A method for pricing American options using infinite linear programming

Sören Christensen

Mathematisches Seminar, CAU Kiel

March 29, 2011

Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,
- $g : E \to \mathbb{R}$ (gain function),

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,
- $g : E \to \mathbb{R}$ (gain function),
- $T \in (0, \infty]$ (time horizon),

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,
- $g : E \to \mathbb{R}$ (gain function),
- $T \in (0, \infty]$ (time horizon),
- $r \geq 0$ (discounting factor).

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,
- $g : E \to \mathbb{R}$ (gain function),
- $T \in (0, \infty]$ (time horizon),
- $r \geq 0$ (discounting factor).

Optimal stopping problem

Maximize

$$E_x(e^{-r\tau} g(X_\tau))$$

over all stopping times $\tau \leq T$ and $x \in E$.

Setting

Let

- $(X_t)_{t\geq 0}$ be a continuous time Markov process with state space E,
- $g : E \to \mathbb{R}$ (gain function),
- $T \in (0, \infty]$ (time horizon),
- $r \geq 0$ (discounting factor).

Optimal stopping problem

Maximize

$$E_x(e^{-r\tau} g(X_\tau))$$

over all stopping times $\tau \leq T$ and $x \in E$.

$$v(t, x) = \sup_{\tau \leq T-t} E_{(t,x)}(e^{-r\tau} g(X_{t+\tau})) \quad \text{value functions.}$$

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - backward induction

Methods of solution

- $T = \infty$, $(X_t)_{t\geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t\geq 0}$ 1-dim. diffusion:
  - backward induction
  - partial differential equation method

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - backward induction
  - partial differential equation method
  - linear programming

Methods of solution

- $T = \infty$, $(X_t)_{t\geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t\geq 0}$ 1-dim. diffusion:
  - backward induction
  - partial differential equation method
  - linear programming
  - integral equation method

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
    - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
    - backward induction
    - partial differential equation method
    - linear programming
    - integral equation method
    - ...

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - backward induction
  - partial differential equation method
  - linear programming
  - integral equation method
  - ...
- $T < \infty$, $(X_t)_{t \geq 0}$ multi-dim. diffusion:

Methods of solution

- $T = \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - analytical solution (free boundary,...)
- $T < \infty$, $(X_t)_{t \geq 0}$ 1-dim. diffusion:
  - backward induction
  - partial differential equation method
  - linear programming
  - integral equation method
  - ...
- $T < \infty$, $(X_t)_{t \geq 0}$ multi-dim. diffusion:
  - Monte-Carlo methods

Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

General theory

For simplicity: $T = \infty$, no time-dependence.

General theory

For simplicity: $T = \infty$, no time-dependence.
Under minimal conditions v is the smallest r-excessive majorant of the gain function g.

General theory

For simplicity: $T = \infty$, no time-dependence.
Under minimal conditions v is the smallest r-excessive majorant of the gain function g.

Characterization of v

$v(x_0)$ is the value to the problem

$$\begin{aligned} \text{min!} \quad & h(x_0) \\ \text{s.t.} \quad & h(x) \geq g(x) \quad \text{for all } x \in E, \\ & h \text{ is } r\text{-excessive.} \end{aligned}$$

Reduction to an infinite linear programming problem

Each positive r-excessive function h can be represented as

$$h(\cdot) = \int_B k_b(\cdot)\pi(db),$$

where B is a compact space (called minimal Martin boundary), $k_b, b \in B$, are the minimal r-excessive functions and π is a measure on B.

Reduction to an infinite linear programming problem

Each positive r-excessive function h can be represented as

$$h(\cdot) = \int_B k_b(\cdot)\pi(db),$$

where B is a compact space (called minimal Martin boundary), $k_b, b \in B$, are the minimal r-excessive functions and π is a measure on B.

infinite linear programming problem

$v(x_0)$ is the value to

$$\begin{aligned} \text{min!} \quad & \int_B k_b(x_0)\pi(db) \\ \text{s.t.} \quad & \int_B k_b(x)\pi(db) \geq g(x) \quad \text{for all } x \in E, \\ & \pi \text{ is a measure on } B. \end{aligned}$$

Approach

1. Fix $n \in \mathbb{N}$ and choose a finite subset $H := \{h_1, ..., h_n\}$ of r-excessive functions (equivalently choose n measures $\pi_1, ..., \pi_n$).

Approach

1. Fix $n \in \mathbb{N}$ and choose a finite subset $H := \{h_1, ..., h_n\}$ of r-excessive functions (equivalently choose n measures $\pi_1, ..., \pi_n$).
2. Solve the *linear semi-infinite programming problem*

$$\begin{aligned} \text{min!} \quad & \sum_{i=1}^{n} \lambda_i h_i(x_0) && \text{(LSIP)} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \lambda_i h_i(x) \geq g(x) \quad \text{for all } x \in E. \end{aligned}$$

Approach

1. Fix $n \in \mathbb{N}$ and choose a finite subset $H := \{h_1, ..., h_n\}$ of r-excessive functions (equivalently choose n measures $\pi_1, ..., \pi_n$).
2. Solve the *linear semi-infinite programming problem*

$$\begin{aligned} \text{min!} \quad & \sum_{i=1}^{n} \lambda_i h_i(x_0) \qquad \text{(LSIP)} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \lambda_i h_i(x) \geq g(x) \quad \text{for all } x \in E. \end{aligned}$$

Proposition

The value of (LSIP) is an upper bound for $v(x_0)$.

de la Vallée-Poussin

How to solve the linear semi-infinite programming problem

$$\begin{aligned} \text{min!} \quad & \sum_{i=1}^{n} \lambda_i h_i(x_0) && \text{(LSIP)} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \lambda_i h_i(x) \geq g(x) \quad \text{for all } x \in E? \end{aligned}$$

de la Vallée-Poussin

How to solve the linear semi-infinite programming problem

$$\text{min!} \quad \sum_{i=1}^{n} \lambda_i h_i(x_0) \qquad \text{(LSIP)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \lambda_i h_i(x) \geq g(x) \quad \text{for all } x \in E?$$

Theoretical background (de la Vallée-Poussin):

One can find $x_1, ..., x_n$ such that the restriction can be substituted by

$$\sum_{i=1}^{n} \lambda_i h_i(x_j) \geq g(x_j) \quad \text{for all } j = 1, ..., n.$$

de la Vallée-Poussin

How to solve the linear semi-infinite programming problem

$$\text{min!} \quad \sum_{i=1}^{n} \lambda_i h_i(x_0) \qquad \text{(LSIP)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \lambda_i h_i(x) \geq g(x) \quad \text{for all } x \in E?$$

Theoretical background (de la Vallée-Poussin):

One can find $x_1, \ldots, x_n$ such that the restriction can be substituted by

$$\sum_{i=1}^{n} \lambda_i h_i(x_j) \geq g(x_j) \quad \text{for all } j = 1, \ldots, n.$$

$\rightarrow$ well-known numerical methods, e.g. cutting plane algorithm.

Outline

1 Setting and existing methods

2 The main idea

3 Instructive example: Brownian motion

4 American options with finite time-horizon

5 American option on multiple assets

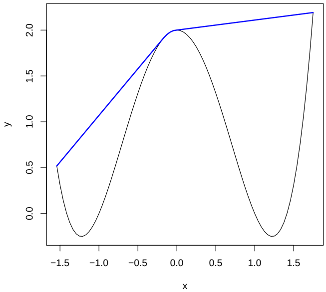
6 Conclusion

Brownian motion

- Let $(X_t)_{t\geq 0}$ be a Brownian motion on $E = [a, b]$, absorbed at a, b.
- $T = \infty$, $r = 0$.

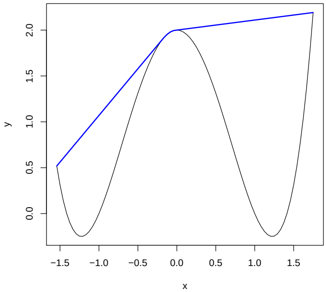
Brownian motion

- Let $(X_t)_{t \geq 0}$ be a Brownian motion on $E = [a, b]$, absorbed at a, b.
- $T = \infty$, $r = 0$.



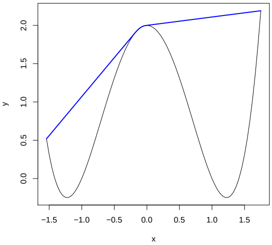
Brownian motion

- Let $(X_t)_{t\geq 0}$ be a Brownian motion on $E = [a, b]$, absorbed at a, b.
- $T = \infty$, $r = 0$.

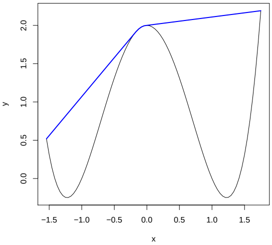


$$\begin{aligned} \text{min!} \quad & \lambda_1 x_0 + \lambda_2 \\ \text{s.t} \quad & \lambda_1 x + \lambda_2 \geq g(x) \quad \text{for all } x \in E. \end{aligned}$$

Brownian motion: Summary

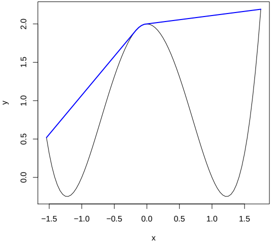


Brownian motion: Summary



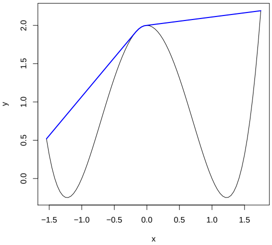
- For defining H: Taking the (minimal) r-harmonic functions $x \mapsto x$ and $\equiv 1$ is no restriction.

Brownian motion: Summary



- For defining H: Taking the (minimal) r-harmonic functions $x \mapsto x$ and $\equiv 1$ is no restriction.
- Optimizing for one point x_0 in the continuation set yields the value function for the whole connection component of the continuation set containing x_0 ($\rightarrow$ maximum principle).

Brownian motion: Summary



- For defining H: Taking the (minimal) r-harmonic functions $x \mapsto x$ and $\equiv 1$ is no restriction.
- Optimizing for one point x_0 in the continuation set yields the value function for the whole connection component of the continuation set containing x_0 ($\rightarrow$ maximum principle).
- One also obtains the optimal stopping time.

Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

Black-Scholes market

- $(X_t)_{t \geq 0}$ 1-dim. diffusion, e.g. geometric Brownian motion $dX_t = rX_t dt + \sigma X_t dW_t$ on $E = (0, \infty)$.
- $T < \infty$, $r > 0$.
- Gain function g (nearly) arbitrary, e.g. put-option $g(x) = (K - x)^+$.

Black-Scholes market

- $(X_t)_{t\geq 0}$ 1-dim. diffusion, e.g. geometric Brownian motion $dX_t = rX_t dt + \sigma X_t dW_t$ on $E = (0, \infty)$.
- $T < \infty$, $r > 0$.
- Gain function g (nearly) arbitrary, e.g. put-option $g(x) = (K - x)^+$.

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

Black-Scholes market

- $(X_t)_{t\geq 0}$ 1-dim. diffusion, e.g. geometric Brownian motion $dX_t = rX_t dt + \sigma X_t dW_t$ on $E = (0,\infty)$.
- $T < \infty$, $r > 0$.
- Gain function g (nearly) arbitrary, e.g. put-option $g(x) = (K - x)^+$.

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

$$h_a(t,x) = E_{(t,x)}(e^{-r(T-t)} 1_{\{X_T \geq a\}})$$

Black-Scholes market

- $(X_t)_{t\geq 0}$ 1-dim. diffusion, e.g. geometric Brownian motion $dX_t = rX_t dt + \sigma X_t dW_t$ on $E = (0, \infty)$.
- $T < \infty$, $r > 0$.
- Gain function g (nearly) arbitrary, e.g. put-option $g(x) = (K - x)^+$.

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

$$\begin{aligned} h_a(t,x) =& E_{(t,x)}(e^{-r(T-t)} 1_{\{X_T \geq a\}}) \\ =& e^{-r(T-t)} \Phi\left(-\frac{\log(x/a) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

Black-Scholes market

- $(X_t)_{t\geq 0}$ 1-dim. diffusion, e.g. geometric Brownian motion $dX_t = rX_t dt + \sigma X_t dW_t$ on $E = (0, \infty)$.
- $T < \infty$, $r > 0$.
- Gain function g (nearly) arbitrary, e.g. put-option $g(x) = (K - x)^+$.

How to choose the finite subset of r-excessive functions?
→ r-harmonic functions!

$$h_a(t,x) = E_{(t,x)}(e^{-r(T-t)} 1_{\{X_T \geq a\}})$$
$$= e^{-r(T-t)} \Phi\left(-\frac{\log(x/a) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

→ choose $a_1, a_2, ..., a_n$ randomly.

Black-Scholes market

$$\text{min!} \quad \sum_{i=1}^{n} \lambda_i h_i(t_0, x_0) \tag{LSIP}$$

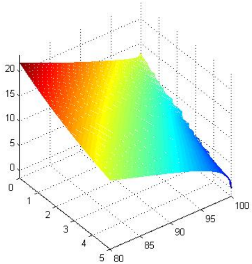
$$\text{s.t} \quad \sum_{i=1}^{n} \lambda_i h_i(t, x) \geq (K - x)^+ \quad \text{for all } x \in E, t \in [0, T].$$

Black-Scholes market

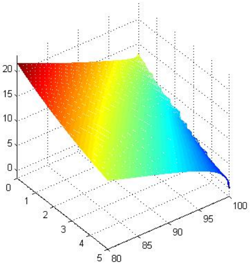
$$\text{min!} \quad \sum_{i=1}^{n} \lambda_i h_i(t_0, x_0) \qquad \text{(LSIP)}$$

$$\text{s.t} \quad \sum_{i=1}^{n} \lambda_i h_i(t, x) \geq (K - x)^+ \quad \text{for all } x \in E, t \in [0, T].$$

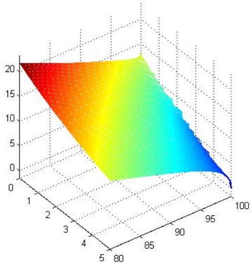
$\rightarrow h^*(t, x) = \sum_{i=1}^{n} \lambda_i^* h_i(t, x)$ approximation of $v(t, x)$.



Results



Results



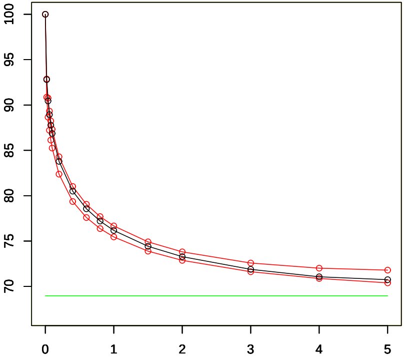
x	$v(0,x)$	RLP	error	time
80	21.606	21.615	0.009	
90	14.919	14.923	0.004	
$x_0 = 100$	9.946	9.951	0.005	9.4s
110	6.435	6.439	0.004	
120	4.061	4.064	0.003	

Stopping boundary

Using the approximated value function h^* one can find the optimal stopping boundary:

Stopping boundary

Using the approximated value function h^* one can find the optimal stopping boundary:



Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

Black-Scholes market with multiple assets

- $(X_t)_{t\geq 0} = (X_t^{(1)}, ..., X_t^{(d)})_{t\geq 0}$ multi-dim. diffusion, e.g. d-dim. geometric Brownian motion on $E = (0,\infty)^d$.
- $T < \infty$, $r > 0$.
- Gain function g, e.g. put-option on the minimum

$$g(x_1, ..., x_d) = \left(K - \min_{i=1,...,d} x_i\right)^+ .$$

Black-Scholes market with multiple assets

- $(X_t)_{t\geq 0} = (X_t^{(1)}, \dots, X_t^{(d)})_{t\geq 0}$ multi-dim. diffusion, e.g. d-dim. geometric Brownian motion on $E = (0,\infty)^d$.
- $T < \infty$, $r > 0$.
- Gain function g, e.g. put-option on the minimum

$$g(x_1, \dots, x_d) = \left(K - \min_{i=1,\dots,d} x_i\right)^+ .$$

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

Black-Scholes market with multiple assets

- $(X_t)_{t\geq 0} = (X_t^{(1)}, ..., X_t^{(d)})_{t\geq 0}$ multi-dim. diffusion, e.g. d-dim. geometric Brownian motion on $E = (0, \infty)^d$.
- $T < \infty$, $r > 0$.
- Gain function g, e.g. put-option on the minimum

$$g(x_1, ..., x_d) = \left(K - \min_{i=1,...,d} x_i\right)^+ .$$

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

$$h_a(t, x) = E_{(t,x)}\left(e^{-r(T-t)} 1_{\{X_T^{(1)}\geq a^{(1)},...,X_T^{(d)}\geq a^{(d)}\}}\right)$$

Black-Scholes market with multiple assets

- $(X_t)_{t\geq 0} = (X_t^{(1)}, ..., X_t^{(d)})_{t\geq 0}$ multi-dim. diffusion, e.g. d-dim. geometric Brownian motion on $E = (0, \infty)^d$.
- $T < \infty$, $r > 0$.
- Gain function g, e.g. put-option on the minimum

$$g(x_1, ..., x_d) = \left(K - \min_{i=1,...,d} x_i\right)^+ .$$

How to choose the finite subset of r-excessive functions?
$\rightarrow$ r-harmonic functions!

$$h_a(t, x) = E_{(t,x)}\left(e^{-r(T-t)} 1_{\{X_T^{(1)} \geq a^{(1)}, ..., X_T^{(d)} \geq a^{(d)}\}}\right)$$

$\rightarrow$ choose $a_1, a_2, ..., a_n$ randomly.

Black-Scholes market with multiple assets

x	interval in [Rogers2002]	RLP	Comp. time
$(80, 80)$	[38.01 , 38.35]	38.30	
$(80, 100)$	[32.23 , 32.60]	32.28	
$x_0 = (100, 100)$	[25.81 , 26.02]	25.86	$42s$
$(100, 120)$	[20.75 , 21.05]	20.77	
$(120, 120)$	[16.98 , 16.98]	17.02	
$(120, 80)$	[31.21 , 31.31]	31.30	

Black-Scholes market with multiple assets

x	interval in [Rogers2002]	RLP	Comp. time
$(80, 80)$	[38.01 , 38.35]	38.30	
$(80, 100)$	[32.23 , 32.60]	32.28	
$x_0 = (100, 100)$	[25.81 , 26.02]	25.86	$42s$
$(100, 120)$	[20.75 , 21.05]	20.77	
$(120, 120)$	[16.98 , 16.98]	17.02	
$(120, 80)$	[31.21 , 31.31]	31.30	

d	interval in [Rogers2002]	RLP	Comp. time
2	[24.87 , 25.16]	24.93	41 s
3	[31.21 , 31.76]	31.41	72 s
5	[39.01 , 39.47]	39.21	103 s
10	[47.99 , 48.33]	48.01	324 s
15	[52.23 , 52.14]*	52.10	612 s

Outline

1. Setting and existing methods
2. The main idea
3. Instructive example: Brownian motion
4. American options with finite time-horizon
5. American option on multiple assets
6. Conclusion

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.
- No discretization of space and time and no Monte-Carlo-elements are necessary.

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.
- No discretization of space and time and no Monte-Carlo-elements are necessary.
- The algorithm is very easy to implement in every common language (1 page of programming code!).

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.
- No discretization of space and time and no Monte-Carlo-elements are necessary.
- The algorithm is very easy to implement in every common language (1 page of programming code!).
- Optimizing for one special starting point gives very accurate approximations of the value function in the continuation set.

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.
- No discretization of space and time and no Monte-Carlo-elements are necessary.
- The algorithm is very easy to implement in every common language (1 page of programming code!).
- Optimizing for one special starting point gives very accurate approximations of the value function in the continuation set.
- The same is true for varying time horizons.

Summary

- The algorithm is based on reducing the ILP-problem connected to optimal stopping to a SILP-problem by choosing finitely many r-excessive functions.
- No discretization of space and time and no Monte-Carlo-elements are necessary.
- The algorithm is very easy to implement in every common language (1 page of programming code!).
- Optimizing for one special starting point gives very accurate approximations of the value function in the continuation set.
- The same is true for varying time horizons.
- The algorithm also works for large dimensions (e.g. ≥ 10), where apart from it only Monte-Carlo methods are applicable.

Final remarks

- The Greeks can be found immediately.

Final remarks

- The Greeks can be found immediately.
- Implied volatilities can be calculated.

Final remarks

- The Greeks can be found immediately.
- Implied volatilities can be calculated.
- Some classes of Lévy models can also be handled.

Final remarks

- The Greeks can be found immediately.
- Implied volatilities can be calculated.
- Some classes of Lévy models can also be handled.
- The algorithm can be used for infinite time horizons, too.