On the existence of stationary deterministic Nash equilibria in stochastic games

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Non-zero-sum stochastic games with perfect information and two players

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- A – action space for Player 1
- B – action space for Player 2
- All three sets are assumed to be Borel sets in Polish spaces

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- K_A – constraint set of Player 1 (Borel subset of $S \times A$)
- $A(s) := \{a \in A : (s, a) \in K_A\}$ – set of admissible actions for Player 1 in state $s \in S$
- K – constraint set of Player 2 (Borel subset of $K_A \times B$)
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- $p(\cdot|\, s, a, b)$– transition probability from K to S (Borel measurable)
- $k^{(i)} : K \to \mathbb{R}$ – cost function from K to $\mathbb{R}$ for Player i $(i = 1, 2)$. (Borel measurable)
- $\alpha \in (0, 1)$ – discount factor

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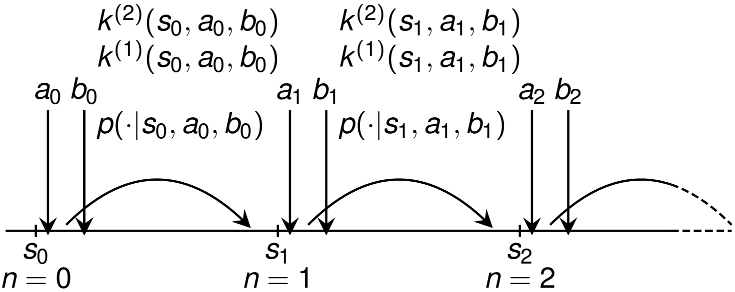

Decision rules

- Decision rule of Player 1 (Markov deterministic)

$$f : S \to A \text{ with } f(s) \in A(s) \text{ for all } s \in S$$

- Decision rule of Player 2 (Markov deterministic)

$$g : K_A \to B \text{ with } g(s, a) \in B(s, a) \text{ for all } (s, a) \in K_A$$

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Strategies

- A sequence $\Pi = (f_n)$ or $P = (g_n)$ of decision rules of the Player 1 or Player 2 is called a strategy of that player.
- A strategy $\Pi = (f_n)$ of Player 1 with $f_n = f$ for all $n \in \mathbb{N}_0$ is called stationary and denoted by f^∞. A stationary strategy g^∞ of Player 2 is analogously defined.

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The total expected discounted cost

- Let us consider the space $\Omega := (S \times A \times B)^\infty$, endowed with the product σ-algebra $\mathcal{F}$.
- For every strategy pair (Π, P) and each initial state $s \in S$, there exist a probability measure $P_{s,\Pi,P}$ and a stochastic process (S_n, A_n, B_n) on $(\Omega, \mathcal{F})$.
- The total expected discounted cost for Player i is

$$V^{(i)}_{\Pi,P}(s) = E_{s,\Pi,P} \sum_{m=0}^{\infty} \alpha^m k^{(i)}(S_m, A_m, B_m)$$

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Nash equilibrium

- A strategy pair (Π^*, P^*) is called a Nash equilibrium pair on S, if
$$V^{(1)}_{\Pi^*,P^*}(s) \leq V^{(1)}_{\Pi,P^*}(s)$$
and
$$V^{(2)}_{\Pi^*,P^*}(s) \leq V^{(2)}_{\Pi^*,P}(s)$$
for all $s \in S$, Π, P.
- In the zero-sum case, every discounted stochastic game with perfect information and finite state and action spaces has a Nash equilibrium pair (that means an optimal strategy pair).
- Unfortunately, this property does not hold in a general case.

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Example

- Given a stationary strategy f^∞ of Player 1. Let

$$v_f^{(2)}(s) := \inf_P V_{f^\infty,P}^{(2)}(s)$$

for all $s \in S$.

- From the Markov decision theory we know that

$$v_f^{(2)}(s) = \inf_{b \in B(s,f(s))} \{k^{(2)}(s,f(s),b) + \alpha \int_S v_f^{(2)}(t) p(dt|\, s,f(s),b)\} \quad (1)$$

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Assumption 1 Assumption 2

- If there exists a decision rule g^* of Player 2 with $g^*(s,a) \in B(s,a)$, so that

$$v_f^{(2)}(s) = k^{(2)}(s, f(s), g^*(s, f(s))) + \alpha \int_S v_f^{(2)}(t) p(dt|\, s, f(s), g^*(s, f(s))) \quad (2)$$

for every decision rule f of Player 1 and all $s \in S$, then it holds

$$V^{(2)}_{f^\infty, g^{*\infty}}(s) \leq V^{(2)}_{f^\infty, P}(s)$$

for every decision rule f and all P, all $s \in S$.

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$$v_{g^*}^{(1)}(s) = \inf_{a \in A(s)} \{k^{(1)}(s,a,g^*(s,a)) + \alpha \int_S v_{g^*}^{(1)}(t) p(dt|\, s,a,g^*(s,a)) \tag{3}$$

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Assumption 1

$$p(\cdot|s, a, b) = p(\cdot|s, a)$$

for all $s \in S$, $a \in A$, $b \in B$. Reference Equation 1

Conclusions under Assumption 1

- $$v_f^{(2)}(s) = \inf_{b \in B(s,f(s))} \{k^{(2)}(s, f(s), b)\} + \alpha \int_S v_f^{(2)}(t) p(dt | s, f(s))$$

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Assumption 1: An inventory problem

- Two producers compete for one kind of material which they need for their own productions.
- The first producer as Player 1 can store his product (Product 1). He chooses that how much he produces and tries to keep the inventory cost as low as possible.
- The second producer as Player 2 produces a kind of perishable product (Product 2).
- We assume that for both producers one unit material per one unit product is needed .

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- $s_n \in S = (-\infty, M]$, s_n is the stock level in the warehouse of the Player 1 at the beginning of period n.
- $a_n \in A(s_n) = [s_n, M_1]$, Player 1 orders $a_n - s_n$ units of material and produces immediately $a_n - s_n$ units of Product 1. a_n can be interpreted as the stock level in the warehouse of Player 1 immediately after producing of Product 1.
- $b_n \in B = [0, M_2]$, Player 2 orders b_n units of material and produces immediately b_n units of Product 2.
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- Let ξ_n be the market demand of Product 1 in period n, for $n = 1, 2, 3 \ldots$. We assume that ξ_n are independent and identically distributed random variables with probability density ψ_1.
- Analogous η_n is the market demand of Product 2 in period n, for $n = 1, 2, 3 \ldots$. We also assume that η_n are independent and identically distributed random variables with probability density ψ_2.

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- In period n the Player 1 sells ξ_n units of Product 1, then at the beginning of period $n+1$ the stock level is $s_{n+1} = a_n - \xi_n$.
- The transition probability

$$p(C|s,a,b) = \int_S \mathbb{I}_C(a-\xi)\varphi_1(\xi)d\xi$$

$\forall$ Borel set $C \subseteq S$

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- Let $C_2(a-s,b)$ be the reward of selling one unit Product 2, h_2 be the loss per unit Product 2, which could not be sold, and g_2 be the shortage cost per one unit Product 2.
- Let $L_2(b) = \int_0^b h_2 \cdot (b-\eta)\varphi_2(\eta)d\eta + \int_b^\infty g_2 \cdot (\eta - b)\varphi_2(\eta)d\eta$.
- The cost function of Player 2 is

$$k^{(2)}(s,a,b) = L_2(b) - C_2(a-s,b) \cdot b.$$

- Choose $g^*(s,a)$ with

$$k^{(2)}(s,a,g^*(s,a)) = \min_{b \in B(s,a)} \{k^{(2)}(s,a,b)\}$$

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Assumption 2

- Let the state space $S = S_1 \times S_2$, each state can be denoted by $s = (s_1, s_2)$, for $s_1 \in S_1$, $s_2 \in S_2$.
- The action set of Player 2 depends only on s_1 and the cost function of him depends only on s_1 and his action b, it means: $B(s, a) = B(s_1)$, $k^2(s, a, b) = k^2(s_1, b)$.
- The transition probability

$$p(\widetilde{S}_1 \times \widetilde{S}_2 | (s_1, s_2), a, b) = p_1(\widetilde{S}_1 | s_1, b) p_2(\widetilde{S}_2 | s_1, s_2, a, b)$$

$\forall$ Borel sets $\widetilde{S}_1, \widetilde{S}_2$ with $\widetilde{S}_1 \subseteq S_1$, $\widetilde{S}_2 \subseteq S_2$.

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Conclusions under Assumption 2

- The decision of Player 1 has no influence on the total cost of Player 2.
- $$v_f^{(2)}(s) = v^{(2)}(s_1) = \inf_{b \in B(s_1)} \{k^{(2)}(s_1, b) + \alpha \int_{S_1} v^{(2)}(\xi) p_1(d\xi | s_1, b)\}$$

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Assumption 2: Example

- We consider a financial product which can be regarded as an American option with expiry time $T = \infty$
- The buyer has the right, but not the obligation to buy or sell an asset for a certain price.
- The seller and buyer are the Player 1 and Player 2, respectively. Both players want to maximize their expected discounted utilities.
- Let the state $s = ((x, y), z)$, where $x \in X$ is the market information of the financial market, $z \in Z$ is the wealth of the seller, $y \in Y = \{0, 1\}$. Especially, $y = 0$ indicates that the option has not been exercised in this period and $y = 1$ means the option has been already exercised.
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- The seller owns $d + 1$ securities. Their prices are dependent on x and are given by the vector $S(x) = (S_0(x), S_1(x), \ldots, S_d(x))^T$.
- The action of the seller is a portfolio $a = h = (h_0, h_1, \ldots, h_d)^T$. The wealth of the portfolio is $z = h^T S(x)$.
- The admissible action set of seller is $A(s) = A(x, z) = \{h \in \Re^{d+1} : h^T S(x) = z\}$. It holds also $z_{n+1} = h_n^T S(x_{n+1})$.

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- The cost function of the seller

$$k^1(z,b) = \begin{cases} -u^1(z) & \text{if} \quad b = 1 \\ 0 & \text{if} \quad b = 0 \end{cases}$$

- The cost function of the buyer

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- The transition probability is

$$p((\widetilde{X}, Y_{n+1}), \widetilde{Z}|(x, y_n), z, h, b) =$$

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- For the total expected discounted cost of the buyer it holds

$$v_f^{(2)}(s) = v^{(2)}(x, y)$$

- $$v^{(2)}(x, 0) = \min_{b \in \{0,1\}} \{k^2(x, b) + \alpha \int_X v^{(2)}(\xi, 0) p_1(d\xi|x) p_2(0|0, b)\}$$

  for $x \in X$

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Assumption 3

- $$k^2(s,a,b) = k^I(s) + k^{II}(a+b)$$
- $$p(S'|s,a,b) = \tilde{p}(S'|a+b)$$

  $\forall$ Borel set $S' \subseteq S$, $\forall s \in S$
- Let $G_\alpha(\beta) = k^{II}(\beta) + \alpha \int\limits_S k^I(t)\tilde{p}(dt|\beta)$,

  $G_\alpha(\beta^*) = \min\limits_{\beta \in \Re} G_\alpha(\beta)$.

  It holds

  $$\beta^* - a \in B(s,a)$$

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Conclusion under Assumption 3

$$v_f^{(2)}(s) = v^{(2)*}(s) = k^I(s) + \frac{1}{1-\alpha} G_\alpha(\beta^*)$$

$\forall s \in S$

Example

Nash Equilibrium

- $X = \{1, 2\}$,
- $A(1) = \{1, 2\}$, $A(2) = \{1\}$
- $B(1, 1) = B(1, 2) = \{1\}$, $B(2, 1) = \{1, 2\}$,
- $p(1, 1, 1; 1) = \frac{2}{3}$, $p(2, 1, 1; 1) = \frac{2}{3}$, $p(1, 2, 1; 1) = \frac{1}{3}$, $p(2, 1, 2; 1) = \frac{1}{3}$,
- $k^{(1)}(2, 1, 1) = -1$, $k^{(1)}(2, 1, 2) = 1$, $k^{(2)}(1, 1, 1) = -1$, $k^{(2)}(1, 2, 1) = 1$ ($k^{(i)}(x, a, b) = 0$ everywhere),
- $\alpha = \frac{3}{4}$.

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Example

Every player has 2 stationary deterministic strategies, namely

- $\Pi_1 = f^{1\infty}$ with $f^1(1) = f^1(2) = 1$ and $\Pi_2 = f^{2\infty}$ with $f^2(1) = 2$, $f^2(2) = 1$ for Player 1.
- $P_1 = g^{1\infty}$ with $g^1(1,1) = g^1(1,2) = g^1(2,1) = 1$ and $P_2 = g^{2\infty}$ with $g^2(1,1) = g^2(1,2) = 1$, $g^2(2,1) = 2$ for Player 2.

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Example

- Let

$$v_{ij}^{(l)}(x) := v_{\alpha,\Pi_i,\mathrm{P}_j}^{(l)}(x), \quad \mathbf{v}_{ij}^{(l)} = \begin{pmatrix} v_{ij}^{(l)}(1) \\ v_{ij}^{(l)}(2) \end{pmatrix}$$

- Then

$$\mathbf{v}_{11}^{(1)} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_{11}^{(2)} = \begin{pmatrix} -3 \\ -2 \end{pmatrix},$$

$$\mathbf{v}_{12}^{(1)} = \begin{pmatrix} \frac{4}{3} \\ \frac{8}{3} \end{pmatrix}, \qquad \mathbf{v}_{12}^{(2)} = \begin{pmatrix} -\frac{8}{3} \\ -\frac{4}{3} \end{pmatrix},$$

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- $$\mathbf{v}_{12}^{(1)} < \mathbf{v}_{22}^{(1)}, \quad \mathbf{v}_{12}^{(2)} > \mathbf{v}_{11}^{(2)}$$
- $$\mathbf{v}_{21}^{(1)} < \mathbf{v}_{11}^{(1)}, \quad \mathbf{v}_{21}^{(2)} > \mathbf{v}_{22}^{(2)}$$
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[3] Heinz-Uwe Küenle *On a discounted inventory game.*

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