

(A3) The financial market is complete, i.e. for every $H \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, there is a trading strategy $\pi \in \mathcal{A}$ s.t. $H = X_T^\pi$.

Examples of financial markets satisfying these assumptions are given next.

Example 2.1 (Black-Scholes Market)**.** A standard Black-Scholes-Market is a continuous-time financial market, given by one bond with price process

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1,$$

where $r \geq 0$ is deterministic and d stocks with price processes given by

$$dS_t^k = S_t^k \Big(b_k dt + \sum_{j=1}^d \sigma_{kj} dW_j(t) \Big), \quad S_0^k = 1,$$

where $(W_1(t), \ldots, W_d(t))$ is a d-dimensional Brownian motion and the drift vector $b = (b_1, ..., b_d)$ and the volatility matrix $(\sigma_{kj})_{k,j=1,\ldots,d}$ are also deterministic. When

$$\xi^\top \sigma \sigma^\top \xi \geq \delta \xi^\top \xi, \quad \text{for all } \xi \in \mathbb{R}^d$$

holds for a $\delta > 0$, then the market is free of arbitrage and complete. (A1) is clearly also satisfied since

$$\frac{X_t^\pi}{S_t^0} = x_0 + \int_0^t \pi_t^\top d\Big(\frac{S_t}{S_t^0}\Big).$$

Example 2.2 (Cox-Ross-Rubinstein Model)**.** The CRR model is a discrete time financial market, given by one bond with price process

$$S_k^0 = (1+i)^k, \quad k = 0, 1, \ldots, N$$

where $i \geq 0$ is deterministic and one stock with price processes given by $S_0 = 1$

$$S_k = Y_1 \cdot \ldots \cdot Y_k, \quad k = 1, \ldots, N$$

where $Y_1, Y_2, \ldots$ are iid random variables taking values in $\{d, u\}$ with $0 < d < u$. When $d < 1 + i < u$ then the market is free of arbitrage and complete. (A1) is also satisfied.

The following Mean-Variance problem has been considered extensively in the last decade:

$$(MV) \quad \begin{cases} \operatorname{Var}[X_T^\pi] \to \min \\ \mathbb{E}[X_T^\pi] = \mu \\ X_0^\pi = x_0, \pi \in \mathcal{A}, \end{cases}$$

where we assume that $x_0 S_T^0 < \mu$. Otherwise a capital of at least μ can be obtained by investing in the bond only which yields zero variance.

We denote the minimal value of this problem by $V(x_0, \mu)$. The solution of the problem is well-known and we do not repeat it here. For solutions in the continuous Black-Scholes setting see [10, 9, 14], in the discrete setting see [12, 3] and in a general semimartingale market see [6].

A first simple observation is the next proposition:

Proposition 2.3. *The mapping $x \mapsto V(x, \mu)$ is strictly decreasing for all $x < (S_T^0)^{-1}\mu$.*

Proof. Suppose $x_1 < x_2 < (S_T^0)^{-1}\mu$ and π_1 is the optimal strategy for initial capital x_1. Define

$$\alpha := \frac{\mu - x_2 S_T^0}{\mu - x_1 S_T^0}.$$

Then $\alpha \in (0,1)$. For initial capital x_2 consider the following strategy π_2: Invest αx_1 of the money to realize strategy $\alpha\pi_1$ and put the remaining money $x_2 - \alpha x_1$ in the bond. Due to linearity of trading we have $X_T^{\pi_2} = \alpha X_1^{\pi_1} + (x_2 - \alpha x_1)S_T^0$. Obviously

$$\mathbb{E}[X_T^{\pi_2}] = \alpha\mu + (x_2 - \alpha x_1)S_T^0 = \mu \quad \text{and} \quad V(x_2,\mu) \leqslant \mathrm{Var}[X_T^{\pi_2}] = \alpha^2 \mathrm{Var}[X_T^{\pi_1}] < V(x_1,\mu)$$

which implies the statement. □

3. Complete Markets do not allow free Cash Flow Streams

In [7] the authors show in a discrete financial market that the optimal strategy for (MV) can be modified s.t. there is the possibility to consume some money and still receive a terminal capital with expected value μ and variance $V(x,\mu)$. The authors conjecture at the end of the paper that this is not possible in a complete financial market but give no proof. They only prove the statement in a complete discrete financial market but the proof is very complicated and blurs the main reasons for this effect. We will show this statement now in our generic complete financial market. In order to stay within the framework of self-financing strategies we modify the definition of a free cash flow strategy.

Definition 3.1. A strategy $\pi \in \mathcal{A}$ is called *free cash flow strategy* for (MV) if

(i) $X_0^\pi = x_0 \in (0, (S_T^0)^{-1}\mu)$,
(ii) $X_T^\pi = X_1 + X_2$ s.t. $\mathbb{E}\, X_1 = \mu$, $\mathrm{Var}[X_1] = V(x_0,\mu)$ and $X_2 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(X_2 > 0) > 0$.

Remark 3.2. Note that we interpret a free cash flow strategy in a different way than it was introduced in [7] where the authors assume that with positive probability a positive amount of money can be consumed or taken out of the market at some time point. We instead want to work with self-financing strategies and assume that the money is not consumed but invested in the bond and then may be consumed at the final time point T (cf. also [8]).

Theorem 3.3. *Financial markets which satisfy (A1)-(A3) do not allow free cash flow strategies.*

Proof. Suppose there is a free cash flow strategy $\pi \in \mathcal{A}$. Hence $X_0^\pi = x_0 < (S_T^0)^{-1}\mu$ and $X_T^\pi = X_1 + X_2$. Since the market is complete, there exist strategies $\pi_1, \pi_2 \in \mathcal{A}$ s.t. $X_T^{\pi_1} = X_1$ and $X_T^{\pi_2} = X_2$. Due to the linearity and no-arbitrage we have

$$X_0^{\pi_1} = x_1, X_0^{\pi_2} = x_2 \quad \text{and} \quad x_1 + x_2 = x_0.$$

Due to the no-arbitrage assumption we must have $x_2 > 0$, thus $x_1 < x_0$. Since $x \mapsto V(x,\mu)$ is strictly decreasing for $x < (S_T^0)^{-1}\mu$ due to Proposition 2.3 we must have $V(x_0,\mu) < \mathrm{Var}[X_1]$ which is a contradiction to the definition of free cash flow strategy. □

4. When can free Cash Flow Strategies appear?

In this section we show that mainly the performance criterion is responsible for the appearance of free cash flow strategies and not the time-inconsistency of the optimal control of (MV). Therefore we introduce the so-called benchmark problem (BM) for some fixed predetermined benchmark $B > 0$:

$$(BM) \quad \begin{cases} \mathbb{E}[(X_T^\pi - B)^2] \to \min \\ \\ X_0^\pi = x_0, \pi \in \mathcal{A} \end{cases}$$

where we assume that $x_0 S_T^0 < B$ for the same reason as before. Moreover we denote the minimal value of this problem by $V(x_0, B)$. The benchmark problem has in contrast to the mean-variance problem no additional constraints and a time-consistent optimal control. Before we address solvability of the benchmark problem and the appearance of free cash flow strategies, we have to clarify how a free cash flow strategy for the benchmark problem is defined:

Definition 4.1. A strategy $\pi \in \mathcal{A}$ is called *free cash flow strategy* for (BM) if

(i) $X_0^\pi = x_0 \in (0, (S_T^0)^{-1}B)$,

(ii) $X_T^\pi = X_1 + X_2$ s.t. $\mathbb{E}[(X_1 - B)^2] = V(x_0, B)$ and $X_2 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(X_2 > 0) > 0$.

In the following we investigate the reason for the appearance of free cash flow strategies for problem (BM). We do this by using two examples.

4.1. **Non-appearance of free cash flow strategies in the Black-Scholes model.** Suppose we are given the Black Scholes Market from Example 2.2 which is free of arbitrage and complete. Following the same lines of ideas as in Section 3 it is easy to see that problem (BM) does not allow free cash flow strategies in this setting. But what is the reason for it? Let us denote

$$\rho := \sigma^{-1}(b - r1),$$

where $1 = (1, \ldots, 1)^\top \in \mathbb{R}^d$. It is well-known (see e.g. [14]) that the optimal strategy $\pi_t^* = \pi^*(t, X_t^{\pi^*})$ of (BM) is of feedback form and given by

$$\pi^*(t, x) = (\sigma\sigma^\top)^{-1}(b - r)(Be^{-(T-t)r} - x).$$

Note that (π_t^*) is time-consistent. When we denote by

$$L_T := \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(- \rho^\top W_T - \frac{1}{2}\rho^\top \rho \cdot T\Big).$$

the density of the equivalent martingale measure $\mathbb{Q}$, then the following proposition holds:

Proposition 4.2. *The optimal terminal wealth of the benchmark problem is given by*

$$X_T^* = \Big(x_0 S_T^0 - B\Big)\frac{L_T}{\mathbb{E}[L_T^2]} + B$$

and it holds for all $t \in [0, T]$ that

$$X_t^* < Be^{-r(T-t)}.$$

Proof. It is easy to check that we can apply the martingale method for quadratic utility functions (see e.g. [9]). So the optimal wealth X_T^* is given by

$$X_T^* = \frac{1}{2}y\frac{L_T}{S_T^0} + B,$$

where y denotes the Lagrange multiplier which can be calculated from the conditional assumption $\mathbb{E}_\mathbb{Q}\left[\frac{X_T^*}{B_T}\right] = x_0$. We obtain

$$y = 2\Big(x_0 S_T^0 - B\Big)\frac{S_T^0}{\mathbb{E}[L_T^2]}$$

and, since $x_0 e^{rT} < B$, it follows that $y < 0$. This implies $X_T^* < B$. It remains to show that $X_t^* < e^{-r(T-t)}B$ holds as well for all $t \in [0, T)$. To see this, observe that $(\frac{X_t^*}{S_t^0})$ is a $\mathbb{Q}$-martingale. Thus

$$\begin{aligned} X_t^* &= S_t^0\,\mathbb{E}_\mathbb{Q}\Big[\frac{X_T^*}{S_T^0}\Big|\mathcal{F}_t\Big] \\ &= B\frac{S_t^0}{S_T^0} + \frac{S_t^0}{S_T^0}\Big(x_0 S_T^0 - B\Big)\frac{L_t}{\mathbb{E}[L_T^2]}\exp\big(\rho^\top\rho\cdot(T-t)\big) < Be^{-r(T-t)} \end{aligned}$$

which proves the statement. □

We see that in the Black-Scholes Market it is never possible to construct a free cash flow strategy since $X_t^* < e^{-r(T-t)}B$ holds for all $t \in [0, T]$, i.e. the optimal wealth will always stay below the benchmark B. This is possible due to the continuity of the wealth process.

Remark 4.3. Note that in an arbitrage-free, complete financial market the benchmark problem with $B \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be reduced to a problem with benchmark 0, since B can be hedged perfectly.

4.2. **The appearance of free cash flow strategies in an incomplete, discrete financial model.** In this section we consider an N-period discrete financial market with d stocks and one riskless bond. We suppose that all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_n)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The riskless bond is given by

$$S_n = (1+i)^n, \quad S_0^0 = 1,$$

where $i > 0$ denotes the deterministic and constant interest rate. The price process of asset k is given by

$$S_{n+1}^k = S_n^k \widetilde{R}_{n+1}^k, \quad S_0^k = s_0^k \in \mathbb{R}_+.$$

The processes (S_n^k) are assumed to be adapted with respect to the filtration $(\mathcal{F}_n)$ for all k. Furthermore we assume that the relative price change $\widetilde{R}_{n+1}^k$ in the time interval $[n, n+1)$ for the risky asset k is almost surely positive for all k and n. We define the so called relative risk process $R_n = (R_n^1, ..., R_n^d)$ by

$$R_n^k := \frac{\widetilde{R}_n^k}{1+i} - 1, \; k = 1, ..., d,$$

and we assume that

(i) the random vectors $R_1, R_2, \dots$ are independent.
(ii) the support of R_n^k is $(0, \infty)$.
(iii) $\mathbb{E}||R_n|| < \infty$ and $\mathbb{E}R_n \neq 0$ for $n = 1, ..., N$.
(iv) The covariance matrix of the relative risk processes $\Big(\mathrm{Cov}(R_n^j, R_n^k)\Big)_{1 \le j,k \le d}$ is positive definite for all $n = 1, ..., N$.

We denote by $(\phi^0, \phi) = (\phi_n^0, \phi_n) = (\phi_n^0, \dots, \phi_n^d)$ a self-financing trading strategy adapted to $(\mathcal{F}_n)$ where ϕ_n^k is the amount of money invested in the k-th asset at time n. The corresponding wealth process is denoted by (X_n^ϕ). It satisfies the recursion

$$X_{n+1}^\phi = (1+i)(X_n^\phi + \phi_n^\top R_{n+1}). \tag{4.1}$$

As in Section 2 we denote by $\mathcal{A}^\phi := \{\phi : \phi \text{ is self-financing}, \; X_N^\phi \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ the set of all *admissible strategies*. In this context problem (BM) reads:

$$(BM) \quad \begin{cases} \mathbb{E}[(X_N^\phi - B)^2] \to \min \\ \\ X_0^\phi = x_0, \phi \in \mathcal{A}^\phi \end{cases}$$

where we assume that $x_0(1+i)^N < B$.

Using the following abbreviations:

$$C_n := \mathbb{E}\left[R_n R_n^\top\right], \qquad l_n := \left(\mathbb{E}\left[R_n\right]\right)^\top C_n^{-1} \mathbb{E}\left[R_n\right],$$
$$d_N := 1, \qquad d_n := d_{n+1}(1 - l_{n+1}),$$

the solution of problem (BM) is given in feedback form $\phi_n^* = \phi_n^*(X_n^*)$ by (see e.g. [3]):

$$\phi_n^*(x) = \Big(B(1+i)^{n-N} - x\Big) C_{n+1}^{-1} \mathbb{E}[R_{n+1}].$$

Note that ϕ_n^* gives the optimal investment in the stocks only and that ϕ_n^* is time consistent. The investment in the bond is then due to the self-financing condition. In the following we show that a free cash flow strategy can appear in this discrete context.

Proposition 4.4. *Let* $\tau := \min\left\{n \in \mathbb{N}_0 | X_n^* > B(1+i)^{-(N-n)}\right\}$ *and define a new strategy* $(\widetilde{\phi}_n)$ *by*

$$\widetilde{\phi}_n := \begin{cases} \phi_n^*, & n < \tau \\ 0 & n \geq \tau \end{cases}$$

Then $(\widetilde{\phi}_n)$ *is a free cash flow strategy.*

Proof. Let X_N^* and $\widetilde{X}_N$ be the terminal wealths which we attain by using the strategies (ϕ_n^*) or $(\widetilde{\phi}_n)$. If $\tau > N$ we obtain $X_N^* = \widetilde{X}_N$. W.r.t. the definition of a free cash flow strategy we define $X_1 1_{\{\tau > N\}} := X_N^* 1_{\{\tau > N\}}$ and $X_2 1_{\{\tau > N\}} = 0$. In case $\tau \leq N$ we have

$$\widetilde{X}_N = X_\tau^*(1+i)^{N-\tau} > B.$$

Note that $\mathbb{P}(\tau \leq N) > 0$ due to assumption (ii) and the structure of ϕ_n^*. Now choose $\widetilde{B} < B$ s.t.

$$\mathbb{E}\left[(X_N^* - B)^2 1_{\{\tau > N\}}\right] + (B - \widetilde{B})^2\, \mathbb{P}(\tau \leq N) = V(x_0, B)$$

and define $X_1 1_{\{\tau \leq N\}} := \widetilde{B} 1_{\{\tau \leq N\}}$ and $X_2 1_{\{\tau \leq N\}} = (X_\tau^*(1+i)^{N-\tau} - \widetilde{B}) 1_{\{\tau \leq N\}}$. Then obviously $\mathbb{E}[(X_1 - B)^2] = V(x_0, B)$ and $X_2 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(X_2 > 0) = \mathbb{P}(\tau \leq N) > 0$ and the statement is shown.

□

Hence, a free cash flow strategy can be constructed as follows: If the wealth-process (X_n^*) exceeds the discounted value of the benchmark B, we put all money in the bond which will at time N yield a terminal wealth which is strictly larger than B. The difference to B can be consumed at time N and the distance to the benchmark is zero. When we suppose that the random variable R_k^i has support $(0, \infty)$ and we invest a positive amount of money in asset i, then by equation (4.1) there is always a positive probability of exceeding the discounted benchmark in the next step. This observation also causes troubles in other respect (see e.g. [4]). In case the supports are bounded and no short-selling is allowed, there are also setups where no free cash flow strategies exist in a discrete and incomplete financial market, see the next example:

Example 4.5. In the above context we choose $d = 1$ and $i = 0$. Moreover we suppose that the random vectors $R_1, ..., R_d$ are identically distributed, s.t.

$$R_1 \sim \mathcal{U}\left(-1 + \frac{1}{10}, 1\right).$$

Then $(\widetilde{\phi}_n)$ is no free cash flow strategy. To show this, we presume that $X_n^{\widetilde{\phi}} < B$ for some $n = 0, ..., N-1$. It follows due to equation (4.1):

$$X_{n+1}^{\widetilde{\phi}} = X_n^{\widetilde{\phi}} + \phi_n^*(X_n^{\widetilde{\phi}}) R_{n+1} = X_n^{\widetilde{\phi}} + (B - X_n^{\widetilde{\phi}}) \frac{\mathbb{E}[R_{n+1}]}{\mathbb{E}[R_{n+1}^2]} R_{n+1}.$$

It is clear that $X_{n+1}^{\widetilde{\phi}} < B$ $\mathbb{P}$-a.s. holds if

$$\mathbb{P}\left(R_{n+1} < \frac{\mathbb{E}[R_{n+1}^2]}{\mathbb{E}[R_{n+1}]}\right) = 1.$$

Since R_{n+1} is uniformly distributed with upper bound 1, it is sufficient to show

$$1 < \frac{\mathbb{E}[R_{n+1}^2]}{\mathbb{E}[R_{n+1}]}.$$

It is easy to check that the above equation holds. Since $X_0 = x_0 < B$ it follows by recursion from the above that $X_k^{\widetilde{\phi}} < B$ for all $k = 1, ..., N$. So $(\widetilde{\phi}_n)$ is no free cash flow strategy since $X_2 \equiv 0$.

4.3. **Consistency with FSD rules out free cash flow strategies.** We have seen in the last section that the performance criterion is the reason for the appearance of free cash flow strategies. Indeed the problem is that the quadratic criterion is not consistent with first order stochastic dominance (FSD). In contrast, most risk measures like e.g. Value-at-Risk (VaR) or Conditional Value-at-Risk (CVaR) are consistent with FSD (see [2], Theorem 4.2). Obviously if a criterion is consistent with FSD then free cash flow strategies cannot exist in any financial market. So as an alternative to the mean-variance problem one could also consider mean-VaR or mean-CVaR problems. Another alternative is the mean-Gini criterion (see e.g. [13]) which has especially been introduced as a FSD consistent alternative to the mean-variance criterion. It even yields the same efficient sets as the mean-variance criterion in the case of normally distributed returns.

The same observation is true for general portfolio problems without mean restriction, like

$$(G) \quad \begin{cases} F(X_T^\pi) \to \max \\ \\ X_0^\pi = x_0, \pi \in \mathcal{A} \end{cases}$$

where F is an arbitrary performance measure mapping a random variable to $\mathbb{R}$. Denote by $V(x_0)$ the maximal value. In this case we call a strategy $\pi \in \mathcal{A}$ a *free cash flow strategy* for (G) if

(i) $X_0^\pi = x_0$,
(ii) $X_T^\pi = X_1 + X_2$ s.t. $F(X_1) = V(x_0)$ and $X_2 \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(X_2 > 0) > 0$.

If F preserves the first order stochastic dominance in a strict sense, i.e. $X \leq_{st} Y$ and $X \overset{d}{\neq} Y$ imply $F(X) < F(Y)$, then no free cash flow strategies are possible in any markets. Note that it has been shown in [5] (Theorem 3) that F preserves FSD if essentially $F(X) = \mathbb{E}\,U(X)$ where U is increasing and concave.

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