Entry-and-exit problems with a finite number of cycles

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References (short selection)

A. Dixit, Entry and Exit Decisions under Uncertainty, *Journal of Political Economy,* **97,** 3 (1989), 620-638.

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Temperature control

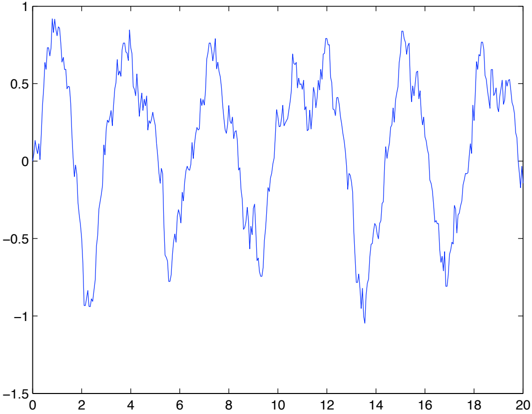

Figure 1: Temperature control

Model description

- $$\mathbb{E}\left[\int_0^\infty e^{-\alpha s} r(X(s), Y(s))\, ds - \sum_{k=1}^N e^{-\alpha\tau_k^{(in)}} c_k^{(in)} - \sum_{k=1}^N e^{-\alpha\tau_k^{(out)}} c_k^{(out)}\right]$$

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- $$dX(t) = \mu(X(t), Y(t))\, dt + \sigma(X(t), Y(t))\, dW(t),$$
  $$X(0) = x_0,\ Y(0) = 0 =: y_0$$

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- $$\begin{aligned} dX(t) &= \mu(X(t), Y(t))\, dt + \sigma(X(t), Y(t))\, dW(t), \\ & X(0) = x_0,\ Y(0) = 0 =: y_0 \end{aligned}$$

- **Special case:** Always the same dynamic, $y \equiv 0$.

Assumptions

- Integrability conditions on $r(x,y)$; $c_k^{(in)} + c_k^{(out)} > 0$, $1 \leq k \leq N$, $c_k^{(in)}, c_k^{(out)} \geq 0$.

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- For $y = 0$ and $y = 1$ there exists a positive and strictly decreasing solution ϕ_y as well as a non-negative strictly increasing solution ψ_y of

$$\boxed{Af(\cdot, y) = \alpha f(\cdot, y),}$$

where A_y, i. e. $A_y f = Af(\cdot, y)$, is the *generator of the diffusion* with coefficients μ_y and σ_y.

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- further technical conditions, s. Appendix

Example

- *Mean reverting processes,* $y \in \{0, 1\}$ fixed,

$$dX_y(t) = \mu_y(1 - \gamma_y X_y(t))dt + \sigma_y(t)\sqrt{X_y(t)}\, dW(t),$$

$X_y(0) = x_0 > 0$ given.

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- ϕ_y and ψ_y can be expressed in terms of well known power series (*Kummer functions*)

- $r_y(x) = k_y x - c_y$ and $f_y(x) := -E_x\left[\int_0^\infty e^{-\alpha s} r(X_y(s), y) ds\right]$

$$\Rightarrow \quad \boxed{f_{r_y}(x) = -\frac{k_y}{\alpha + \gamma_y \mu_y} x - \frac{1}{\alpha}\left(\frac{k_y \mu_y}{\alpha + \gamma_y \mu_y} - c_y\right)}$$

Linear programming imbedding

Notation:

- For a feasible sequence of stopping times $\tau_k^{(in)}$ and $\tau_k^{(out)}$, $1 \leq k \leq N$, we define:

$$\tau_{2k} := \tau_{N-k+1}^{(in)} \qquad \text{and} \qquad \tau_{2k-1} := \tau_{N-k+1}^{(out)}.$$

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where

$$\begin{aligned} g_k^{(in)}(x,0) &= f_r(x,0) - f_r(x,1) - c_k^{(in)}, \\ g_k^{(out)}(x,1) &= f_r(x,1) - f_r(x,0) - c_k^{(out)}, \end{aligned}$$

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and, $y = 0, 1$,

$$f_r(x,y) := (1-y)f_0(x) + yf_1(x).$$

Linear programming imbedding (continued)

- Let the process (Z_t), $Z(0) = 2N =: z_0$, decrease by 1 at each finite intervention time τ_k. Consider test functions $f \in \mathcal{C}_c^2((x_e, x_r) \times \{0, 1\} \times \{0, \dots, 2N\})$:

Linear programming imbedding (continued)

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- $\tilde{A}f(x,y,z) := \dfrac{\sigma^2(x,y)}{2}\dfrac{\partial^2 f}{\partial x^2}(x,y,z) + \mu(x,y)\dfrac{\partial f}{\partial x}(x,y,z)$

Linear programming imbedding (continued)

- The occupation measure $\mu^{\tilde{\tau}}$ and the intervention measures $\nu_k^{\tilde{\tau}}$:

$$\mu^{\tilde{\tau}}(G) := E\left[\int_0^\infty e^{-\alpha s} I_G(X(s), Y(s), Z(s))ds\right]$$

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and

$$\nu_k^{\tilde{\tau}}(G) := E\left[\int_0^\infty e^{-\alpha s} I_G(X(s), Y(s-), Z(s-)) I_k(Z(s-) d\lambda_{\tilde{\tau}}(s))\right]$$

where

$$\lambda_{\tilde{\tau}}(t) := \sum_{k=1}^{2N} I_{[0,t]}(\tau_k) = \sum_{k=1}^{2N} I_{[\tau_k,\infty)}(t).$$

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Reformulate the *finite switching problem* as an (infinite dimensional) **linear program**

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$$\begin{cases} \text{Maximize} \quad \sum_{k=1}^{2N} \int g_k(x,y)\,\nu_k(dx \times dy) \end{cases}$$

Linear programming imbedding (continued)

Reformulate the *finite switching problem* as an (infinite dimensional) **linear program**

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\begin{cases}
\text{Maximize} & \displaystyle\sum_{k=1}^{2N} \int g_k(x,y)\, \nu_k(dx \times dy) \\
\text{Subject to} & \displaystyle f(x_0,y_0,z_0) = -\int [\tilde{A}f(x,y,z) - \alpha f(x,y,z)]\, \mu(dx \times dy \times dz) \\
& \displaystyle\quad -\sum_{k=1}^{2N} \int [f(x,1-y,k-1) - f(x,y,k)] \nu_k(dx \times dy), \quad \forall f \in C_c^2,
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& \displaystyle\quad -\sum_{k=1}^{2N} \int [f(x,1-y,k-1) - f(x,y,k)]\nu_k(dx \times dy), \quad \forall f \in C_c^2, \\
& \displaystyle\int 1\,\nu_k(dx) \leq 1, \qquad k = 0,\ldots,2N, \\
& \displaystyle\int 1\,\mu(dx) \leq 1/\alpha, \\
& \mu, \nu_k \text{ measures, } \mathit{supp}(\nu_1 \times \cdots \times \nu_{2N}) \subset \Xi_{\bar{w}}, \\
& \text{feasible set of switching locations (s. Appendix).}
\end{cases}
$$

Auxiliary LP

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Step 3. Define $f(x,y) := \tilde{\psi}(x) I_{\{0\}}(y) + \tilde{\phi}(x) I_{\{1\}}(y)$, $\tilde{\psi}$, $\tilde{\phi}$ resp., a mollified version of ψ_0, ϕ_1 resp., and $f_k(x,y,z) := f(x,y) I_{\{k\}}(z)$. Consider

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$$\begin{cases} \text{Maximize} & \displaystyle\sum_{j=1}^{2N} \int g_j(x,y)\, \nu_j(dx \times dy) \\ \text{Subject to} & \displaystyle -\sum_{j=1}^{2N} \int [f_k(x, 1-y, j-1) - f_k(x,y,j)] \nu_j(dx \times dy) \\ & \qquad\qquad = f_k(x_0, y_0, z_0), \quad k = 1, \ldots, 2N, \\ & \nu_j \text{ are finite measures with } \text{supp}(\nu_1 \times \cdots \times \nu_{2N}) \subset \Xi_R. \end{cases}$$

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\begin{array}{rcll}
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\int \psi(x_2)\,\nu_2(dx_2) & - & \int \psi(x_3)\,\nu_3(dx_3) & = & 0 \\
\int \phi(x_3)\,\nu_3(dx_3) & - & \int \phi(x_4)\,\nu_4(dx_4) & = & 0 \\
 & \vdots & & & \\
\int \phi(x_{2N-1})\,\nu_{2N-1}(dx_{2N-1}) & - & \int \phi(x_{2N})\,\nu_{2N}(dx_{2N}) & = & 0 \\
\int \psi(x_{2N})\,\nu_{2N}(dx_{2N}) & & & = & \psi(x_0).
\end{array}
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$$a_{ij}(x_j) = \begin{cases} \phi(x_j), & i = 2k-1, j = i, k = 1, \dots, N, \\ -\phi(x_j), & i = 2k-1, j = i+1, k = 1, \dots, N, \\ \psi(x_j), & i = 2k, j = i, k = 1, \dots, N, \\ -\psi(x_j), & i = 2k, j = i+1, k = 1, \dots, N, \\ 0, & \text{otherwise.} \end{cases}$$

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Define $b := (0, \ldots, 0, \psi(x_0))^T$. Then the system of constraints of the auxiliary LP takes the form:

$$\sum_{j=1}^{2N} \int a_{ij}(x_j)\, \nu_j(dx_j) = b_j, \qquad i = 1, \ldots, 2N.$$

Theorem 1. Let $n \in \mathbb{N}$ and for $j = 1, \ldots, n$, let $(S_j, \mathcal{X}_j)$ be measurable spaces; let $g_j : S_j \to \mathbb{R}$ and $a_{ij} : S_j \to \mathbb{R}$, $1 \leq i \leq n$, be measurable functions and $b \in \mathbb{R}^n$. Define $\hat{A}(s_1, \ldots, s_n) := (a_{ij}(s_j))_{1 \leq i,j \leq n}$ and let $\hat{A}_j$ be the matrix $\hat{A}$ with its jth column replaced by b; denote the $(n-1)$ variables $(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$ by $\bar{s}_j$. For finite measures ν_j on $(S_j, \mathcal{X}_j)$, $1 \leq j \leq n$, let ν be the corresponding product measure and $\hat{a} := \int \det \hat{A}(s) d\nu(s)$. Assume $\nu_1, \ldots, \nu_n$ are feasible measures for the linear program

Nonlinear optimization problem

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\begin{cases}
\text{Maximize} & \displaystyle\sum_{j=1}^{n} \int g_j(s_j)\, \nu_j(ds_j) \\
\text{Subject to} & \displaystyle\sum_{j=1}^{n} \int a_{ij}(s_j)\, \nu_j(ds_j) = b_j, \quad i = 1, \dots, n \\
& \nu_j \text{ a finite measure on } (S_j, \mathcal{X}_j), \;\; j = 1, \dots, n.
\end{cases}
$$

Nonlinear optimization problem

If $\det\left(\hat{A}\right) > 0$ and $\hat{a} < \infty$ then

$$\sum_{j=1}^{n} \int g_j(s_j)\,\nu_j(ds_j) \le \sup_{s\in S} \left\{ (\det \hat{A}(s))^{-1} \sum_{j=1}^{n} g_j(s_j) \det \hat{A}_j(\bar{s}_j) \right\}.$$

If the supremum is achieved at a point $s^* = (s_1^*, \ldots, s_n^*)$, then for $j = 1, \ldots, n$, the measures ν_j^* which are concentrated on $\{s_j^*\}$ are optimal for the linear program.

Nonlinear optimization problem

Theorem 2. Under the technical conditions referred to above, the stochastic multiple-intervention problem of maximizing (1) over admissible sets of stopping times $\tilde{\tau} \in \mathcal{A}_1$ is equivalent to the nonlinear optimization problem of maximizing

$$J_{nl}(x) = (\det \hat{A}(x))^{-1} \sum_{j=1}^{2N} g_j(x_j) \det \hat{A}_j(\bar{x}_j)$$

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over $x \in S$. Using an optimizer $x^* = (x_1^*, \ldots, x_{2N}^*)$ of J_{nl}, the intervention times

$$\tau_k^* = \inf\{t : (X(t-), Z(t-)) = (x_k^*, k)\}, \qquad k = 1, \ldots, 2N,$$

are optimal for the stochastic problem.

Special cases: $N = 1$ and $N = 2$

This optimality result can be best understood by considering the simple examples of a stochastic problem in which one is limited to a single increase and decrease in production and two increases and two decreases of production levels.

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Example: ONE CYCLE
The problem of interest consists of a single decision to open up production and one to close out production.

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Example: One Cycle
The problem of interest consists of a single decision to open up production and one to close out production. To be consistent with the indexing of the stopping times, let τ_2 denote the time and x_2 denote the level at which production is started (hence there are two decisions to be made) and similarly τ_1 is the time and x_1 the level for closing down production (the last decision). The restricted auxiliary linear program for this single cycle problem is therefore

Special cases: $N = 1$ and $N = 2$

$$\begin{cases} \text{Maximize} & \int g_1(x_1)\,\nu_1(dx_1) \;+\; \int g_2(x_2)\,\nu_2(dx_2) \\ \text{Subject to} & \int \phi(x_1)\,\nu_1(dx_1) \;-\; \int \phi(x_2)\,\nu_2(dx_2) \;=\; 0, \\ & \qquad\qquad\qquad\qquad\quad \int \psi(x_2)\,\nu_2(dx_2) \;=\; \psi(x_0). \end{cases}$$

Special cases: $N = 1$ and $N = 2$

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\begin{cases}
\text{Maximize} & \int g_1(x_1)\,\nu_1(dx_1) + \int g_2(x_2)\,\nu_2(dx_2) \\
\text{Subject to} & \int \phi(x_1)\,\nu_1(dx_1) - \int \phi(x_2)\,\nu_2(dx_2) = 0, \\
 & \int \psi(x_2)\,\nu_2(dx_2) = \psi(x_0).
\end{cases}
$$

From this we see that $\det\hat{A}(x_1, x_2) = \phi(x_1)\psi(x_2)$, $\det\hat{A}_1(x_2) = \psi(x_0)\phi(x_2)$ and $\det\hat{A}_2(x_1) = \phi(x_1)\psi(x_0)$. The nonlinear function to be optimized is therefore

$$
J_{nl}(x_1, x_2) := \frac{g_1(x_1)\phi(x_2) + \phi(x_1)g_2(x_2)}{\phi(x_1)\psi(x_2)} \cdot \psi(x_0).
$$

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Example: Two Cycles

Now consider the case where production is started up, then mothballed, then restarted and finally closed down resulting in two cycles.

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Now consider the case where production is started up, then mothballed, then restarted and finally closed down resulting in two cycles. Let τ_4 and x_4 denote the time and level of initial start-up, τ_3 and x_3 the time and level at which mothballing occurs and similarly for τ_2, x_2, τ_1 and x_1 for the second cycle.

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$$\begin{cases} \text{Max.} & \int g_1(x_1)\,\nu_1(dx_1) + \int g_2(x_2)\,\nu_2(dx_2) + \int g_3(x_3)\,\nu_3(dx_3) + \int g_4(x_4)\,\nu_4(ds_4) \\ \text{S.t.} & \int \phi(x_1)\,\nu_1(dx_1) - \int \phi(x_2)\,\nu_2(dx_2) = 0, \\ & \int \psi(x_2)\,\nu_2(dx_2) - \int \psi(x_3)\,\nu_3(dx_3) = 0, \\ & \int \phi(x_3)\,\nu_3(dx_3) - \int \phi(x_4)\,\nu_4(dx_4) = 0, \\ & \int \psi(x_4)\,\nu_4(dx_4) = \psi(x_0). \end{cases}$$

Special cases: $N = 1$ and $N = 2$

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\begin{aligned}
J_{nl}(x) \;=\; & \Big[g_1(x_1)\phi(x_2)\psi(x_3)\phi(x_4) + \phi(x_1)g_2(x_2)\psi(x_3)\phi(x_4) \\
& + \phi(x_1)\psi(x_2)g_3(x_3)\phi(x_4) + \phi(x_1)\psi(x_2)\phi(x_3)g_4(x_4) \Big] \\
& \cdot \frac{\psi(x_0)}{\phi(x_1)\psi(x_2)\phi(x_3)\psi(x_4)} \,.
\end{aligned}
$$

Iterative Solution Approach

The structure of the nonlinear function enables an efficient iterative approach to be employed in computing an optimal solution $x^* = (x_1^*, \ldots, x_{2N}^*)$. To easily describe this method, we reconsider the single-cycle and double-cycle examples of the previous section.

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Observe the function J_{nl} of (19) can be rewritten as

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The first-order conditions for optimality imply

$$\psi(x_2)\left[g_2'(x_2) + \frac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi'(x_2)\right] - [g_2(x_2) + \frac{g_1(x_1^*)}{\phi(x_1^*)} \cdot \phi(x_2)]\psi'(x_2) = 0.$$

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$$+$$

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& + \frac{\psi(x_0)}{\psi(x_4)} \cdot \frac{\phi(x_4)}{\phi(x_3)} \cdot \frac{\psi(x_3)}{\psi(x_2)} \cdot \frac{\phi(x_2)}{\phi(x_1)} \cdot g_1(x_1).
\end{aligned}
$$

Iterative Solution Approach

Theorem 3. A maximizer $x^* = (x_1^*, \dots, x_{2N}^*)$ of

$$J_{nl}(x) = (\det \hat{A}(x))^{-1} \sum_{j=1}^{2N} g_j(x_j) \det \hat{A}_j(\bar{x}_j)$$

is obtained by sequentially solving $2N$ one-dimensional nonlinear optimization problems of the form $\frac{h_j(x_1^*, \dots, x_{j-1}^*, x_j)}{\theta_j(x_j)}$ for $j = 1, \dots, 2N$,

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$$h_j(x_1, \dots, x_j) = g_j(x_j) + \frac{\theta_{j-1}(x_j)}{\theta_{j-1}(x_{j-1})} g_{j-1}(x_{j-1}) + \cdots + \prod_{i=1}^{j-1} \frac{\theta_i(x_{i+1})}{\theta_i(x_i)} \cdot g_1(x_1).$$

Example

N	a_N^*	b_N^*	*value*
1	0.125109	1.58976	3.98052
2	0.321411	1.47036	4.49382
3	0.354076	1.44612	4.59896
⋮	⋮	⋮	⋮
8	0.363173	1.43929	4.62853
9	0.363177	1.43929	4.62854
10	0.363178	1.43929	4.62855

Table 1: Trigger prices for exit a_N^* and entry b_N^* and Values as functions of remaining cycles N for a mean reverting process ($\gamma_0 = \gamma_1 = 1$, $\mu_0 = \mu_1 = 0.1$, $\sigma_0 = \sigma_1 = 0.3$) when $\alpha = 0.04$, $c^{(in)} = 2$, $c^{(out)} = 0.2$, $k_0 = 0$, $k_1 = 1$, $c_{fixed\,0} = 0$, $c_{fixed\,1} = 0.8$, and $x_0 = 0.8$.

Example

σ	MR a_0^*	MR b_0^*	MR *value*	GBM a_0^*	GBM b_0^*	GBM *value*
0.1@	0.4599	1.0495	2.2477	0.6278	1.1265	2.3482
0.15	0.4457	1.1607	2.7126	0.5820	1.2319	3.8760
0.2@	0.4195	1.2608	3.3005	0.5452	1.3361	5.3543
0.25	0.3910	1.3526	3.9509	0.5144	1.4412	6.7427
0.3@	0.3632	1.4393	4.6286	0.4880	1.5481	8.0211

Table 2: Trigger prices (in \$/lb) for exit (a_0^*) and entry (b_0^*) and values (in \$10M units) as functions of σ for mean-reverting ($\mu = 0.1$, $\gamma = 1$) and geometric Brownian motion ($\mu = 0$) price models; $x_0 = 0.8$, $\alpha = 0.04$.

Notation and technical conditions:

- CONDITION 1. The coefficients μ and σ of the diffusion X and the income rate function r are such that

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s}[|r(X(s),0)| + |r(X(s),1)|]\,ds\right] < \infty.$$

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- CONDITION 2. For $y = 0, 1$, the eigenvalue problem $Af(\cdot, y) = \alpha f(\cdot, y)$ has both a positive, strictly decreasing solution ϕ_y and a non-negative, strictly increasing solution ψ_y.

- CONDITION 3.

(a) For $k = 1, \ldots, N$, there exists some values $x_k^{(in)} > \bar{w}$ and $x_k^{(out)} < \bar{w}$ in (x_l, x_r) such that $g_k^{(in)}(x_k^{(in)}, 0) > 0$ and $g_k^{(out)}(x_k^{(out)}, 1) > 0$.

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(c) For $k = 1, \ldots, N$, $\lim_{x \searrow x_l} \frac{g_k^{(out)}(x, 1)}{\phi_1(x)} = 0$.

Appendix

- $\Xi_{\bar{w}} = \{(x_1, y_1, x_2, y_2 \ldots, x_{2N}, y_{2N}) : x_{2j-1} \leq \bar{w}, y_{2j-1} = 1, x_{2j} \geq \bar{w}, y_{2j} = 0, j = 1, 2, \ldots, N\}$

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- For $a > x_e$ and $b < x_r$ define
  $\Xi_R = \{(x_1, y_1, x_2, y_2, \ldots, x_{2n}, y_{2N}) : x_{2k-1} \in [a, \bar{w}], y_{2k-1} = 1, x_{2k} \in [\bar{w}, b], y_{2k} = 0, k = 1, 2, \ldots, N\}$

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- CONDITION 4. $X(0) = x_0 \leq \bar{w}$.

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- CONDITION 5. A sequence of stopping times

$$\left(\tau_k^{(in)}, \tau_k^{(out)}\right)_{1 \leq k \leq N}, \ \tau_1^{(in)} \leq \tau_1^{(out)} \leq \tau_2^{(out)} \leq \cdots,$$

is an admissible sequence of switching times, if for each k,

$X\left(\tau_k^{(in)}\right) \geq \bar{w}$ on the set $\left\{\tau_k^{(in)} < \infty\right\}$ and $X\left(\tau_k^{(out)}\right) \leq \bar{w}$

on the set $\left\{\tau_k^{(out)} < \infty\right\}$.

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- These optimal hitting levels are characterized as a maximizing point for a high-dimensional nonlinear function and can be efficiently and iteratively determined as the solutions of successive **1-dimensional nonlinear maximization problems**.