Time-delayed backward stochastic differential equations: the theory and applications

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BSDEs

The backward stochastic differential equation:

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T.$$

- A solution is a pair (Y, Z).
- The terminal condition ξ is an exogenous random variable,
- The generator f at time $s \in [0, T]$ depends on the current value of the solution and an additional source of randomness.

BSDEs with time-delayed generators

The backward stochastic differential equation with a time-delayed generator: [1][2]

$$Y(t) \;=\; \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T.$$

- The generator f at time $s \in [0, T]$ depends on the past values of the solution

$$Y_s := (Y(s+u))_{-T \leq u \leq 0}, \qquad Z_s := (Z(s+u))_{-T \leq u \leq 0}.$$

[1] Ł. Delong, P. Imkeller, 2010. Backward stochastic differential equations with time delayed generators - results and counterexamples. The Annals of Applied Probability.

[2] Ł. Delong, P. Imkeller, 2010. On Malliavin's differentiability of BSDE with time delayed generators driven by Brownian motions and Poisson random measures. Stochastic Processes and their Applications.

Time-delayed BSDEs

The time-delayed backward stochastic differential equation:

$$Y(t) = \xi(Y_T, Z_T) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T.$$

- The terminal condition ξ and the generator f depend on the past values of the solution

$$Y_s := (Y(s+u))_{-T \le u \le 0}, \qquad Z_s := (Z(s+u))_{-T \le u \le 0}.$$

Time-delayed BSDEs

- Existence and uniqueness of a solution,
- Linear time-delayed BSDEs,
- Properties of the solutions.

- Applications to portfolio management,
- Applications to pricing, hedging of insurance and financial contracts,
- Applications to pricing principles and recursive utilities.

The existence and uniqueness of a solution

Existence and uniqueness

(A1) The generator and the terminal condition are product measurable, $\mathbb{F}$-adapted and Lipschitz continuous, in the sense that for a probability measure α on $[-T,0]\times\mathcal{B}([-T,0])$ and with constants K_1, K_2 we have:

$$\mathbb{E}\Big[|f(\omega,t,Y_t,Z_t)-f(\omega,t,\tilde{Y}_t,\tilde{Z}_t)|^2\Big]$$
$$\leq \quad K_1\mathbb{E}\Big[\sup_{0\leq u\leq t}|Y(u)-\tilde{Y}(u)|^2+\int_{-T}^{0}|Z(t+u)-\tilde{Z}(t+u)|^2\alpha(du)\Big],$$

and

$$\mathbb{E}[|\xi(\omega,Y_T,Z_T)-\xi(\omega,\tilde{Y}_T,\tilde{Z}_T)|^2]$$
$$\leq \quad K_2\mathbb{E}\Big[\sup_{0\leq u\leq T}|Y(u)-\tilde{Y}(u)|^2+\int_{0}^{T}|Z(u)-\tilde{Z}(u)|^2du\Big],$$

for any square integrable processes (Y,Z),$(\tilde{Y},\tilde{Z})$,

Existence and uniqueness

(A2) $\mathbb{E}\big[\int_0^T |f(\omega, t, 0, 0, 0)|^2 dt\big] < \infty$,

(A3) $\mathbb{E}[|\xi(\omega, 0, 0, 0)|^2] < \infty$,

(A4) $f(\omega, t, ., .) = 0$ for $\omega \in \Omega$ and $t < 0$.

The result on existence and uniqueness

Theorem

Assume that **(A1)-(A4)** *hold. For a sufficiently small time horizon T or for a sufficiently small Lipschitz constant K_1, and for a sufficiently small Lipschitz constant K_2, the time-delayed backward stochastic differential equation has a unique solution.*

Linear time-delayed BSDEs

Linear time-delayed BSDEs

$$Y(t) = \xi + \int_t^T \int_0^s KY(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

1. $K < 0 \Rightarrow$ there exists a unique solution:

$$\begin{aligned} Y(t) &= \mathbb{E}[\xi]\frac{e^{\sqrt{-K}t} + e^{-\sqrt{-K}t}}{e^{\sqrt{-K}T} + e^{-\sqrt{-K}T}} \\ &\quad + \frac{1}{2}\int_0^t Z(s)\big(e^{\sqrt{-K}(t-s)} + e^{-\sqrt{-K}(t-s)}\big)dW(s), \\ Z(t) &= \frac{2M(t)}{e^{\sqrt{-K}(T-t)} + e^{-\sqrt{-K}(T-t)}}, \\ \xi &= \mathbb{E}[\xi] + \int_0^T M(t)dW(t). \end{aligned}$$

Linear time-delayed BSDEs

$$Y(t) = \xi + \int_t^T \int_0^s KY(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

2. $K > 0$ and $T\sqrt{K} < \frac{\pi}{2}$ $\Rightarrow$ there exists a unique solution:

$$\begin{aligned} Y(t) &= \mathbb{E}[\xi]\frac{\cos(t\sqrt{K})}{\cos(T\sqrt{K})} + \int_0^t Z(s)\cos((t-s)\sqrt{K})dW(s), \\ Z(t) &= \frac{M(t)}{\cos((T-t)\sqrt{K})}, \\ \xi &= \mathbb{E}[\xi] + \int_0^T M(t)dW(t). \end{aligned}$$

Linear time-delayed BSDEs

$$Y(t) = \xi + \int_t^T \int_0^s KY(u)duds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T$$

4. $K > 0,\ T\sqrt{K} = \frac{\pi}{2},\ \mathbb{E}[\xi] \neq 0\ \Rightarrow$ there exists no solution.

Linear time-delayed BSDEs

$$Y(t) = \xi + \int_t^T \int_0^s KY(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

$$Z(t) = \frac{M(t)}{\cos((T-t)\sqrt{K})}\mathbf{1}\{t > 0\}, \quad \xi = \mathbb{E}[\xi] + \int_0^T M(t)dW(t).$$

5. $K > 0,\ T\sqrt{K} = \frac{\pi}{2},\ \mathbb{E}[\xi] = 0$ and $\mathbb{E}[\int_0^T |Z(s)|^2 ds] = +\infty \ \Rightarrow$ there exists no solution.

6. $K > 0,\ T\sqrt{K} = \frac{\pi}{2},\ \mathbb{E}[\xi] = 0$ and $\mathbb{E}[\int_0^T |Z(s)|^2 ds] < \infty \ \Rightarrow$ there exist multiple solutions:

$$Y(t) = Y(0)\cos(t\sqrt{K}) + \int_0^t Z(s)\cos((t-s)\sqrt{K})dW(s).$$

Linear time-delayed BSDEs

$$Y(t) = \xi - \int_t^T \frac{1}{s}\int_0^s \beta Y(u)du ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

- There exists a unique solution.

The case of $\beta > 0$:

$$\begin{aligned} Y(t) &= \mathbb{E}[\xi]\frac{I_0(2\sqrt{\beta t})}{I_0(2\sqrt{\beta T})} + \int_0^t Z(s)\psi(s,t)dW(s), \\ Z(t) &= \frac{M(t)}{2I_0(2\sqrt{\beta T})\sqrt{\beta t}K_1(2\sqrt{\beta t}) + 2K_0(2\sqrt{\beta T})\sqrt{\beta t}I_1(2\sqrt{\beta t})}, \\ \psi(s,t) &= I_0(2\sqrt{\beta t})K_1(2\sqrt{\beta s}) + K_0(2\sqrt{\beta t})I_1(2\sqrt{\beta s}), \\ \xi &= \mathbb{E}[\xi] + \int_0^T M(s)dW(s). \end{aligned}$$

Linear time-delayed BSDEs

$$Y(t) = \xi + \int_t^T \int_0^s KZ(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

- There exists a unique solution:

$$\begin{aligned} Y(t) &= \mathbb{E}^{\mathbb{Q}}[\xi|\mathcal{F}_t] + (T-t)K\int_0^t Z(s)ds, \\ \xi &= \mathbb{E}^{\mathbb{Q}}[\xi] + \int_0^T Z(s)dW^{\mathbb{Q}}(s), \\ \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} &= \exp\left(K\int_0^T (T-s)dW(s) - K^2\frac{1}{2}\int_0^T (T-s)^2 ds\right). \end{aligned}$$

Linear time-delayed BSDEs

$$Y(t) = \xi - \int_t^T \frac{1}{s}\int_0^s \beta Z(u)du ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T$$

- There exists a unique solution:

$$\begin{aligned} Y(t) &= \mathbb{E}^{\mathbb{Q}}[\xi|\mathcal{F}_t] - \beta \ln(\frac{T}{t})\int_0^t Z(s)ds, \\ \xi &= \mathbb{E}^{\mathbb{Q}}[\xi] + \int_0^T Z(s)dW^{\mathbb{Q}}(s), \\ \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} &= \exp\Big(-\int_0^T \beta \ln(\frac{T}{s})dW(s) - \frac{1}{2}\int_0^T \big(\beta \ln(\frac{T}{s})\big)^2 ds\Big). \end{aligned}$$

Properties of the solutions

Properties of the solutions

- A comparison principle may not hold, a comparison holds up to a stopping time,
- If the terminal condition is bounded/non-negative, a solution may be unbounded/negative,
- A solution exists, but a measure solution may not exists, a measure solution exists up to a stopping time,
- A solution may not be conditionally invariant with respect to the terminal condition.

Malliavin's differentiability

A special case of a BSDE with a time-delayed generator:

$$
\begin{aligned}
Y(t) = \xi &+ \int_t^T f\Big(\omega, s, \int_{-T}^0 Y(s+v)\alpha(dv), \int_{-T}^0 Z(s+v)\alpha(dv)\Big) ds \\
&- \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T.
\end{aligned}
$$

Malliavin's differentiability

(A5) $\mathbb{E}\big[\int_{[0,T]}|D_s\xi|^2 ds\big] < \infty$,

(A6) the mapping $(y,z,u)\mapsto f(\omega,t,y,z,u)$ is continuously differentiable in (y,z,u), with uniformly bounded and continuous partial derivatives f_y, f_z, f_u,

(A7) for $(t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ we have that $f(\cdot,t,y,z,u)\in\mathbb{D}^{1,2}(\mathbb{R})$

$$\mathbb{E}\Big[\int_{[0,T]}\int_0^T\big|D_sf(\cdot,t,0,0,0)\big|^2dtds\Big]<\infty,$$

$$\big|D_sf(\omega,t,\hat{y},\hat{z},)-\big|D_sf(\omega,t,\tilde{y},\tilde{z})\big|\leq K\big(|\hat{y}-\tilde{y}|+|\hat{z}-\tilde{z}|\big).$$

Malliavin's differentiability

Theorem

Assume that **(A1)-(A7)** *hold. For a sufficiently small time horizon T or for a sufficiently Lipschitz constant K there exist unique solutions (Y, Z) and (Y^s, Z^s) to the time-delayed BSDEs*

$$\begin{aligned} Y(t) &= \xi \\ &+ \int_t^T f\Big(\omega, s, \int_{-T}^0 Y(s+v)\alpha(dv), \int_{-T}^0 Z(s+v)\alpha(dv)\Big) ds \\ &- \int_t^T Z(s)dW(s), \quad 0 \le t \le T, \end{aligned}$$

Malliavin's differentiability

$$
\begin{aligned}
Y^s(t) &= D_s\xi \\
&\quad + \int_t^T f^s(r, Y_r^s, Z_r^s)dr - \int_t^T Z^s(r)dW(r), \quad 0 \le s \le t \le T,
\end{aligned}
$$

$$
\begin{aligned}
f^s(r, Y_r^s, Z_r^s) &= D_s f\Big(\omega, r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv)\Big) \\
&+ \quad f_y\Big(\omega, r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv)\Big) \\
&\qquad \cdot \int_{-T}^0 Y^s(r+v)\alpha(dv) \\
&+ \quad f_z\Big(\omega, r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv)\Big) \\
&\qquad \cdot \int_{-T}^0 Z^s(r+v)\alpha(dv).
\end{aligned}
$$

Malliavin's differentiability

- *The solution* $(Y^s(t), Z^s(t))$ *is a version of* $(D_s Y(t), D_s Z(t))$.
- *The process* $(D_t Y(t))^{\mathcal{P}}$ *is a version of* $Z(t)$.

Applications to portfolio management, pricing and hedging

Financial market

The bank account:

$$\frac{dB(t)}{B(t)} = r(t)dt, \quad 0 \le t \le T,$$

$$dr(t) = a(t)dt + b(t)dW(t), \quad 0 \le t \le T, \quad r(0) = r_0.$$

The bond:

$$\frac{dD(t)}{D(t)} = \mu(t)dt + \sigma(t)dW(t), \quad 0 \le t \le T.$$

The wealth process

The portfolio process:

$$dX(t) = \pi(t)\big(\mu(t)dt+\sigma(t)dW(t)\big)+\big(X(t)-\pi(t)\big)r(t)dt, \quad 0 \leq t \leq T.$$

The terminal liability:

$$X(T) = \xi.$$

The discounted portfolio process:

$$dY(t) = Z(t)dW^{\mathbb{Q}}(t), \quad 0 \leq t \leq T.$$

The discounted terminal liability:

$$Y(T) = \tilde{\xi}.$$

Hedging problems

- Common financial instruments: European and Asian options, swaps, swaptions, puts on bonds, caps/floors,
- $\xi = (L(T_0, T) - K)^+$, $\xi = (D(T_0, T) - K)^+$,

$$Y(t) = \tilde{\xi}(W_T) - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \leq t \leq T.$$

- The terminal pay-off depends on the investment portfolio process or the applied investment strategy,

$$Y(t) = \tilde{\xi}(W_T, Y_T, Z_T) - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \leq t \leq T.$$

Option Based Portfolio Insurance

- Invest x euros,
- Protect the initial investment and earn an additional return,
- Buy a bond paying x at maturity and a call option on λ units of a "benchmark" process S

$$xD(0) + C(x\lambda S(T) - x) = x,$$

- But λ units of a "benchmark" process S and a put option on λ units of a "benchmark" process S

$$xS(0) + P(x - x\lambda S(T)) = x.$$

Option Based Portfolio Insurance

- $\xi = X(T) = X(0) + (\lambda S(T) - X(0))^+$,

$$Y(t) = e^{-\int_0^T r(s)ds}\left(Y(0) + (Y(0)\lambda S(T) - Y(0))^+\right) - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \le t \le T,$$

- We expect to find multiple solutions of the time-delayed BSDE.

The applications of time-delayed BSDEs

- Portfolio management problems,
- Participating contracts,

  "*...the benefits are based on the return on a specified pool of assets held by the insurer...*"

  "*...The best estimate should be based on the current assets held by the undertaking. Future changes of the asset allocation should be taken into account...*""

  Solvency II Directive
- Variable annuities,
- Asset-liability management problems in insurance.

Ratchet option

The liability:

$$\xi = \gamma \max\{X(0)e^{gT}, X(t_1)e^{g(T-t_1)}, ..., X(t_{n-1})e^{g(T-t_{n-1})}, X(T)\}.$$

The time-delayed BSDE:

$$\begin{aligned} Y(t) &= \gamma \max\Big\{ Y(0)e^{-\int_0^T r(s)ds + gT}, Y(t_1)e^{-\int_{t_1}^T r(s)ds + g(T-t_1)}, ..., \\ &\qquad Y(t_{n-1})e^{-\int_{t_{n-1}}^T r(s)ds + g(T-t_{n-1})}, Y(T)\Big\} \\ &\quad - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \leq t \leq T. \end{aligned}$$

Ratchet option

Proposition

Let $\mathcal{B}$ denote the following set

$$\mathcal{B} = \Big\{\omega \in \Omega : \gamma \max\left\{e^{gT} D(0), e^{g(T-t_1)} D(t_1), \ldots, e^{g(T-t_{n-1})} D(t_{n-1}), 1\right\} > 1\Big\}.$$

1. *If* $\mathbb{P}(\mathcal{B}) = 0, \gamma e^{gT} D(0) = 1$ *then there exist multiple solutions* (Y, Z)*, which differ in* $Y(0)$*, of the form*

$$\begin{aligned} Y(t) &= \gamma Y(0) e^{gT - \int_0^t r(s)ds} D(t), \quad 0 \le t \le T, \\ Y(0) e^{gT - \int_0^T r(s)ds} &= Y(0) + \int_0^T Z(s) dW^{\mathbb{Q}}(s). \end{aligned}$$

Ratchet option

Definition

2. *If* $\mathbb{P}(\mathcal{B}) = 0, \gamma e^{gT} D(0) < 1, \gamma = 1$ *then there exist multiple solutions* (Y, Z)*, which differ in* $(Y(0), \big(\tilde{\eta}(t_{m+1})\big)_{m=0,1,..n-1})$*, of the form*

$$
\begin{aligned}
Y(t_0) &= Y(0), \\
Y(t_m) &= \gamma \max_{k=0,1,\dots,m} \left\{ Y(t_k) e^{-\int_{t_k}^{t_m} r(u)du} e^{g(T-t_k)} \right\} D(t_m) \\
&\quad + \mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_{m+1}) | \mathcal{F}_{t_m}], \\
Y(t_{m+1}) &= \gamma \max_{k=0,1,\dots,m} \left\{ Y(t_k) e^{-\int_{t_k}^{t_{m+1}} r(u)du} e^{g(T-t_k)} \right\} D(t_{m+1}) \\
&\quad + \tilde{\eta}(t_{m+1}) \\
&= Y(t_m) + \int_{t_m}^{t_{m+1}} Z(s) dW^{\mathbb{Q}}(s), \\
Y(t) &= \mathbb{E}^{\mathbb{Q}}[Y(t_{m+1}) | \mathcal{F}_t], \quad t_m \leq t \leq t_{m+1}, m = 0, 1, \dots, n-1
\end{aligned}
$$

Ratchet option

- Multi-period Option Based Insurance Portfolio,
- Semi-static hedging,
- The solution is derived solely from the time-delayed BSDE.

Ratchet option

The liability:

$$\xi = \gamma \sup_{s\in[0,T]} \{X(s)e^{g(T-s)}\}.$$

The time-delayed BSDE:

$$\begin{aligned} Y(t) &= \gamma \sup_{0\le t\le T} \{Y(t)e^{-\int_t^T r(s)ds+g(T-t)}\} \\ &\quad -\int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0\le t\le T. \end{aligned}$$

Ratchet option

Proposition

Let $\mathcal{C}$ denote the following set

$$\mathcal{C} = \Big\{\omega \in \Omega : \gamma \sup_{0\leq t\leq T} \{e^{g(T-t)}D(t)\} > 1\Big\}.$$

1. *If* $\mathbb{P}(\mathcal{C}) = 0, \gamma e^{gT} D(0) = 1$ *then there exist multiple solutions* (Y, Z)*, which differ in* $Y(0)$*, of the form*

$$\begin{aligned} Y(t) &= \gamma Y(0) e^{gT - \int_0^t r(s)ds} D(t), \quad 0 \leq t \leq T, \\ Y(0)e^{gT-\int_0^T r(s)ds} &= Y(0) + \int_0^T Z(s) dW^{\mathbb{Q}}(s). \end{aligned}$$

Ratchet option

Proposition

2. *If* $\mathbb{P}(\mathcal{C}) = 0, \gamma e^{gT} D(0) < 1, \gamma = 1$ *then there exist multiple solutions* (X, π)*, which differ in* $(X(0), U)$*, of the form*

$$\begin{aligned} X(t) &= X(0) + \int_0^t \pi(s)\frac{dD(s)}{D(s)} + \int_0^t (X(s) - \pi(s))\frac{dB(s)}{B(s)}, \\ \pi(t) &= \gamma \sup_{0\le s\le t}\{X(s)e^{g(T-s)}\}D(t) \\ &\quad + \frac{U(t)}{S(t)}\Big(X(t) - \gamma \sup_{0\le s\le t}\{X(s)e^{g(T-s)}\}D(t)\Big)\mathbf{1}\{S(t) > 0\} \end{aligned}$$

with the process S *defined as*

$$dS(t) = U(t)\frac{dD(t)}{D(t)} + (S(t) - U(t))\frac{dB(t)}{B(t)}, \quad S(0) = s > 0.$$

Return smoothing

The liability:

$$\xi = X(0)\beta S + \gamma \frac{1}{T}\int_0^T e^{\int_s^T r(u)du} X(s)ds.$$

The time-delayed BSDE:

$$Y(t) = \beta Y(0)\tilde{S} + \gamma \frac{1}{T}\int_0^T Y(s)ds - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \leq t \leq T.$$

Return smoothing

Proposition

If $\beta\mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \gamma = 1$ then there exist multiple solutions (Y, Z), which differ in $Y(0)$, of the form

$$Y(t) = Y(0) + \int_0^t Z(s)dW^{\mathbb{Q}}(s), \quad 0 \le t \le T,$$

with the $\mathbb{F}$-predictable control

$$Z(t) = \frac{1}{1 - \gamma + \gamma\frac{t}{T}} M(t), \quad 0 \le t \le T,$$

and the process M derived from the martingale representation

$$\beta Y(0)\tilde{S} = \beta Y(0)\mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \int_0^T M(t)dW^{\mathbb{Q}}(t).$$

Return smoothing

A real-life product from the UK market:

$$\begin{aligned}\xi &= X(0)\Big(1+\max\Big\{g,\beta\Big(\frac{X(1)}{X(0)}-1\Big)\Big\}\Big)\\ &\quad\cdot\Big(1+\max\Big\{g,\frac{\beta}{2}\Big(\frac{X(1)}{X(0)}+\frac{X(2)}{X(1)}-2\Big)\Big\}\Big)\ldots\\ &\quad\cdot\Big(1+\max\Big\{g,\frac{\beta}{T}\Big(\frac{X(1)}{X(0)}+\frac{X(2)}{X(1)}+\ldots+\frac{X(T)}{X(T-1)}-T\Big)\Big\}\Big).\end{aligned}$$

Minimum withdrawal rates

Minimum guaranteed withdrawal benefit with an annuity conversion option:

$$\begin{aligned} dX(t) &= \pi(t)(\mu(t)dt + \sigma(t)dW(t)) + (X(t) - \pi(t))r(t)dt \\ &\quad -\gamma(\sup_{s\in[0,t]} X(s))dt, \quad 0 \le t \le T, \\ X(T) &= La(T), \end{aligned}$$

$$a(T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_T^{\infty} e^{-\int_T^s r(u)du} ds \,|\, \mathcal{F}_T\Big].$$

Minimum withdrawal rates

The time-delayed BSDE:

$$Y(t) = L\tilde{a}(T) + \int_t^T \gamma\Big(\sup_{u\in[0,s]} \big\{ Y(u)e^{-\int_u^s r(v)dv} \big\}\Big) ds - \int_t^T Z(s)dW^{\mathbb{Q}}(s), \quad 0 \le t \le T.$$

- For a sufficiently small γ or T there exists a unique non-zero solution.

Numerical schemes

- Picard iterations,
- Random walk approximations,
- Forward-backward structure of a solution,
- Convergence?

Minimum withdrawal rates

Minimum guaranteed withdrawal benefit with an annuity conversion option:

$$
\begin{aligned}
dX(t) &= \pi(t)(\mu(t)dt + \sigma(t)dW(t)) + (X(t) - \pi(t))r(t)dt \\
&\quad -\gamma \sup_{s\in[0,t]} \{X(s)\}dt, \\
X(T) &= \gamma \sup_{0\le s\le T} \{X(s)\}a(T).
\end{aligned}
$$

$$
a(T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_T^{\infty} e^{-\int_T^s r(u)du} ds | \mathcal{F}_T\Big].
$$

Applications to pricing principles and recursive utilities

Prices and utilities

- We can model a price or a utility as a solution to a BSDE:

$$dY(t) = g(t, Y(t), Z(t))dt + Z(t)dW(t).$$

- The generator defines a local (subjective) valuation rule:

$$\begin{aligned} \mathbb{E}[dY(t)|\mathcal{F}_t] &= g(t, Y(t), Z(t))dt, \\ Var[dY(t)|\mathcal{F}_t] &= Z^2(t)dt. \end{aligned}$$

A habit process

$$dY(t) = -u(c(t))dt - \beta \int_0^t Y(s)ds\,dt + Z(t)dW(t)$$

- $\beta > 0$ an anticipation effect,
- $\beta < 0$ a disappointment effect.

Pricing principles

$$Y(t) = \mathbb{E}[e^{-\beta(T-t)}\xi|\mathcal{F}_t]$$

- The local valuation rule:

$$\mathbb{E}[dY(t)|\mathcal{F}_t] = \beta Y(t)dt,$$

- The price changes proportionally to the last price (a local expected value principle).

Pricing principles

- The price changes proportionally to the average of the past prices,
- The local valuation rule:

$$\mathbb{E}[dY(t)|\mathcal{F}_t] = \frac{1}{t}\int_0^t \beta Y(s)ds\,dt,$$

- A time-delayed BSDE with a generator of a moving average type.

Pricing principles

The global valuation rule:

$$Y(t) = \phi(t,t,T)V(t) - \int_0^t V(s)\phi'(s,t,T)ds,$$

$$V(t) = \mathbb{E}[\xi|\mathcal{F}_t].$$

Recursive utilities

A habit process:

$$dY(t) = -u(c(t))dt - \beta \frac{1}{t} \int_0^t Y(s)ds dt + Z(t)dW(t).$$

Thank you very much

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