

On the Optimality of Kelly Strategies

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Where to invest?

The central question for investment management practitioners is: where should I put my money?

Ideas such as the utility of wealth, and even expected portfolio returns (and risks) take a backseat.

Yet, in finance theory, the optimal asset allocation appears as less important than the utility of wealth. This is particularly true in the duality/martingale approach, where the optimal asset allocation is often obtained last.

Kelly Strategies

As a result of the differences between investment management theories and practice, a number of techniques and tools have been created to speed up the derivation of an optimal or near-optimal asset allocation.

Fractional Kelly strategies are one such technique.

The Kelly (criterion) portfolio invest in the growth maximizing portfolio.

- ▶ In continuous time, this represents an investment in the optimal portfolio under the logarithmic utility function.
- ▶ The Kelly criterion comes from the signal processing/gambling literature (see Kelly [7]).
- ▶ See MacLean, Thorpe and Ziemba [11] for a thorough view of the Kelly criterion.

A number of “great” investors from Keynes to Buffet can be viewed as Kelly investors. Others, such as Bill Gross probably are.

The Kelly criterion portfolio has a number of interesting properties but it is inherently risky.

To reduce the risk while keeping some of the nice properties, MacLean and Ziemba proposed the concept of *fractional Kelly investment*:

- ▶ Invest a fraction k of the wealth in the Kelly portfolio;
- ▶ invest a fraction $1 - k$ in the risk-free asset.

Fractional Kelly strategies are

- ▶ optimal in the Merton world of lognormal asset prices and with a globally risk-free asset;
- ▶ not optimal when the lognormality assumption is removed.

Three questions:

- ▶ Why are fractional Kelly strategies not optimal outside of the Merton model?
- ▶ How is an optimal strategy constructed?
- ▶ Can we improve the definition of fractional Kelly strategies to guarantee optimality?

In this talk, we will use the term Kelly strategies to refer to:

- ▶ The Kelly portfolio.
- ▶ Fractional Kelly investment.

Back to Basics: Kelly Strategies in the Merton Model

Key ingredients:

- ▶ Probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$;
- ▶ \mathbb{R}^m -valued (\mathcal{F}_t) -Brownian motion $W(t)$;
- ▶ Price at time t of the i th security is $S_i(t)$, $i = 0, \dots, m$;

The dynamics of the money market account and of the m risky securities are respectively given by:

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = s_0 \quad (1)$$

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{k=1}^N \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, \dots, m \quad (2)$$

where $r, \mu \in \mathbb{R}^m$ and $\Sigma := [\sigma_{ij}]$ is a $m \times m$ matrix.

Assumption

The matrix Σ is positive definite.

Let $\mathcal{G}_t := \sigma(S(s), 0 \leq s \leq t)$ be the sigma-field generated by the security process up to time t .

An *investment strategy* or *control process* is an \mathbb{R}^m -valued process with the interpretation that $h_i(t)$ is the fraction of current portfolio value invested in the i th asset, $i = 1, \dots, m$.

- ▶ Fraction invested in the money market account is $h_0(t) = 1 - \sum_{i=1}^m h_i(t)$.

Definition

An \mathbb{R}^m -valued control process $h(t)$ is in class $\mathcal{A}(T)$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0, t]) \otimes \mathcal{G}_t\}_{t \geq 0}$ and is càdlàg;
2. $P\left(\int_0^T |h(s)|^2 ds < +\infty\right) = 1, \quad \forall T > 0;$
3. the Doléans exponential χ_t^h , given by

$$\chi_t^h := \exp \left\{ \gamma \int_0^t h(s)' \Sigma dW_s - \frac{1}{2} \gamma^2 \int_0^t h(s)' \Sigma \Sigma' h(s) ds \right\} \quad (3)$$

is an exponential martingale, i.e. $\mathbf{E} [\chi_T^h] = 1$

We say that a control process $h(t)$ is *admissible* if $h(t) \in \mathcal{A}(T)$.

Taking the budget equation into consideration, the wealth, $V(t)$, of the asset in response to an investment strategy $h \in \mathcal{H}$ follows the dynamics

$$\frac{dV(t)}{V(t)} = rdt + h'(t) (\mu - r\mathbf{1}) dt + h'(t)\Sigma dW_t \quad (4)$$

with initial endowment $V(0) = v$ and where $\mathbf{1} \in \mathbb{R}^m$ is the m -element unit column vector.

The objective of an investor with a fixed time horizon T , no consumption and a power utility function is to maximize the expected utility of terminal wealth:

$$J(t, h; T, \gamma) = \mathbf{E}[U(V_T)] = \mathbf{E}\left[\frac{V_T^\gamma}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma \ln V_T}}{\gamma}\right]$$

with risk aversion coefficient $\gamma \in (-\infty, 0) \cup (0, 1)$.

Define the value function Φ corresponding to the maximization of the auxiliary criterion function $J(t, x, h; \theta; T; v)$ as

$$\Phi(t) = \sup_{h \in \mathcal{A}} J(t, h; T, \gamma) \quad (5)$$

By Itô's lemma,

$$e^{\gamma \ln V(t)} = v^\gamma \exp \left\{ \gamma \int_0^t g(h(s); \gamma) ds \right\} \chi_t^h \quad (6)$$

where

$$g(h; \gamma) = -\frac{1}{2} (1 - \gamma) h' \Sigma \Sigma' h + h' (\mu - r \mathbf{1}) + r$$

and the Doléans exponential χ_t^h is defined in (3)

We can solve the stochastic control problem associated with (5) by a change of measure argument (see exercise 8.18 in [8]).

Define a measure \mathbb{P}_h on (Ω, \mathcal{F}_T) via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} := \chi_T^h \quad (7)$$

For $h \in \mathcal{A}(T)$,

$$W_t^h = W_t - \gamma \int_0^t \Sigma' h(s) ds$$

is a standard \mathbb{P}_h -Brownian motion.

The control criterion under this new measure is

$$I(t, h; T, \gamma) = \frac{v^\gamma}{\gamma} \mathbf{E}_t^h \left[\exp \left\{ \gamma \int_t^T g(h(s); \theta) ds \right\} \right] \quad (8)$$

where $\mathbf{E}_t^h[\cdot]$ denotes the expectation taken with respect to the measure \mathbb{P}_h at an initial time t .

Under the measure \mathbb{P}_h , the control problem can be solved through a pointwise maximisation of the auxiliary criterion function $I(v, x; h; t, T)$.

The optimal control h^* is simply the maximizer of the function $g(x; h; t, T)$ given by

$$h^* = \frac{1}{1-\gamma} (\Sigma \Sigma')^{-1} (\mu - r \mathbf{1})$$

which represents a position of $\frac{1}{1-\gamma}$ in the Kelly criterion portfolio.

The value function $\Phi(t)$, or optimal utility of wealth, equals

$$\Phi(t) = \frac{v^\gamma}{\gamma} \exp \left\{ \gamma \left[r + \frac{1}{2(1-\gamma)} (\mu - r\mathbf{1})' (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1}) \right] (T - t) \right\}$$

Substituting (9) into (3), we obtain an exact form for the Doléans exponential χ_t^* associated with the control h^* :

$$\chi_t^* := \exp \left\{ \frac{\gamma}{1-\gamma} (\mu - r\mathbf{1})' \Sigma^{-1} W(t) - \frac{1}{2} \left(\frac{\gamma}{1-\gamma} \right)^2 (\mu - r\mathbf{1})' (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1}) t \right\} \quad (9)$$

We can easily check that χ_t^* is indeed an exponential martingale. Therefore h^* is an admissible control.

In the Merton model, fractional Kelly strategies appear naturally as a result of a classical Fund Separation Theorem:

Theorem (Fund Separation Theorem)

Any portfolio can be expressed as a linear combination of investments in the Kelly (log-utility) portfolio

$$h^K(t) = (\Sigma\Sigma')^{-1}(\mu - r) \quad (10)$$

and the risk-free rate. Moreover, if an investor has a risk aversion γ , then the proportion of the Kelly portfolio will equal $\frac{1}{1-\gamma}$.

Insights from the Merton Model: Change of Measure and Duality

The measure \mathbb{P}_h defined in (7) is also used in the martingale/duality approach to dynamic portfolio selection (see for example [14] and references therein).

In complete market, the change of measure technique is equivalent to the martingale approach: the change of measure approach relies on the optimal asset allocation to identify the equivalent martingale measure, the martingale approach works in the opposite direction.

Further Insights from the Merton Model...

The level of risk aversion γ dictates the choice of measure \mathbb{P}_h .
Two special cases are worth mentioning.

Case 1: The Physical Measure

The measures \mathbb{P}_h is the physical measure \mathbb{P} in the limit as $\gamma \rightarrow 0$, that is in the log utility or Kelly criterion case. This observation forms the basis for the 'benchmark approach to finance' (see [13]).

Case 2: *The Dangers of Overbetting*

A well established and somewhat surprising folk theorem holds that “*when an agent overbets by investing in twice the Kelly portfolio, his/her expected return will be equal to the risk-free rate.*” (see for instance [15] and [11]).

Why is that???

The risk aversion of an agent who invest into twice the Kelly portfolio must be $\gamma = \frac{1}{2}$.

- ▶ Hence, the measures \mathbb{P}_h coincides with the equivalent martingale measure \mathbb{Q} !
- ▶ Under this measure, the portfolio value discounted at the risk-free rate is a martingale.

In the setting of the Merton model and its lognormally distributed asset prices, the definition of Fractional Kelly allocations guarantees optimality of the strategy.

However, Fractional Kelly strategies are no longer optimal as soon as the lognormality assumption is removed (see Thorpe in [11]).

This situation suggests that the definition of Fractional Kelly strategies could be broadened in order to guarantee optimality. We can take a first step in this direction by revisiting the ICAPM (see Merton[12]) in which the drift rate of the asset prices depend on a number of Normally-distributed factors.

Getting Kelly Strategies to Work for Factor Models

Key ingredients:

- ▶ Probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$;
- ▶ \mathbb{R}^N -valued (\mathcal{F}_t) -Brownian motion $W(t)$, $N := n + m$;
- ▶ $S_i(t)$ denotes the price at time t of the i th security, with $i = 0, \dots, m$;
- ▶ $X_j(t)$ denotes the level at time t of the j th factor, with $j = 1, \dots, n$.

For the time being, we assume that the factors are observable.

The dynamics of the money market account is given by:

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A'_0 X(t)) dt, \quad S_0(0) = s_0 \quad (11)$$

The dynamics of the m risky securities and n factors are

$$\begin{aligned} dS(t) &= D(S(t)) (a + AX(t)) dt + D(S(t)) \Sigma dW(t), \\ S_i(0) &= s_i, \quad i = 1, \dots, m \end{aligned} \quad (12)$$

and

$$dX(t) = (b + BX(t)) dt + \Lambda dW(t), \quad X(0) = x \quad (13)$$

where $X(t)$ is the \mathbb{R}^n -valued factor process with components $X_j(t)$ and the market parameters a , A , b , B , $\Sigma := [\sigma_{ij}]$, $\Lambda := [\Lambda_{ij}]$ are vectors and matrices of appropriate dimensions.

Assumption

The matrices $\Sigma\Sigma'$ and $\Lambda\Lambda'$ are positive definite.

- ▶ The objective of an investor remains the maximization of criterion (15) at a fixed time horizon T ;
- ▶ The wealth $V(t)$ of the portfolio in response to an investment strategy $h \in \mathcal{A}(T)$ is now factor-dependent with dynamics:

$$\frac{dV(t)}{V(t)} = (a_0 + A'_0 X(t)) dt + h'(t)(\hat{a} + \hat{A}X(t))dt + h'(t)\Sigma dW_t \quad (14)$$

with $\hat{a} := a - a_0\mathbf{1}$, $\hat{A} := A - \mathbf{1}A'_0$, and initial endowment $V(0) = v$.

The expected utility of terminal wealth $J(t, x, h; T, \gamma)$ is factor dependent:

$$J(t, x, h; T, \gamma) = \mathbf{E}[U(V_T)] = \mathbf{E}\left[\frac{V_T^\gamma}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma \ln V_T}}{\gamma}\right]$$

By Itô's lemma,

$$e^{\gamma \ln V(t)} = v^\gamma \exp\left\{\gamma \int_0^t g(X_s, h(s); \theta) ds\right\} \chi_t^h \quad (15)$$

where

$$g(x, h; \gamma) = -\frac{1}{2}(1-\gamma)h'\Sigma\Sigma'h + h'(\hat{a} + \hat{A}x) + a_0 + A_0'x \quad (16)$$

and the exponential martingale χ_t^h is still given by (3).

Applying the change of measure argument, we obtain the control criterion under the measure \mathbb{P}_h

$$I(t, x, h; T, \gamma) = \frac{v^\gamma}{\gamma} \mathbf{E}_{t,x}^h \left[\exp \left\{ \gamma \int_t^T g(X_s, h(s); \theta) ds \right\} \right] \quad (17)$$

where $\mathbf{E}_{t,x}^h [\cdot]$ denotes the expectation taken with respect to the measure \mathbb{P}_h and with initial conditions (t, x) .

The \mathbb{P}_h -dynamics of the state variable $X(t)$ is

$$dX(t) = (b + BX(t) + \gamma \Lambda \Sigma' h(t)) dt + \Lambda dW_t^h, \quad t \in [0, T] \quad (18)$$

Then value function Φ associated with the auxiliary criterion function $I(t, x; h; T, \gamma)$ is defined as

$$\Phi(t, x) = \sup_{h \in \mathcal{A}(T)} I(t, x; h; T, \gamma) \quad (19)$$

We use the Feynman-Kač formula to write down the HJB PDE:

$$\frac{\partial \Phi}{\partial t}(t, x) + \frac{1}{2} \text{tr}(\Lambda \Lambda' D^2 \Phi(t, x)) + H(t, x, \Phi, D\Phi) = 0 \quad (20)$$

subject to terminal condition

$$\Phi(T, x) = \frac{v^\gamma}{\gamma} \quad (21)$$

and where

$$H(s, x, r, p) = \sup_{h \in \mathbb{R}} \left\{ (b + Bx + \gamma \Lambda \Sigma' h)' p - \gamma g(x, h; \gamma) r \right\} \quad (22)$$

for $r \in \mathbb{R}$ and $p \in \mathbb{R}^n$.

To obtain the optimal control in a more convenient format and derive the value function $\Phi(t, x)$ more easily, we find it convenient to consider the logarithmically transformed value function $\tilde{\Phi}(t, x) := \frac{1}{\gamma} \ln \gamma \Phi(t, x)$ with associated HJB PDE

$$\frac{\partial \tilde{\Phi}}{\partial t}(t, x) + \inf_{h \in \mathbb{R}^m} L_t^h(t, x, D\tilde{\Phi}, D^2\tilde{\Phi}) = 0 \quad (23)$$

where

$$\begin{aligned} L_t^h(s, x, p, M) &= (b + Bx + \gamma \Lambda \Sigma' h(s))' p + \frac{1}{2} \text{tr}(\Lambda \Lambda' M) \\ &\quad + \frac{\gamma}{2} p' \Lambda \Lambda' p - g(x, h; \gamma) \end{aligned} \quad (24)$$

for $r \in \mathbb{R}$ and $p \in \mathbb{R}^n$ and subject to terminal condition

$$\tilde{\Phi}(T, x) = \ln v \quad (25)$$

This is in fact a risk-sensitive asset management problem (see [1], [9], [6]).

Solving the optimization problem in (24) gives the optimal investment policy $h^*(t)$

$$h^*(t) = \frac{1}{1-\gamma} (\Sigma\Sigma')^{-1} \left[\hat{a} + \hat{A}X(t) + \gamma\Sigma\Lambda'D\tilde{\Phi}(t, X(t)) \right] \quad (26)$$

The solution to HJB PDE (23) is

$$\tilde{\Phi}(t, x) = \frac{1}{2}x'Q(t)x + x'q(t) + k(t)$$

where $Q(t)$ satisfies a matrix Riccati equation, $q(t)$ satisfies a vector linear ODE and $k(t)$ is an integral. As a result,

$$h^*(t) = \frac{1}{1-\gamma} (\Sigma\Sigma')^{-1} \left[\hat{a} + \hat{A}X(t) + \gamma\Sigma\Lambda' (Q(t)X(t) + q(t)) \right] \quad (27)$$

Substituting (26) into (3), we obtain an exact form for the Doléans exponential χ_t^* associated with the control h^* :

$$\begin{aligned} \chi_t^* := & \exp \left\{ \frac{\gamma}{1-\gamma} \int_0^t \left[\hat{a} + \hat{A}X(s) + \gamma \Sigma \Lambda' (Q(s)X(s) + q(s)) \right]' \right. \\ & \times (\Sigma \Sigma')^{-1} \Sigma dW(s) \\ & - \frac{1}{2} \left(\frac{\gamma}{1-\gamma} \right)^2 \int_0^t \left(\hat{a} + \hat{A}X(s) + \gamma \Sigma \Lambda' (Q(s)X(s) + q(s)) \right)' \\ & \left. \times (\Sigma \Sigma')^{-1} \left(\hat{a} + \hat{A}X(s) + \gamma \Sigma \Lambda' (Q(s)X(s) + q(s)) \right)' ds \right\} \quad (28) \end{aligned}$$

We can interpret the Girsanov kernel

$$\frac{\gamma}{1-\gamma} \Sigma' (\Sigma \Sigma')^{-1} \left[\hat{a} + \hat{A}X(s) + \gamma \Sigma \Lambda' (Q(s)X(s) + q(s)) \right]$$

as the projection of the factor-dependent returns

$$\hat{a} + \hat{A}X(s) + \gamma \Sigma \Lambda' (Q(s)X(s) + q(s))$$

on the subspace spanned by the asset volatilities Σ , scaled by $\frac{\gamma}{1-\gamma}$.

This observation also implies that the appropriate Girsanov kernel in a complete (factor-free) setting is

$$\frac{\gamma}{1-\gamma} \Sigma' (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1})$$

with a similar interpretation.

In the ICAPM, a new view of Fractional Kelly investing emerges:

Theorem (ICAPM Fund Separation Theorem)

Any portfolio can be expressed as a linear combination of investments into two funds' with respective risky asset allocations:

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1} \left(\hat{a} + \hat{A}X(t) \right) \\ h^I(t) &= (\Sigma\Sigma')^{-1} \Sigma\Lambda' (Q(t)X(t) + q(t)) \end{aligned} \quad (29)$$

and respective allocation to the money market account given by

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1} \left(\hat{a} + \hat{A}X(t) \right) \\ h_0^I(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1} \Sigma\Lambda' (Q(t)X(t) + q(t)) \end{aligned}$$

Moreover, if an investor has a risk aversion γ , then the respective weights of each mutual fund in the investor's portfolio equal $\frac{1}{1-\gamma}$ and $\frac{\gamma}{1-\gamma}$, respectively.

In the factor-based ICAPM,

$$(\Sigma\Sigma')^{-1} [\hat{a} + \hat{A}X(t)] \quad (30)$$

represents the Kelly (log utility) portfolio and

$$(\Sigma\Sigma')^{-1} \Sigma\Lambda' (Q(t)X(t) + q(t)) \quad (31)$$

is the 'intertemporal hedging portfolio' identified by Merton.

This mutual fund theorem raises some questions as to the practicality of the intertemporal hedging portfolio as an investment option.

Beyond Factor Models: Random Coefficients

The change of measure approach still works well in a partial observation case where we would need to estimate the coefficients of the factor process through a Kalman filter.

However, this presupposes that we know the form of the factor process.

A more general approach would be to assume that the coefficients of the asset price dynamics are random (See Bjørk, Davis and Landén [14] and references therein).

Key ingredients:

- ▶ The “full” underlying probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$;
- ▶ The filtration $\mathcal{F}_t^W := \sigma(W(s)), 0 \leq s \leq t$ generated by an m -dimensional Brownian motions driving the asset returns (augmented by the \mathbb{P} -null sets);
- ▶ The filtration $\mathcal{F}_t^S := \sigma(S(s)), 0 \leq s \leq t$ generated by the m asset price (also augmented by the \mathbb{P} -null sets);

The dynamics of the money market account is given by:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = s_0 \quad (32)$$

where $r \in \mathbb{R}^+$ is a bounded \mathcal{F}_t^S -adapted process. We will denote by $Z(t, T)$ the discount factor:

$$Z(t, T) = \exp \left\{ - \int_t^T r(s)ds \right\} \quad (33)$$

The dynamics of the m risky securities and n factors can be expressed as:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sum_{k=1}^m \sigma_{ik}(t)dW_k(t), \quad S_i(0) = s_i, \quad i = 1, \dots, m \quad (34)$$

where $\mu(t) = (\mu_1(t), \dots, \mu_m(t))'$ is an \mathcal{F}_t -adapted process and the volatility $\Sigma(t) := [\sigma_{ij}(t)]$, $i = 1, \dots, m$, $j = 1, \dots, m$ is an \mathcal{F}_t^S -adapted process. More synthetically,

$$dS(t) = D(S(t))\mu(t)dt + D(S(t))\Sigma(t)dW(t) \quad (35)$$

Note that no Markovian structure is either assumed or required.

Assumption

We assume that the matrix $\Sigma(t)$ is positive definite $\forall t$.

The critical step in this approach is to establish a projection onto the observable filtration.

Once this is done, the partial observation problem related to the filtration \mathcal{F} can be rewritten as an equivalent complete observation problem with respect to the filtration \mathcal{F}^S .

This effectively changes a utility maximization problem set in an incomplete market into a standard utility maximization problem set in a complete market.

Define the process $Y(t)$ by:

$$dY(t) = \Sigma_t^{-1} D(S_t)^{-1} dS_t = \Sigma_t^{-1} \mu_t dt + \Sigma_t dW(t) \quad (36)$$

For any \mathcal{F} -adapted process X , define the filter estimate process \hat{X} as the optional projection of X onto the filtration \mathcal{F}^S :

$$\hat{Y}_t = \mathbf{E}^{\mathbb{P}} \left[Y_t | \mathcal{F}_t^S \right]$$

Next, define the innovation process U by

$$dU(t) = dZ(t) - (\widehat{\Sigma^{-1}\mu}(t))dt = dZ(t) - \Sigma^{-1}\hat{\mu}(t)dt \quad (37)$$

where the second equality follows from the observability of Σ_t .

Note that the innovation process $U(t)$ is a standard \mathcal{F}^S -Brownian motion (see for example Lipster and Shiryaev [10]).

As a result, we can rewrite (35) using the filter estimate for α and the innovation process U obtained with respect to the filtration \mathcal{F}^S :

$$dS(t) = D(S(t))\hat{\mu}(t)dt + D(S(t))\Sigma(t)dU(t) \quad (38)$$

The remainder follows from a standard martingale argument.

Definition

The *Girsanov kernel* is the vector process φ_t given by

$$\varphi(t) = \Sigma^{-1}(t)(\hat{\mu}(t) - r\mathbf{1}) \quad (39)$$

for all t .

Note that the Girsanov kernel $\varphi(t)$ and the *market price of risk* vector $\lambda(t)$ are related by $\lambda(t) = -\varphi(t)$.

Assumption

We assume that the Girsanov kernel satisfies the Novikov condition

$$\mathbf{E} \left[e^{\frac{1}{2} \int_0^T \|\varphi(t)\|^2 dt} \right] < \infty \quad (40)$$

and the integrability condition

$$\mathbf{E} \left[e^{\frac{1}{2} \int_0^T \left(\frac{\gamma}{1-\gamma} \right)^2 \|\varphi(t)\|^2 dt} \right] < \infty \quad (41)$$

Definition

The equivalent martingale measure \mathbb{Q} is defined by the Radon-Nikodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_t := \exp \left\{ - \int_0^t \varphi'(s) dU(s) - \frac{1}{2} \int_0^t \varphi'(s) \varphi(s) ds \right\},$$

on \mathcal{F}_t^S .

- ▶ It follows from Assumption 4 that L_t is a true martingale.
- ▶ Moreover, \mathbb{Q} is unique: this is a direct consequence of the use of filtered estimates in (38).

The objective is to maximize the expected utility of terminal wealth:

$$J(t, h; T, \gamma) = \mathbf{E}^{\mathbb{P}} \left[\frac{V_T^\gamma}{\gamma} \right]$$

subject to the budget constraint

$$\mathbf{E}^{\mathbb{Q}} [K_{0,T} V_T] = v = \mathbf{E}^{\mathbb{P}} [K_{0,T} L_T V_T] \quad (42)$$

Next, form the Lagrangian

$$\begin{aligned} \mathcal{L}(h, \lambda; T) &= \frac{1}{\gamma} \mathbf{E}^{\mathbb{P}} [V_T^\gamma] - \lambda \left(\mathbf{E}^{\mathbb{P}} [K_{0,T} L_T V_T] - v \right) \\ &= \int_{\Omega} \frac{1}{\gamma} V_T^\gamma - \lambda (K_{0,T}(\omega) L_T(\omega) V_T(\omega) - v) d\mathbb{P}(\omega) \end{aligned}$$

This separable problem can be maximized for each ω . The first order condition leads to

$$V_T^* = [\lambda K_{0,T} L_T]^{-\frac{1}{1-\gamma}}$$

Substituting in the budget equation (42), we get

$$\lambda^{-\frac{1}{1-\gamma}} \mathbf{E}^{\mathbb{P}} \left[(K_{0,T} L_T)^{-\frac{\gamma}{1-\gamma}} \right] = v$$

and as a result,

$$V_T^* = \frac{v}{H_0} K_{0,T}^{-\frac{1}{1-\gamma}} L_T^{-\frac{1}{1-\gamma}} \quad (43)$$

with

$$H_0 = \mathbf{E}^{\mathbb{P}} \left[(K_{0,T} L_T)^{-\frac{\gamma}{1-\gamma}} \right]$$

Therefore, the optimal expected utility is equal to:

$$U_0 = \mathbf{E}^{\mathbb{P}} \left[\frac{(V_T^*)^\gamma}{\gamma} \right] = H_0^{1-\gamma} \frac{V}{\gamma} \quad (44)$$

Observe that H_0 can be expressed as

$$H_0 = \mathbf{E}^{\mathbb{P}} \left[L_T^0 \exp \left\{ \int_0^T \left(r_t + \frac{1}{2(1-\gamma)} \|\varphi(t)\|^2 \right) dt \right\} \right]$$

where

$$L_t^0 := \exp \left\{ \frac{\gamma}{1-\gamma} \int_0^t \varphi'(s) dU(s) - \frac{1}{2} \left(\frac{\gamma}{1-\gamma} \right)^2 \int_0^t \varphi'(s) \varphi(s) ds \right\} \quad (45)$$

is a true martingale. This leads us to the following definition:

Definition

(i). The measure \mathbb{Q}^0 is defined by the Radon-Nikodým derivative

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}} = L_t^0, \quad \text{on } \mathcal{F}_t^S$$

(ii). The process H is defined by

$$H_t = \mathbf{E}^0 \left[\exp \left\{ \frac{\gamma}{1-\gamma} \int_t^T \left(r_s + \frac{1}{2(1-\gamma)} \|\varphi(s)\|^2 \right) ds \right\} \middle| \mathcal{F}_t \right] \quad (46)$$

where the expectation $\mathbf{E}^0[\cdot]$ is taken with respect to the \mathbb{Q}^0 measure.

Finally, we recall Proposition 4.1 in Bjørk, Davis and Landén [14]:

Proposition

The following hold:

- (i). The optimal wealth process $V^*(t)$ is given by

$$V^*(t) = \frac{H(t)}{H(0)} (K_{0,t} \bar{L}_t)^{\frac{-1}{1-\gamma}} x$$

- (ii). The optimal weight vector h^* is given by

$$h^*(t) = \frac{1}{1-\gamma} (\Sigma_t \Sigma_t')^{-1} (\mu_t - r\mathbf{1}) + \sigma_H(t) (\Sigma_t^{-1})' \quad (47)$$

where σ_H is the volatility term of H , i.e. H has dynamics of the form

$$H(t) = \mu_H(t)dt + \sigma_H(t)dU(t)$$

In the limit as $\gamma \rightarrow 0$, the optimal wealth process $\bar{V}^*(t)$ is given by

$$V^*(t) = (K_{0,t} \bar{L}_t)^{\frac{-1}{1-\gamma}} x$$

and the optimal investment is the Kelly portfolio,

$$h^*(t) = (\Sigma_t \Sigma_t')^{-1} (\mu_t - r\mathbf{1}) \tag{48}$$

With the choice $\gamma = \frac{1}{2}$, and

$$h^*(t) = 2(\Sigma_t \Sigma_t')^{-1} (\mu_t - r\mathbf{1}) + \sigma_H(t) (\Sigma_t^{-1})' \quad (49)$$

As a result,

$$V^*(t) = \frac{H(t)}{H(0)} (K_{0,t} \bar{L}_t)^2 x = x \exp \left\{ \int_t^T r(s) ds \right\}$$

as expected.

However, there is still something unsatisfactory about this result: the optimal strategy depends on the volatility of the process $H(t)$:

$$H_t = \mathbf{E}^0 \left[\exp \left\{ \frac{\gamma}{1-\gamma} \int_t^T \left(r_s + \frac{1}{2(1-\gamma)} \|\varphi(s)\|^2 \right) ds \right\} \middle| \mathcal{F}_t \right]$$

As a result, getting some intuition on the intertemporal hedging term is 'difficult'.

Conclusion and Next Steps




- ▶ Get stronger statements of equivalence between the martingale/duality approach and the change of measure technique in incomplete markets;
- ▶ Study the practicality of intertemporal hedging portfolio as an investment option;
- ▶ Add jumps.

Thank you!



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Any question?

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