Integro Differential Equations for Option Pricing in Lévy-Models

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PIDEs for option pricing: a 'universal approach'

X Markov pr.

European	$E\big(g(X_T)\big)$	$\partial_t v - \mathcal{G}\, v = 0,\ v(0) = g$
barrier	$E\big(g(X_T)\mathbb{1}_{\tau_D > T}\big)$	boundary value pb.
lookback	$E\big(g(\sup_{t \le T} X_t)\big)$	
American	$sup_{\tau \le T} E\big(g(X_\tau)\big)$	free boundary problem

Fundamental link:

option prices $\longleftrightarrow$ PDEs

Original pricing formula of Black & Scholes (1973):

stock $\cong$ continuous-time random walk,
geometric Brownian motion

option price + replicating portfolio: via PDE (Heat eq.)

Lévy-model

Model: $S_t = S_0 \, e^{L_t}$, $L_t = \sigma B_t + \mu t + L_t^d$

- L^d pure jump process
- Generalizes Black-Scholes
- Models directly the log-returns $\log(S_{t+h}/S_t)$

Lévy-Khintchine formula $E \, e^{iuL_t} = e^{t\kappa(iu)}$

$$\kappa(iu) = -\frac{\sigma}{2}u^2 + ibu + \int \left(e^{iuy} - 1 - iuh(y) \right) F(dy)$$

- Approach to Lévy-models: via **Fourier transform**

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- Approach to Lévy-models: via **Fourier transform**

Calculate expected values via Fourier transform

Faire pice of claim g

$$\Pi_0^g = E[g(L_T)] .$$

Suitable regularity conditions $\Longrightarrow$

$$E[g(L_T)] = \int g(x) P^{L_T}(\mathrm{d}x) \overset{\text{Parseval}}{=} \frac{1}{(2\pi)} \int \hat{g}(u)\, \mathrm{e}^{T\kappa(-iu)}\, \mathrm{d}u .$$

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Option pricing via Fourier methods

Europ. options $\Longrightarrow$ *Raible (2000), Carr and Madan (1999)*

barrier options $\Longrightarrow$ Wiener-Hopf factorization,
Levendorski et al.
Eberlein, G., Papapantoleon

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Expectation via PIDEs

$$\Pi_t^g = E[g(L_T)\,|\mathcal{F}_t] = E[g(L_{T-t}+x)]\,\big|_{x=L_t} =: \Gamma_{T-t}\,g(x)\big|_{x=L_t}$$

$$\partial_t u(t,x) = -\lim_{h\downarrow 0}\frac{\Gamma_h - 1}{h}\,u(t,x) = -\,\mathcal{G}\,u(t,x)$$

by the definition of the inf. generator $\mathcal{G}$ of L i.e.

$$\partial_t u + \mathcal{G}\,u = 0, \qquad u(T,x) = g(x)$$

Expectation via PIDEs

Altogether

$$\Pi_t^g = E[g(L_{T-t} + x)]\,\Big|_{x=L_t} =: u(t, L_t)$$

$$\partial_t u + \mathcal{G}\, u = 0, \qquad u(T, x) = g(x)$$

with infinitesimal generator $\mathcal{G} = \frac{\sigma}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{G}^{\,\mathrm{jump}}$

$$\mathcal{G}^{\,\mathrm{jump}}\varphi(x) = \int \Big(\varphi(x+y) - \varphi(x) - h(y)\varphi'(x)\Big) F(\mathrm{d}y)$$

More general situation

Problem: Barrier option Lack of regularity $\Longrightarrow$ previous argument doesn't apply

Aim: establish Feynman-Kac formula

i.e. Show that $\exists!$ solution u of

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= 0 \\ u(0) &= g \end{aligned}$$

as weak solution and

$$u(T-t, L_t) = E(g(L_T)\,|L_t)$$

Efficient numerical method to solve PIDEs: Wavelet-Galerkin-method

A series of papers of Schwab et al.

- Matache, Petersdorff, Schwab: *Fast deterministic pricing of options on Lévy driven assets*. M2AN (2004)
- Matache, Nitsche, Schwab: *Wavelet Galerkin pricing of American options on Lévy driven assets*. Quant. Finance (2005)
- Winter: *Wavelet Galerkin schemes for option pricing in multidimensional Lévy models.* PhD thesis 18221, ETH Zürich (2009)
- Hilber, Reich, Schwab, Winter: *Numerical methods for Lévy processes*. Fin. Stoch (2009)

Hilbert space approach

Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ and $\mathcal{A} : V \to V^*$ linear with bilinear form

$$a(u, v) := (\mathcal{A}\, u)(v) = \langle \mathcal{A}\, u, v \rangle_{V^* \times V}.$$

Continuity	$a(u, v) \leq C_1 \|u\|_V \|v\|_V$	$(u, v \in V)$,
Gårding inequality	$a(u, u) \geq C_2 \|u\|_V^2 - C_3 \|u\|_H^2$	$(u \in V)$

with $C_1, C_2 > 0$ and $C_3 \geq 0$.

Hilbert space approach

Continuity and Gårding inequality $\Longrightarrow$

Theorem

For $f \in L^2\big(0, T; V^*\big)$ and $g \in H$,

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned}$$

has a unique solution $u \in W^1$.

$$u \in W^1 \;:\Leftrightarrow\; u \in L^2\big(0, T; V\big) \text{ with } \dot{u} \in L^2\big(0, T; V^*\big)$$

Variational formulation $\rightarrow$ Approximation scheme

Variational form   for all $\varphi \in C_0^\infty\big(0, T\big)$ and all $v \in V$:

$$- \int_0^T \langle u(t), v\rangle_H \, \varphi'(t) \, \mathrm{d}t + \int_0^T a\big(u(t), v\big)\varphi(t) \, \mathrm{d}t = \int_0^T \langle f(t), v\rangle_{V^* \times V} \, \varphi(t) \, \mathrm{d}t$$

Finite Element Approximation: $V^N \subset V^{N+1} \subset \ldots V$ finite dimensional with $\cup_{n \in \mathbb{N}} V^N$ dense in V

$\Longrightarrow$ system of linear ODEs.

Combine PIDE method and Fourier approach

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Lack of regularity

$\Longrightarrow$ characterize price via weak solution of the PIDE

Lévy-model

$\Longrightarrow$ Fourier transform: study the symbol of the operator

PIDEs in the light of Fourier transform

Fourier transform of $\mathcal{A}\, u$:

$\mathcal{A}$ Pseudo Diff. Op . $\mathcal{F}\big(\mathcal{A}\, u\big) = A\,\mathcal{F}(u)$

Symbol
$$\begin{aligned} A(\xi) &= \frac{\sigma}{2}\xi^2 + ib\xi - \int \big(\mathrm{e}^{-i\xi y} - 1 - i\xi h(y) \big) F(\mathrm{d}y) \\ &= -\kappa(-i\xi) \end{aligned}$$

Sobolev index

Bilinear form via Symbol for $u, v \in C_0^\infty$

$$a(u,v) = \int (\mathcal{A}\, u) v = \frac{1}{2\pi} \int A\, \hat{u}\, \overline{\hat{v}} \quad \text{(Parseval)}$$

Sobolev-Index of A is $\alpha > 0$, if

Continuity condition $\left|A(\xi)\right| \le C_1 (1 + |\xi|)^\alpha$

Gårding condition $\Re(A(\xi)) \ge C_2 (1 + |\xi|)^\alpha - C_3 (1 + |\xi|)^\beta$

for $\xi \in \mathbb{R}$ with $C_1, C_2 > 0$ and $C_3 \ge 0$, $0 \le \beta < \alpha$.

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Sobolev index and Sobolev spaces

$\Longrightarrow$ Continuity & Gårding inequality in Sobolev-Slobodeckii space $H^{\alpha/2}$.

Sobolev-Slobodeckii space $H^s := \{u \in L^2 \mid \|u\|_{H^s} < \infty\}$ with norm

$$\|u\|_{H^s}^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, \mathrm{d}\xi$$

Sobolev index and parabolic PIDEs

Gelfand triplet $H^s \hookrightarrow L^2 \hookrightarrow (H^s)^* = H^{-s}$.

Let $\mathcal{A}$ PDO with Symbol A with Sobolev index α.

Theorem

Let $s = \alpha/2$. For $f \in L^2\big((0,T); H^{-s}\big)$ and $g \in L^2$

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned}$$

$\exists!$ solution $u \in W^1$ i.e. $u \in L^2(0,t;H^s)$ with $\dot{u} \in L^2(0,t;H^{-s})$.

Stochastic representation

If A is the symbol of a Lévy-Process L with with Sobolev index α

$\Longrightarrow$ the solution $u \in W^1\left(0, T; H^{\alpha/2}, L^2\right)$ of has the stochastic representation

$$u(T-t, L_t) = E\left(g(L_T) - \int_t^T f(T-s, L_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right)$$

Sobolev index for Lévy processes

Distribution	Sobolev index α
Brownian motion (+ drift)	2
GH, NIG	1
CGMY, no drift	Y
CGMY, with drift, $Y \geq 1$	Y
CGMY, with drift, $Y < 1$	–
Variance Gamma	–

Sobolev index $\alpha < 2 \Longrightarrow$ Blumenthal-Getoor index $\beta = \alpha$.

For option pricing: drift condition $\Longrightarrow$ Require $\alpha \geq 1$.

Pricing barrier options

Digital barrier option payoff 1 only, if $S_t < H \; \forall t \leq T$

$$\mathbb{1}_{\sup_{t \leq T} L_T < B} = \mathbb{1}_{\tau_{(-\infty, B]} < T}$$

with $\tau_{(-\infty,B]}$ = first exit time of L out of $(-\infty, B]$.

$\rightsquigarrow$ Feynman-Kac formula for sol. to boundary value pb.

Feynman-Kac formula for digital barrier option

Let $\eta > 0$ with

(A1) Exponential moments $\int_{|x|>1} e^{-\eta x} F(dx) < \infty$.

(A2) Continuity $\big|A(z)\big| \leq C_1\big(1 + |z|\big)^{\alpha}$.

(A3) Gårding $\Re\big(A(z)\big) \geq C_2\big(1 + |z|\big)^{\alpha} - C_3\big(1 + |z|\big)^{\beta}$.

For all $z \in U_{-\eta} := \mathbb{R} - i[0, \eta)$ with constants $C_1,\ C_2 > 0,\ C_3 \geq 0$ and $0 \leq \beta < \alpha$.

PIDEs for digital barrier option

Digital barrier option: payoff $\mathbb{1}_{\{\tau_{\overline{D}} > T\}}$

Fair price $\Pi_t = \underbrace{E\left(\mathbb{1}_{\{T < \tau_{t,\overline{D}}\}} \middle| \mathcal{F}_t\right)}_{u(T-t, L_t)} \mathbb{1}_{\{t < \tau_{\overline{D}}\}}$

Theorem

Let (A1)–(A3) with $\alpha \geq 1$ *and* $\eta > 0$*. Then* $u \in W^1$ *is the unique solution of*

$$\partial_t u + \mathcal{A}\, u = 0 \quad \text{in } \left(\widetilde{H}_\eta^{\alpha/2}(D)\right)^*$$
$$u(0) = g$$

PIDE for digital barrier options: numerics

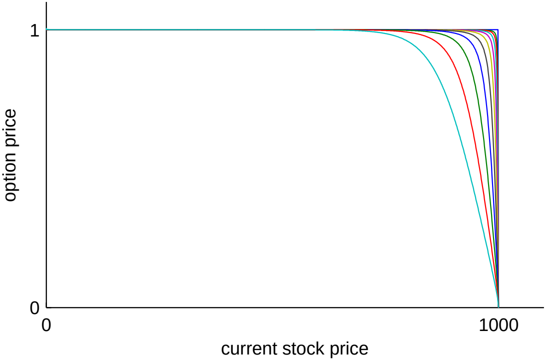

Figure: Price of digital barrier option maturity $(1/2)^k$, $k = 0, \ldots 9$ in a CGMY model.

Distribution of the supremum process

Distr. func. $F^{\overline{L}_T}$ of the supremum $\overline{L}_T = \sup_{0 \le t \le T} L_t$ i.e.

$$F^{\overline{L}_T}(x) = P(\overline{L}_T < x) = P\big(T < \tau_{(-\infty,x]}\big) = E\big(\mathbb{1}_{T<\tau_{(-\infty,0]}} \big| L_0 = -x\big)$$

$$\Longrightarrow \boxed{F^{\overline{L}_T}(x) = u^{\mathrm{digi},0}(T,-x)}\,.$$

$\Longrightarrow$ theor. result: regularity of $F^{\overline{L}_T}$

$\Longrightarrow$ building block for other option prices:
e.g. lookback option

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Lookback-Option

Faire price

$$V_0(S_0) = E\Big(\sup_{0\le t\le T} S_t - K\Big)^+$$

$$= \int (S_0\, \mathrm{e}^x - K)^+ \, F^{\overline{L}_T}(\mathrm{d}x)$$

$$V_0(S_0) = S_0\left(\int_{k-\log(S_0)}^{\infty} \big(1 - u^{\mathrm{digi},0}(T,-x)\big)\, \mathrm{e}^x\, \mathrm{d}x + (1 - K/S_0)^+\right).$$

Lookback-Option

Faire price
$$V_0(S_0) = E\Big(\sup_{0\le t\le T} S_t - K\Big)^+$$
$$= \int (S_0\, e^x - K)^+ \, F^{\overline{L}_T}(dx)$$

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Numerics

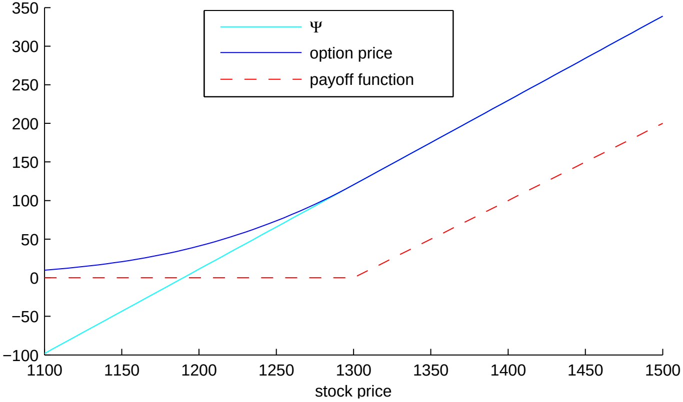

Figure: $\Psi(S_0) = E \max_{0 \leq t \leq T} S_t - K$, maturity 1 year, strike 1300.

Feynman-Kac formula for boundary value pb

Boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \qquad \text{on open set } D \subset \mathbb{R}\,,\; D \neq \mathbb{R} \\ u(0) &= g \end{aligned}$$

Aim: Stoch. representation of solution u.

Precise formulation: $\widetilde{H}^{\alpha/2}_\eta(D) = \{u \in H^{\alpha/2}_\eta(\mathbb{R}) \mid u|_{D^c} \equiv 0\}$

Find $u \in W^1$ with

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \quad \text{in } \left(\widetilde{H}^{\alpha/2}_\eta(D)\right)^* \\ u(0) &= g \end{aligned}$$

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Pénalisation du domaine

For $\lambda > 0$ let $u^\lambda \in W^1$ solution of

$$\partial_t u^\lambda + \mathcal{A}\, u \; + \lambda \mathbb{1}_{D^c} u^\lambda = f \qquad \text{in } (H_\eta^{\alpha/2}(\mathbb{R}))^*$$
$$u^\lambda(0) = g$$

u^λ has the stochastic representation

$$u_\lambda(T-t, L_t) = E\Big(g(L_T)\, e^{-\int_t^T \lambda \mathbb{1}_{D^c}(L_{u-})\,\mathrm{d}u} - \int_t^T f(T-s, L_s)\, e^{-\int_t^s \lambda \mathbb{1}_{D^c}(L_{u-})\,\mathrm{d}u}\, \mathrm{d}s \,\Big|\, \mathcal{F}_t \Big).$$

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Pénalisation du domaine

For $\lambda \uparrow \infty$ converge a.s.

$$\underbrace{u^\lambda(T-t, L_t)}_{\displaystyle\downarrow} = \underbrace{E\Big(g(L_T)\, e^{-\int_t^T \lambda \mathbb{1}_{D^c}(L_{h-})\,dh} \,\Big|\, \mathcal{F}_t\Big)}_{\displaystyle\downarrow}$$

$$u(T-t, L_t) = E\Big(g(L_T)\, \mathbb{1}_{\{T<\tau_{\overline{D}}\}} \,\Big|\, \mathcal{F}_t\Big),$$

with stopping time $\tau_{\overline{D},t}$ first exit time after t out of $\overline{D}$

$$\text{and } u \text{ solves} \quad \partial_t u + \mathcal{A}\, u = f \quad \text{in } \big(\widetilde{H}_\eta^{\alpha/2}(D)\big)^*, \qquad u(0) = g$$