Optimising Proportional Reinsurance Using a Worst Case Scenario Approach

by

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1 Literature Review

1. Merton (1976), Aase (1984), and many others: risky asset modelled as jump diffusion.

2. Korn and Wilmott (2002), Korn (2005), Korn and M. (2005), M. (2006), Korn and Steffensen (2007) → Determine worst case bounds for the performance of optimal investment.

3. Minimising Probability of Ruin: usually involves PIDE.

4. Brown (1995) and Liu/Ma (2009)

2 Introduction
– The Worst Case Scenario Approach

1. Korn and Wilmott (2002) $\rightarrow$ Determine worst case bounds for the performance of optimal investment.

For simplicity: One bond, one stock, at most one crash in $[0, T]$ with a height of k with $0 < k_* \leq k \leq k^* < 1$. Security prices in "normal times":

$$\begin{array}{rcll} dP_{0,0}(t) & = & P_{0,0}(t)\, r_0\, dt\,, & P_{0,0}(0) = 1\,, \quad \text{"bond"} \\ dP_{0,1}(t) & = & P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW_0(t)\right], & P_{0,1}(0) = p_1\,, \quad \text{"stock"} \end{array}$$

At crash time: stock price falls by a factor of $k \in [k_*, k^*]$.

Consequence: The wealth process $X_0^\pi(t)$ at crash time τ satisfies:

$$\begin{array}{rcl} X_0^\pi(\tau-) & = & (1-\pi(\tau))\, X_0^\pi(\tau-) + \pi(\tau) X_0^\pi(\tau-) \\ \Longrightarrow & & (1-\pi(\tau))\, X_0^\pi(\tau-) + \pi(\tau) X_0^\pi(\tau-)\,(1-k) \\ & = & \boxed{\boldsymbol{(1-\pi(\tau)k)\cdot X_0^\pi(\tau-) = X_0^\pi(\tau)}.} \end{array}$$

Thus: Following the portfolio process $\pi(\cdot)$ if a crash of size k happens at time τ leads to a final wealth of

$$X^{\pi}(T) = (1 - \pi(\tau)k) \cdot X_0^{\pi}(T),$$

if $X_0^{\pi}(\cdot)$ denotes the wealth process in the model without any crash.

Hence:

- "High" values of $\pi(\cdot)$ lead to a high final wealth if no crash occurs at all, but to a high loss at the crash time.
- "Low" values of $\pi(\cdot)$ lead to a low final wealth if no crash occurs at all, but to a small loss (or even no loss at all!) at the crash time.

Moral: We have two competing aspects ("Hedging vs. Return") for two different scenarios ("Crash or not") and are therefore faced with a **balanced problem between crash impact minimization and return maximization.**

Aim: Find the best uniform **worst case bound**, e.g. solve

$$\sup_{\pi(\cdot)\in A_0(x)} \inf_{\substack{0\le\tau\le T\\ k\in K}} \mathbb{E}\left[U\left(X^{\pi}\left(T\right)\right)\right],$$

where the final wealth satisfies $X^{\pi}(T) = (1-\pi(\tau)k)\, X_0^{\pi}(T)$ in the case of a crash of size k at stopping time τ. Moreover, $K = \{0\} \cup [k_*, k^*]$.

Note: To avoid bankruptcy we require $\pi(t) < \frac{1}{k^*}$ for all $t \in [0, T]$.

Important Remarks: (!)

- We do **not (!)** compare two different strategies **scenario-wise.** Typically, two different strategies have two different worst case scenarios!

- The worst case bounds do **not** depend on the probability of the worst case!

Two extreme strategies (in the logarithmic utility case):

1. **"Playing safe":**
   $\pi(t) \equiv 0 \Longrightarrow$ worst case scenario: no crash (!), leading to the following worst case bound of

$$WCB_0 = \mathbb{E}\left[\ln\left(X^0(T)\right)\right] = \ln(x) + r_0 T.$$

2. **"Optimal investment in the crash–free world":**
   $\pi(t) \equiv \pi_0^* = \frac{\mu_0 - r_0}{\sigma_0^2} \Longrightarrow$ worst case scenario: a crash of maximum size k^* (at any arbitrary time (!)), leading to the following worst case bound of

$$WCB_{\pi_0^*} = \mathbb{E}\left[\ln\left(X^{\pi_0^*}(T)\right)\right] = \ln(x) + r_0 T + \frac{1}{2}\left(\frac{\mu_0 - r_0}{\sigma_0}\right)^2 T + \ln\left(1 - \pi_0^* k^*\right).$$

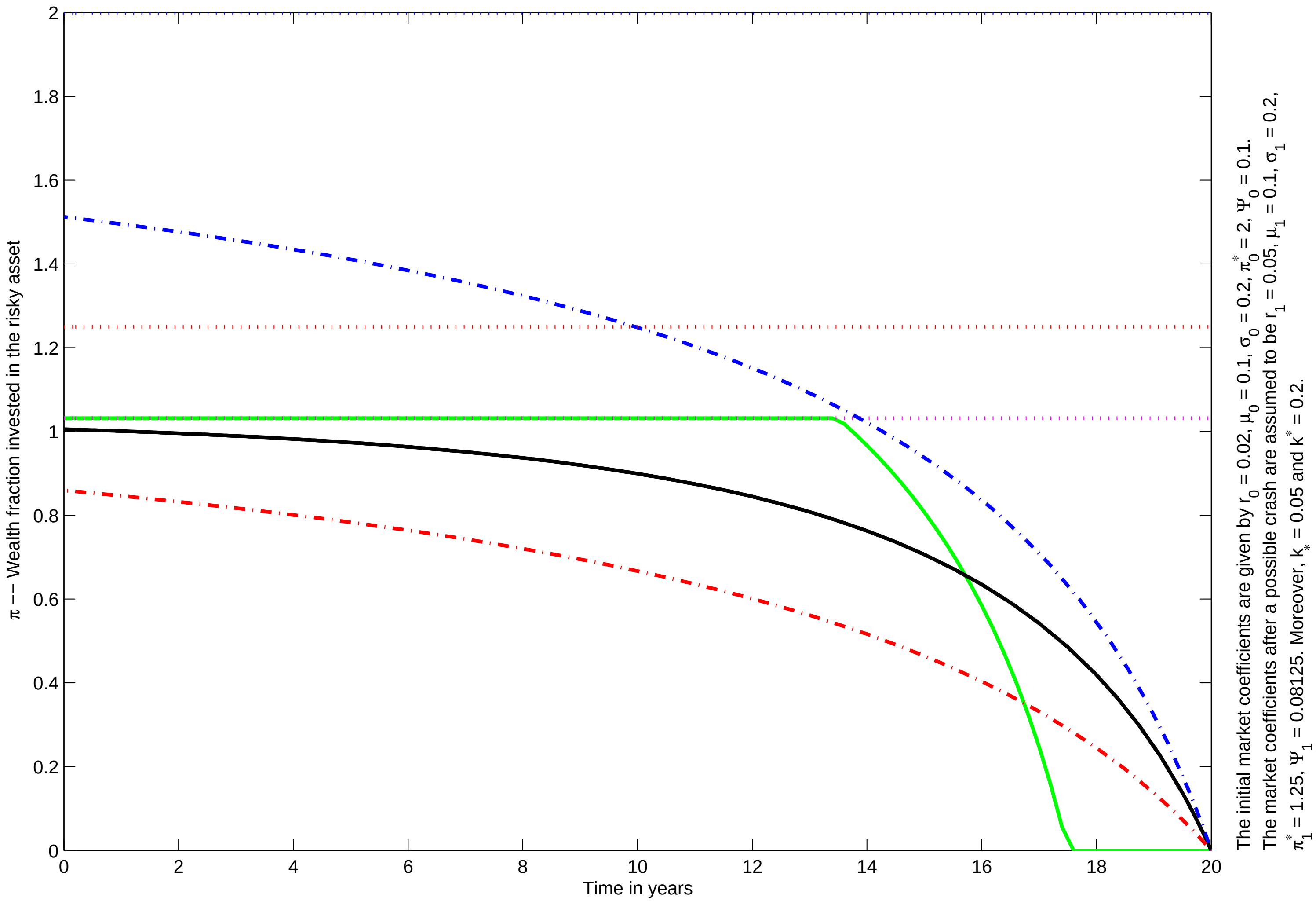
Insights:

- it depends on the time to maturity which one of the above strategies is better.

- strategy 1 takes too few risk to be good if no crash occurs while strategy 2 is too risky to perform well if a crash occurs $\Longrightarrow$ the optimal strategy should balance this out!

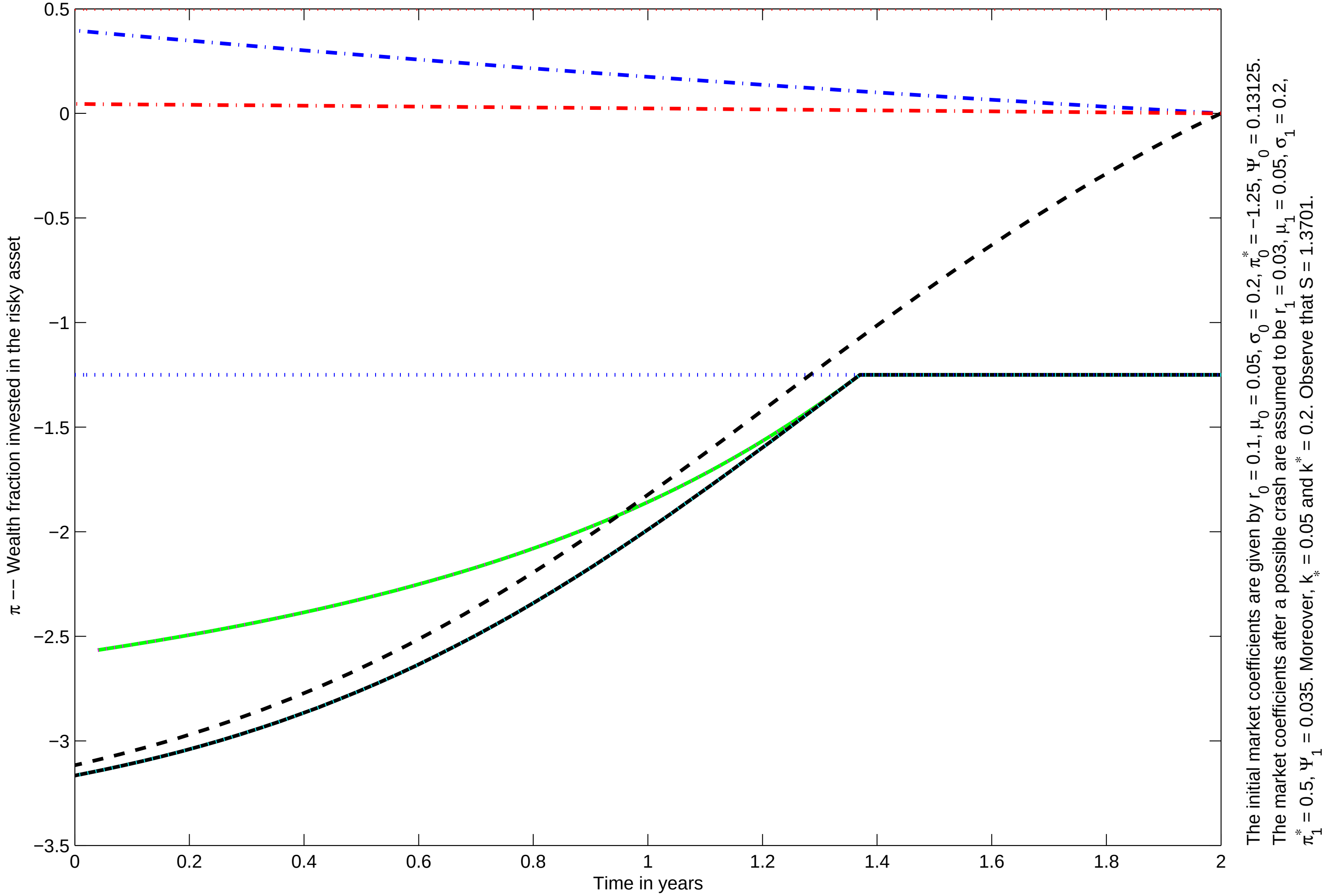
- a constant portfolio process **cannot** be the optimal one.

Example for $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\widehat{\pi} = \bar{\pi}$ (black line), $\widehat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), $\pi_0^* = 2$ (blue dotted line), and π_1^* (red dotted line).

Example $\Psi_1 < r_0$ and $\pi_0^* < 0$



This graphic shows $\widehat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\widehat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\widehat{\phi}_0$ (blue dash–dotted line), $\widehat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Definition 2.1

For $i = 0, 1$ let us name

1. *the* **optimal portfolio strategy in market i**, *assuming that no crash will happen, by*

$$\pi_i^* := \frac{\mu_i - r_i}{\sigma_i^2}.$$

2. *Moreover,*

$$\Psi_i := r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i}\right)^2 = r_i + \frac{\sigma_i^2}{2}\left(\pi_i^*\right)^2$$

will be called the **utility growth potential** *or* **earning potential of market i**.

With these definitions, one has for an arbitrary admissible portfolio strategy $\pi(t)$

$$\begin{aligned} \nu_\pi(t,x) &:= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \\ &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right]. \end{aligned}$$

3 Optimising Proportional Reinsurance

3.1 The Model

$$\begin{aligned} dR(t) &= (\pi - \mu)\, dt - \nu\, dW(t) - \beta\, dN^P(t) \\ dI(t) &= -\pi \cdot p(t) \cdot [1+\varepsilon]\, dt + \mu \cdot p(t)\, dt + \nu \cdot p(t)\, dW(t) + \beta \cdot p(t)\, dN^P(t) \\ dR^p(t) &= dR(t) + dI(t) \\ &= \pi\, [1 - p(t) \cdot [1+\varepsilon]]\, dt - \mu \cdot [1-p(t)]\, dt - \nu \cdot [1-p(t)]\, dW(t) \\ &\quad -\beta \cdot [1-p(t)]\, dN^P(t)\,, \end{aligned} \tag{1}$$

For simplicity, it will be assumed that all insurance policies will be terminated at time T and that no claim is possible thereafter. Notice that after observing the maximal number of possible claims N is above dynamics will become

$$\begin{aligned} dR(t) &= (\pi - \mu)\, dt - \nu\, dW(t) \\ dI(t) &= -\pi \cdot p(t) \cdot [1+\varepsilon]\, dt + \mu \cdot p(t)\, dt + \nu \cdot p(t)\, dW(t) \\ dR^p(t) &= dR(t) + dI(t) \\ &= \pi\, [1 - p(t) \cdot [1+\varepsilon]]\, dt - \mu \cdot [1-p(t)]\, dt - \nu \cdot [1-p(t)]\, dW(t)\,. \end{aligned} \tag{2}$$

Thus, this set up can be considered as a regime–switching model.

3.2 A Verification Theorem

The aim is to maximize the expected utility of final reserves, that is

$$\sup_{p\in\mathcal{A}} \inf_{N\in\mathcal{B}} \mathbb{E}\left[U\left(R^p\left(T\right)\right)\right] , \qquad (3)$$

where $\mathcal{A}$ and $\mathcal{B}$ are the sets of admissible controls for p and N, respectively. More specific, $\mathcal{A}$ is the set of all predictable processes with respect to the σ–algebra generated by the reserve process and the poisson process which determines how many claims are still possible. $\mathcal{B}$ denotes the set of all Poisson processes. With this, the optimal value function $V^n(t,r)$ is given by

$$V^n(t,r) = \sup_{p\in\mathcal{A}} \inf_{N\in\mathcal{B}} \mathbb{E}^{t,r,n}\left[U\left(R^p\left(T\right)\right)\right] ,$$

where $\mathbb{E}^{t,r,n}$ is the conditional expectation given that $X(t) = r$ and given that there are at most n claims possible left.

Moreover, define the differential operator

$$\begin{aligned}\mathcal{L}^p \nu(t,r) &= \nu_t(t,r) + \nu_r(t,r) \cdot [\pi \{1 - p(t) \cdot [1+\varepsilon]\} - \mu \cdot (1-p(t))] + \\ &\quad - \nu_{rr}(t,r) \cdot \frac{\nu^2}{2} \cdot (1-p(t))^2\end{aligned}$$

for any function $\nu \in C^{1,2}$. We are now in the position to formulate the main theorem of this paper. This theorem is the corresponding theorem to Theorem 2 in Korn and Steffensen (2007).

Theorem 3.1 (Verification Theorem)
Let be

$$\begin{aligned} \mathcal{A}'_n(t,r) &= \{p \mid p \in \mathcal{A}, 0 \le \mathcal{L}^p \nu^n(t,r)\} \\ \mathcal{A}''_n(t,r) &= \left\{p \mid p \in \mathcal{A}, 0 \le \nu^{n-1}\left(t, r - \beta \cdot [1 - p^n(t)]\right) - \nu^n(t,r)\right\} \end{aligned}$$

for $n \in \mathbb{N}$ *and any* $\nu \in C^{1,2}$*. If there exists a polynomially bounded* $C^{1,2}$*–solution of*

$$\begin{aligned} 0 &\le \sup_{p \in \mathcal{A}''(t,r)} \left[\mathcal{L}^p \nu^n(t,r)\right], \\ 0 &\le \sup_{p \in \mathcal{A}'(t,r)} \left[\nu^{n-1}\left(t, r - \beta \cdot [1 - p^n(t)]\right) - \nu^n(t,r)\right], \\ 0 &= \sup_{p \in \mathcal{A}''(t,r)} \left[\mathcal{L}^p \nu^n(t,r)\right] \cdot \sup_{p \in \mathcal{A}'(t,r)} \left[\nu^{n-1}\left(t, r - \beta \cdot [1 - p^n(t)]\right) - \nu^n(t,r)\right], \\ \nu^n(T,r) &= U(r), \end{aligned}$$

and if

$$\begin{aligned} q^n(t,r) &= \arg\sup_{p \in \mathcal{A}''(t,r)} \left[\mathcal{L}^p \nu^n(t,r)\right], \\ \theta^n(t,r) &= \inf_{\tau\,:\,\tau \ge t} \left[\nu^{n-1}\left(\tau, R^p(\tau) - \beta \cdot [1 - p^n(\tau)]\right) - \nu^n\left(\tau, R^p(\tau)\right)\right], \end{aligned}$$

where $R^p(\tau) = r$ *and* τ *is a stopping time, then* $(q^n(t,r), \theta^n(t,r))$ *is a pair of admissible control functions.*

3.3 Characterisation of the Solution

As in Korn and Steffensen (2007), it is possible to characterize the solution in the worst case scenario optimum. That is, the value function has to satisfy the following conditions

$$\begin{aligned} V_t^n(t,r) &= -V_r^n(t,r)\cdot\left[\pi\left\{1-p^n\cdot[1+\varepsilon]\right\}-\mu\cdot(1-p^n)\right]+ \\ &\quad + V_{rr}^n(t,r)\cdot\frac{\nu^2}{2}\cdot(1-p^n)^2 \qquad (4) \\ V^n(t,r) &= V^{n-1}\left(t,r-\beta\cdot[1-p^n]\right), \qquad (5) \end{aligned}$$

where $n\in\{1,2,\ldots,N\}$ is the maximum number of large claims still possible at time t. Note that the case $n=0$ means that no further large claims are expected. In this case, the solution is known to be

$$p^0 = 1-\frac{\mu-\pi\,(1+\varepsilon)}{\nu^2}\cdot\frac{V_r^0}{V_{rr}^0}\,. \qquad (6)$$

Using the above two equations, one arrives after some reformulations at the following partial differential equation

$$\begin{aligned} p_t^n \;=\; & \frac{1}{\beta}\left(p^n - p^{n-1}\right)\left[\pi\,(1+\varepsilon) - \mu - \frac{V_{rr}^{n-1}}{V_r^{n-1}} \cdot \frac{\nu^2}{2}\left(2 - p^n - p^{n-1}\right)\right] \\ & + p_{rr}^n \cdot \frac{\nu^2}{2} \cdot (1 - p^n)^2 - p_r^n\left[\pi\,(1 - p^n\,(1+\varepsilon)) - \mu \cdot (1 - p^n)\right] \\ & + \frac{V_{rr}^{n-1}}{V_r^{n-1}} \cdot \nu^2 \left(p_r^n + \frac{\beta}{2}\,(p_r^n)^2\right) \cdot (1 - p^n)^2 \,. \end{aligned} \tag{7}$$

which describes the optimal reinsurance level given that there are still n large claims possible.

4. A Special Case

4.1 The Model

We consider a simple model for non–life reinsurance, i.e. a model consisting of a constant continuously paying premium π and claims which are modelled by a Poisson process N^P. Moreover, we suppose that all claims have constant claim size β. Furthermore, let us assume that the insurance company can reinsure the fraction $p(t) \in [0, 1]$ of its business. To do so, the insurance company has to pay $\pi \cdot p(t)\,[1+\varepsilon]$, where $\varepsilon \geq 0$ is the load or premium. $\varepsilon = 0$ is also known as cheap reinsurance compared to $\varepsilon > 0$ which is called non–cheap reinsurance. Thus, the dynamics of the reserve process $R(t)$, the reinsured part $I(t)$, and the net reserve process $R^p(t)$ are given by

$$\begin{aligned} dR(t) &= \pi\,dt - \beta\,dN^P \\ dI(t) &= -\pi \cdot p(t) \cdot [1+\varepsilon]\;dt + \beta \cdot p(t)\,dN^P \\ dR^p(t) &= dR(t) + dI(t) \\ &= \pi\,[1 - p(t) \cdot [1+\varepsilon]]\;dt - \beta \cdot [1 - p(t)]\;dN^P\,, \qquad (8) \end{aligned}$$

4.2 Example – Logarithmic Utility

The aim is to maximize the expected utility of final reserves, that is

$$\sup_{p\in\mathcal{A}} \inf_{N\in\mathcal{B}} \mathbb{E}\left[\ln\left(R^p\left(T\right)\right)\right] . \tag{9}$$

Having in mind that the value function is additive in this case (and not multiplicative), one can derive that

$$p_t^n(t) \;=\; \frac{\pi}{\beta}\cdot[1+\varepsilon]\cdot\left[p^n(t)-p^{n-1}(t)\right] ,$$

where the boundary condition is $p^n(T)=1$ for all $n\in\{1,2,\ldots,N\}$. First, let us consider the case $n=0$. It means that no claim will be made, therefore reinsurance is not necessary (that is $p^0(t)\equiv 0$). Using this, the case $n=1$ gives the following differential equation

$$p_t^1(t)=\frac{\pi}{\beta}\cdot[1+\varepsilon]\cdot p^1(t) .$$

Clearly,

$$p^1(t)=\exp\left(-\frac{\pi}{\beta}\cdot[1+\varepsilon]\cdot[T-t]\right) .$$

Since π, β, and ε are non–negative, $p^1(t) \leq 1$. It is obvious that $p^1(t) \geq 0$. Moreover,

$$\begin{aligned} p_t^2(t) &= \frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot \left[p^2(t) - p^1(t)\right] \\ &= \frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot \left[p^2(t) - \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)\right] . \end{aligned}$$

Thus, the solution is

$$p^2(t) = \left\{\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t] + 1\right\} \cdot \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) .$$

It is now straightforward to verify that

$$\begin{aligned} p^n(t) &= \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) \cdot \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)^k \\ &= p^1(t) \cdot \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)^k \qquad (10) \end{aligned}$$

for $n \in \{1, 2, \ldots, N\}$. This can be rewritten as

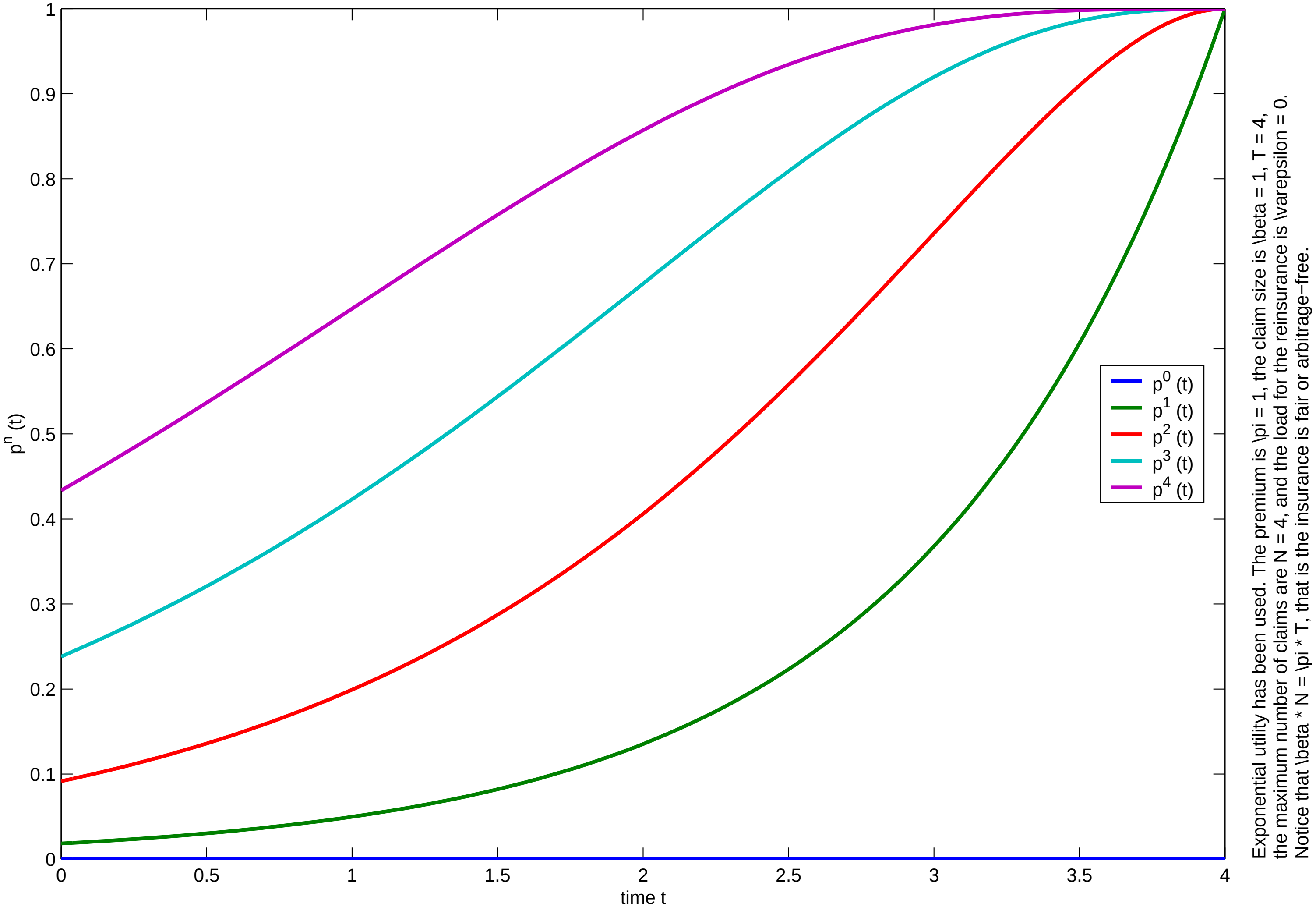
$$p^n(t) = \frac{1}{(n-1)!} \cdot \left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)^{n-1} \cdot p^1(t) + p^{n-1}(t)$$

for $n \in \{1, 2, \ldots, N\}$, where $0! = 1$ by convention. It is clear that $p^n(t) \geq 0$ for any finite n. Note also that it is straightforward from equation (10) to see that

$$\begin{aligned} p^n(t) &= \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) \cdot \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)^k \\ &< \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right)^k \\ &= \exp\left(-\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) \cdot \exp\left(\frac{\pi}{\beta} \cdot [1+\varepsilon] \cdot [T-t]\right) \\ &= 1 \end{aligned}$$

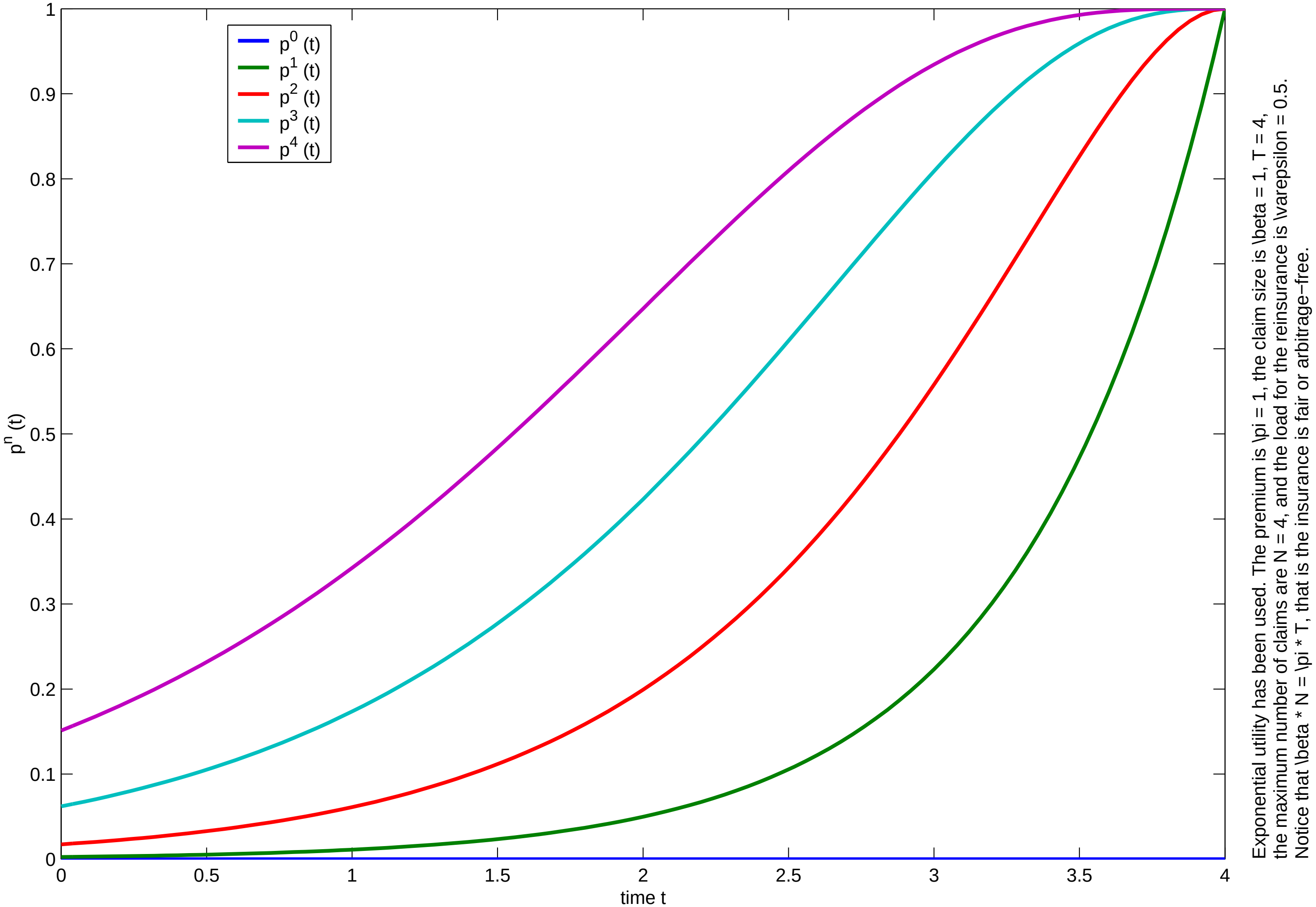
for any finite n.

Example $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$



This graphic shows the worst case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

Example $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$

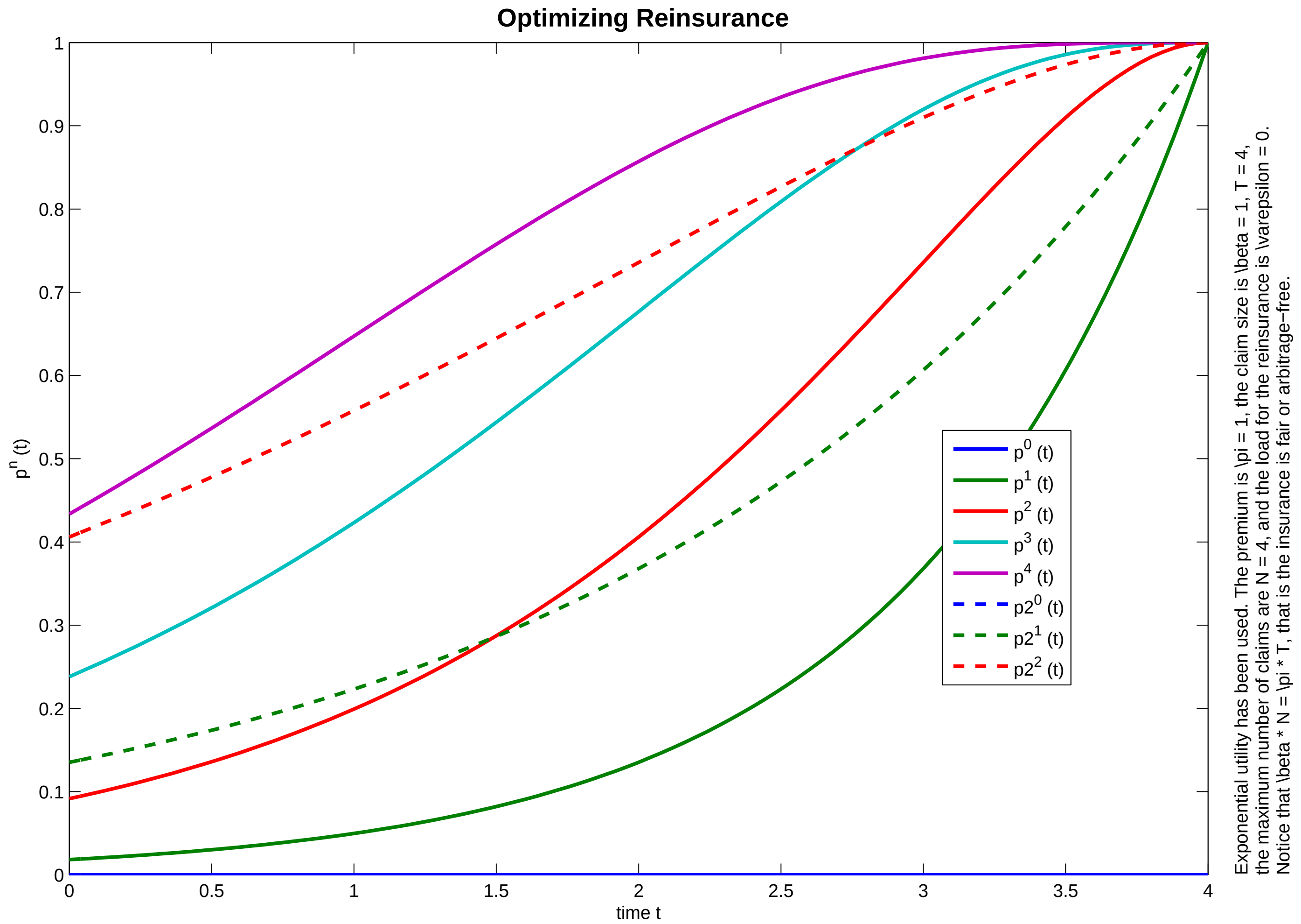


This graphic shows the worst case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

4.3 Comparing Different Business Strategies

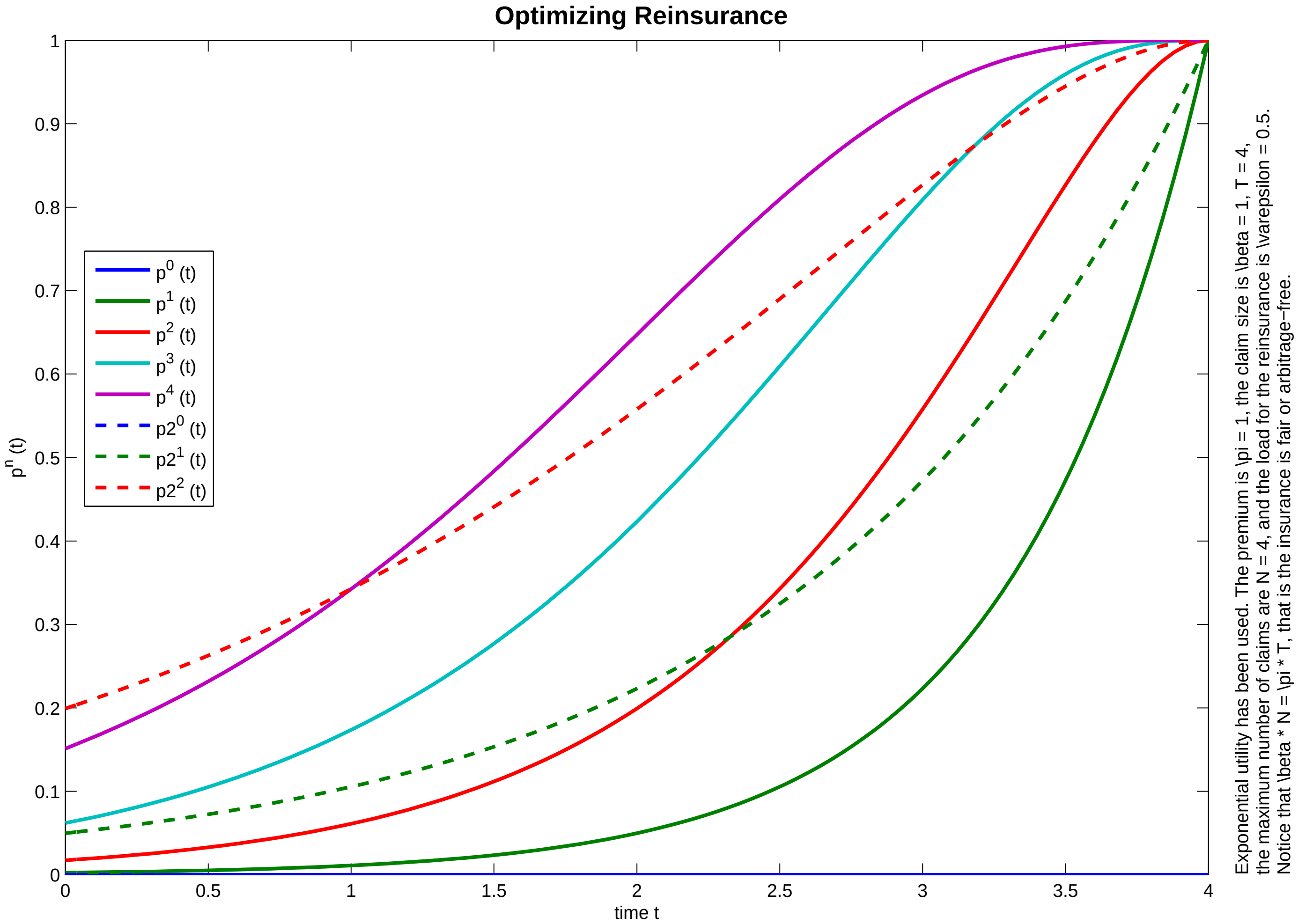
Observe that it is possible to follow different business strategies. One basic strategy is to concentrate on large businesses, that is the claim size β is large and the number of possible claims N is small. On the other hand, it is possible to concentrate on small businesses, that is the claim size β is small and the number of possible claims N is large. If the theory of mean–variance portfolios (Markowitz) applies here (don't put all your eggs in one basket), the later strategy is less risky and should therefore require less reinsurance. However, this is not the case for the optimal worst case scenario reinsurance strategy.

Example $\pi = 1$, $T = 4$, and $\varepsilon = 0$



This graphic shows the worst case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$ (solid lines) and $\beta = 2$, $N = 2$ (dashed lines).

Example $\pi = 1$, $T = 4$, and $\varepsilon = 0.5$



This graphic shows the worst case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$ (solid lines) and $\beta = 2$, $N = 2$ (dashed lines).

Outlook

- Include investment opportunity,
- Consider excess loss reinsurance,
- Compare with the minimising probability of ruin approach,
- Relax the condition of an upper number N,
- Generalise the worst case stochastic differential game approach.