Further calculations for the McKean stochastic game for a spectrally negative Lévy process: from a point to an interval *

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*Based on joint work with Kees van Schaik (Univ. of Manchester)

Outline

- Lévy processes

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- McKean stochastic game: point to interval

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Spectrally negative Lévy process (SNLP): no upwards jumps and paths not monotone.

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$$\mathbb{E}_x[e^{-q(\sigma\wedge\tau)}\left(f(X_{\sigma\wedge\tau})+\delta 1_{\{\sigma<\tau\}}\right)]$$

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So: American put option which can be terminated by seller for penalty δ.

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$$\inf_\sigma \sup_\tau Z_x(\tau, \sigma) = \sup_\tau \inf_\sigma Z_x(\tau, \sigma) = Z_x(\tau^*, \sigma^*).$$

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Mordecki (2002) proved $\tau^* = \tau_{k^*}^- = \inf\{t \geq 0 : X_t \leq k^*\}$. Here

$$k^* = \log(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}]),$$

where $\mathbf{e}_q$ is independent exponentially(q) distributed and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$.

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where $\mathbf{e}_q$ is independent exponentially(q) distributed and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$. Also

$$U(x) = \frac{\mathbb{E}\left[\left(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}] - e^{x+\underline{X}_{\mathbf{e}_q}}\right)^+\right]}{\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}]}.$$

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(Lower) Value of McKean stochastic game:

$$V(x) := \sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathbb{E}_x \left[e^{-q(\tau\wedge\sigma)} \left((K - e^{X_{\tau\wedge\sigma}})^+ + \delta 1_{\{\sigma<\tau\}} \right) \right].$$

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Aim: Find "optimal" σ^*, τ^* such that

$$\begin{aligned} V(x) &= \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-q(\tau \wedge \sigma)} \left((K - e^{X_{\tau \wedge \sigma}})^+ + \delta 1_{\{\sigma < \tau\}} \right) \right] \\ &= \mathbb{E}_x \left[e^{-q(\tau^* \wedge \sigma^*)} \left((K - e^{X_{\tau^* \wedge \sigma^*}})^+ + \delta 1_{\{\sigma^* < \tau^*\}} \right) \right]. \end{aligned}$$

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Taking $\sigma = \infty$, we see that $V(x) \leq U(x)$.

Laplace exponent

From now on, X is a SNLP.

Laplace exponent ψ of X is defined for $\lambda \geq 0$

$$\mathbb{E}[e^{\lambda X_t}] = e^{\psi(\lambda)t} \quad \forall t \geq 0.$$

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Equivalent measure: for $c > 0$

$$\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{cX_t - \psi(c)t}.$$

Under $\mathbb{P}^c$, X is still a SNLP.

Scale functions

Scale function for $q \geq 0$ $W^{(q)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ "defined" as continuous function satisfying

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q} \quad \forall \lambda > \Phi(q).$$

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Extend to $\mathbb{R}$ by: $W^{(q)}(y) = 0$ when $y < 0$.
Define $Z^{(q)} : \mathbb{R} \to \mathbb{R}_{\geq 1}$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy.$$

Exit problems

Two-sided exit problems: let

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For $a < x < b$

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$$\mathbb{E}_x[e^{-q\tau_a^-} \mathbf{1}_{\{\tau_a^- < \tau_b^+\}}] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}.$$

McKean Optimal Stopping Problem for SNLP

Recall

$$U(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau}(K - e^{X_\tau})^+]$$

and $\tau^* = \inf\{t > 0 : X_t < k^*\}$,
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Using scale functions (and Wiener–Hopf) we find

$$U(x) = KZ^{(q)}(x - k^*) - e^x Z_1^{(q)}(x - k^*),$$

where

$$k^* = \log\left(K \frac{q}{\Phi(q)} \frac{\Phi(q) - 1}{q - \psi(1)}\right)$$

and where Z_1 plays the role of Z under $\mathbb{P}^1$.

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When $\delta \geq U(\log K)$ it can be shown that $\sigma^* = \infty$, which implies $V = U$. When $\delta < U(\log K)$ we always have $U \neq V$, since $V(x) \leq (K - e^x)^+ + \delta$ implies

$$V(\log K) \leq \delta < U(\log K).$$

"Solution" to McKean stochastic game (B. and Kyprianou (2008))

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where x^* *uniquely solves*

$$Z^{(q)}(\log K - x) - Z_1^{(q-\psi(1))}(\log K - x) = \frac{\delta}{K},$$

and $y^* \geq \log K$.

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for $x \leq \log K$.
When $\sigma = 0$ *it holds that* $y^* > \log K$.

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$$f_\delta(x) = \sup_\tau \mathbb{E}_x[e^{-q\tau}(K - e^{X_\tau})\mathbf{1}_{\{\tau \leq T_{\log K}\}} + \delta e^{-qT_{\log K}}\mathbf{1}_{\{T_{\log K} < \tau\}}]$$

i.e. the optimal value for the maximiser provided the minimiser only exercises when X hits $\log K$.

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Some properties of f_δ

Recall

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and value function of game

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for $x \le \log K$.

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Lemma

Suppose $\sigma > 0$ and $0 < \delta \le U(\log K)$. The function f_δ is differentiable on $\mathbb{R}\backslash\{\log K\}$. Furthermore, $f_\delta = V$ on $(-\infty, \log K]$, $f_\delta \ge V$ on $\mathbb{R}$ and $f'_\delta(\log K+)$ is a strictly decreasing continuous function of δ.

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Proof makes use of scale function results for exit problems for SNLPs.

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Theorem

Suppose $\sigma > 0$*. When* $\Pi \neq 0$*, then there exists a unique* $\delta_0 \in (0, U(\log K))$ *such that an optimal stopping time for the minimiser is given by*

- $T_{\log K}$ *when* $\delta \in [\delta_0, U(\log K)]$*,*
- $T_{[\log K, y^*(\delta)]}$ *for some* $y^*(\delta) > \log K$ *when* $\delta \in (0, \delta_0)$*.*

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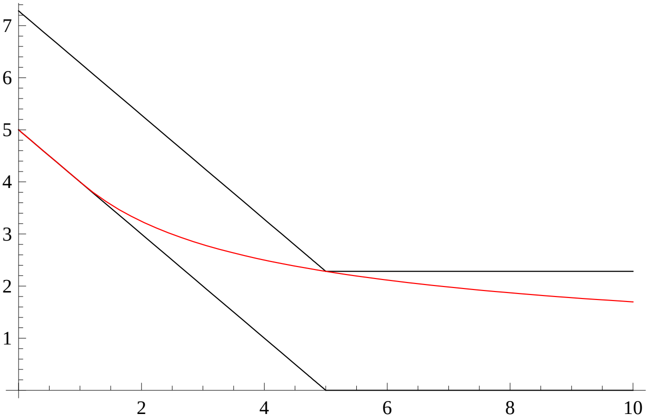
- $T_{[\log K, y^*(\delta)]}$ *for some* $y^*(\delta) > \log K$ *when* $\delta \in (0, \delta_0)$.

Proof mostly based on the previous lemma.

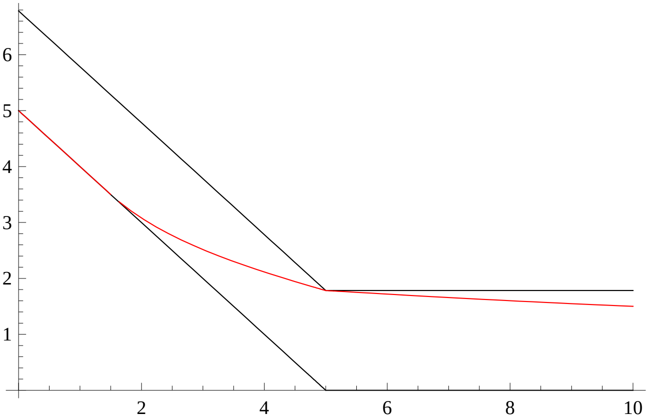
$\sigma > 0$

For $\delta = U(\log K)$ we have $f'_\delta(\log K+) < 0$.



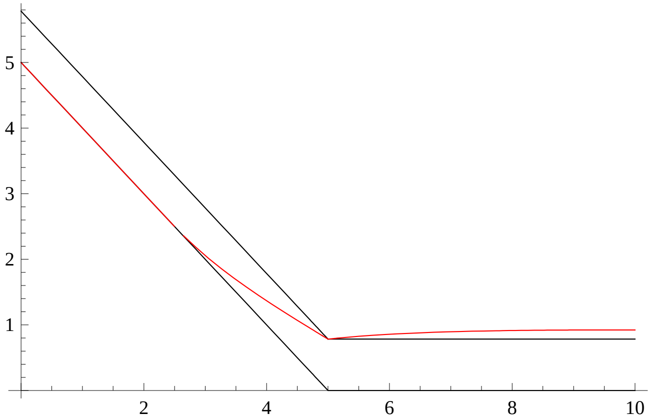
$\sigma > 0$

For $\delta \in (\delta_0, U(\log K))$ we have $f'_\delta(\log K+) < 0$.



$\sigma > 0$

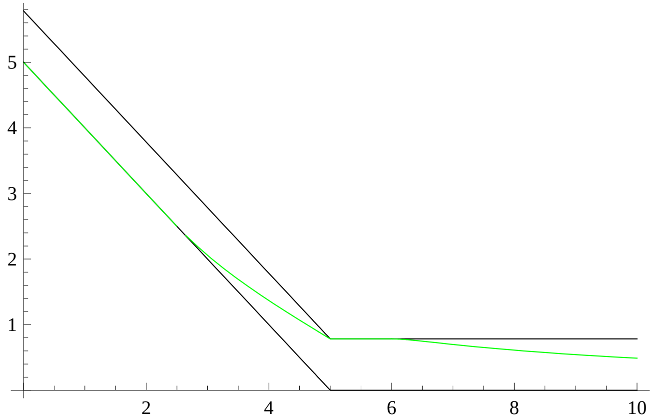
For $\delta < \delta_0$ we have $f'_\delta(\log K+) > 0$



Upper bound violated $\Longrightarrow y^* > \log K$

$\sigma > 0$

$\delta < \delta_0$, actual value function:



Equations for δ_0 and y^*

Define

$$w_\delta(x) = \begin{cases} V(x) & \text{for } x < \log K \\ \delta & \text{for } x \geq \log K. \end{cases}$$

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Theorem

Suppose $\Pi \neq 0$.

(i) *Suppose* $\sigma > 0$. *Then* δ_0 *is the unique solution on* $(0, U(\log K))$ *to*

$$\int_{t<0} \int_{u<t} (w_\delta(t + \log K) - \delta) e^{-\Phi(q)(t-u)} \Pi(du) dt = \frac{\delta q}{\Phi(q)}.$$

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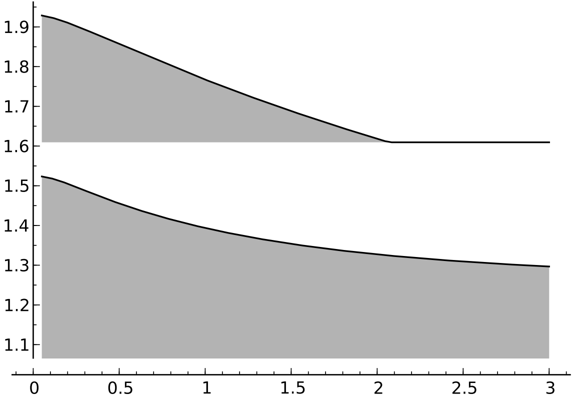
Proof makes use of compensation formula, potential measure of X, and smooth/continuous fit

Stopping regions as a function of σ

When $\sigma > 0$ and exponential jumps, stopping region for minimiser (top one) and maximiser (bottom one) as a function of σ.



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- Solution of the McKean stochastic game for SNLP.
- Stopping region for maximiser of the form $(-\infty, x^*]$.
- Stopping region for the minimiser $\{\log K\}$ when $\sigma > 0$ and $\delta < \delta_0$.
- Otherwise, stopping region for the minimiser $[\log K, y^*]$, with $y^* > \log K$ solution to equation involving scale functions.

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