Optimal investment under multiple defaults: a BSDE-decomposition approach

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Multiple defaults times and marks

On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$:

• **Reference filtration** $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$: default-free information

Progressive information provided, when they occur, by:

• a family of n **random times** $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ associated to a family of n **random marks** $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$.

- τ_i default time of name $i \in \mathbb{I}_n = \{1, \ldots, n\}$.
- The mark ζ_i, valued in E Borel set of $\mathbb{R}^p$, represents a jump size at τ_i, which cannot be predicted from the reference filtration, e.g. the loss given default.

Progressive enlargement of filtrations

The **global market information** is defined by:

$$\mathbb{G} \;=\; \mathbb{F} \vee \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^n,$$

where $\mathbb{D}^i$ is the default filtration generated by the observation of τ_i and ζ_i when they occur, i.e.

$$\mathbb{D}^i \;=\; (\mathcal{D}^i_t)_{t \geq 0}, \qquad \mathcal{D}^i_t \;=\; \sigma\{\mathbf{1}_{\tau_i \leq s}, s \leq t\} \vee \sigma\{\zeta_i \mathbf{1}_{\tau_i \leq s}, s \leq t\}.$$

$\rightarrow \mathbb{G} = \mathbb{F} \vee \mathbb{F}^\mu$, where $\mathbb{F}^\mu$ is the filtration generated by the jump random measure $\mu(dt, de)$ associated to (τ_i, ζ_i).

Successive defaults

For simplicity of presentation, we assume that

$$\tau_1 \leq \ldots \leq \tau_n$$

Remark. The general multiple random times case for $(\tau_1, \ldots, \tau_n)$ can be derived from the ordered case by considering the filtration generated by the corresponding ranked times $(\hat{\tau}_1, \ldots, \hat{\tau}_n)$ and the index marks ι_i, $i = 1, \ldots, n$ so that $(\hat{\tau}_1, \ldots, \hat{\tau}_n) = (\tau_{\iota_1}, \ldots, \tau_{\iota_n})$.

Notation: For $k = 0, \ldots, n$,

$$\boldsymbol{\tau}_k = (\tau_1, \ldots, \tau_k) \quad \text{valued in} \quad \Delta_k = \{(\theta_1, \ldots, \theta_k) : 0 \leq \theta_1 \leq \ldots \leq \theta_k\}$$

$$\boldsymbol{\zeta}_k = (\zeta_1, \ldots, \zeta_k) \quad \text{valued in} \quad E^k,$$

with the convention $\boldsymbol{\tau}_0 = \emptyset$, $\boldsymbol{\zeta}_0 = \emptyset$.

Decomposition of $\mathbb{G}$-adapted and predictable processes

Lemma

Any $\mathbb{G}$-adapted process Y is represented as:

$$Y_t = \sum_{k=0}^{n} 1_{\{\tau_k \le t < \tau_{k+1}\}} Y_t^k(\boldsymbol{\tau}_k, \boldsymbol{\zeta}_k), \tag{1}$$

where Y_t^k is $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable.

Remarks. • A similar decomposition result holds for $\mathbb{G}$-predictable processes: $< \leftrightarrow \leq$, and Y^k is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable in (1).

• Extension of Jeulin-Yor result (case of single random time without mark).

• We identify Y with the $n+1$-tuple $(Y^0, \ldots, Y^n)$.

• **Portfolio of N assets** with $\mathbb{G}$-adapted value process S:

$$S_t = \sum_{k=0}^{n} 1_{\{\tau_k \leq t < \tau_{k+1}\}} S_t^k(\boldsymbol{\tau}_k, \boldsymbol{\zeta}_k),$$

where $S^k(\boldsymbol{\theta}_k, \mathbf{e}_k)$, $\theta_k = (\theta_1, \ldots, \theta_k) \in \Delta_k$, $\mathbf{e}_k = (e_1, \ldots, e_k) \in E^k$, indexed $\mathbb{F}$-adapted process valued in $\mathbb{R}_+^N$, represents the assets value given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and marks at default $\boldsymbol{\zeta}_k = \mathbf{e}_k$.

Change of regimes with jumps at defaults

- Dynamics of $S = S^k$ between $\tau_k = \theta_k$ and $\tau_{k+1} = \theta_{k+1}$:

$$dS_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = S_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) * \left(b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)dW_t\right),$$

where W is a m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \geq N$.

- Jumps at $\tau_{k+1} = \theta_{k+1}$:

$$S_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}) = S_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) * \left(\mathbf{1}_N + \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e_{k+1})\right),$$

γ^k vector-valued in $[-1, \infty)^N$.

Admissible control strategies

• A trading strategy in the N-assets portfolio is a $\mathbb{G}$-predictable process $\pi = (\pi^0, \ldots, \pi^n)$:

$$\pi^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \quad \text{is valued in} \quad A^k \quad \text{closed convex set of } \mathbb{R}^N,$$

denoted $\pi^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k; A^k)$, and representing the amount invested given the past default events $(\boldsymbol{\tau}_k, \boldsymbol{\zeta}_k) = (\boldsymbol{\theta}_k, \mathbf{e}_k)$, $k = 0, \ldots, n$, and until the next default time.

▶ The set of *admissible controls*: $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{F}}^0 \times \ldots \times \mathcal{A}_{\mathbb{F}}^n$, where $\mathcal{A}_{\mathbb{F}}^k$ includes some integrability conditions

Wealth process

- Given an admissible trading strategy $\pi = (\pi^k)_{k=0,\ldots,n}$, the controlled wealth process is given by:

$$X_t = \sum_{k=0}^{n} 1_{\{\tau_k \leq t < \tau_{k+1}\}} X_t^k(\boldsymbol{\tau}_k, \boldsymbol{\zeta}_k), \quad t \geq 0,$$

where X^k is the wealth process with an investment π^k in the assets of price S^k given the past defaults events $(\boldsymbol{\tau}_k, \boldsymbol{\zeta}_k)$.

▶ Dynamics between $\tau_k = \theta_k$ and $\tau_{k+1} = \theta_{k+1}$:

$$dX_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = \pi_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \big(b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) dt + \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) dW_t \big).$$

▶ Jumps at default time $\tau_{k+1} = \theta_{k+1}$:

$$X_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}) = X_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \mathbf{e}_k, e_{k+1}).$$

Value function

- **Value function** of the optimal investment problem:

$$V_0(x) \quad = \quad \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\Big[U(X_T^{x,\pi})\Big], \quad x \in \mathbb{R}.$$

where U is an utility function.

Remark. One can also deal with running gain function, involving e.g. utility from consumption, and utility-based pricing with credit derivative.

Usual global approach

• Write the dynamics of assets and wealth process in the global filtration $\mathbb{G}$
→ Jump-Itô controlled process under $\mathbb{G}$ in terms of W and μ (random measure associated to $(\tau_k, \zeta_k)_k$).

• Use a martingale representation theorem for (W, μ) w.r.t. $\mathbb{G}$ under intensity hypothesis on the default times

▶ Derive the dynamic programming Bellman equation in the $\mathbb{G}$ filtration
→ BSDE with jumps or Integro-Partial-differential equations: Ankirchner et al. (09), Lim and Quenez (10), Jeanblanc et al (10).

Our solutions approach

• Find a suitable decomposition of the $\mathbb{G}$-control problem on each default scenario $\rightarrow$ sub-control problems in the $\mathbb{F}$-filtration

- by relying on the $\mathbb{F}$-decomposition of $\mathbb{G}$-processes,
- density hypothesis on the defaults

▶ Backward system of BSDEs in Brownian filtration

- Get rid of the jump terms and overcome the technical difficulties in BSDEs with jumps
- Existence, uniqueness and characterization results in a general formulation under weaker conditions

▶ Explicit description of the optimal strategies and impact of the defaults

Density hypothesis on defaults

- There exists $\alpha_T(\boldsymbol{\theta}, \mathbf{e})$, $\mathcal{F}_T \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$-measurable, s.t.

$$\textbf{(DH)} \qquad \mathbb{P}\big[(\boldsymbol{\tau}, \zeta) \in d\boldsymbol{\theta} d\mathbf{e} \big| \mathcal{F}_T\big] \;=\; \alpha_T(\boldsymbol{\theta}, \mathbf{e}) d\boldsymbol{\theta} \eta(d\mathbf{e})$$

where $d\boldsymbol{\theta} = d\theta_1 \ldots d\theta_n$ is the Lebesgue measure on $\mathbb{R}^n$, and $\eta(d\mathbf{e}) = \eta_1(de_1) \prod_{k=1}^{n-1} \eta_{k+1}(\mathbf{e}_k, de_{k+1})$.

Comments on density hypothesis

• Standard hypothesis in the theory of initial enlargement of filtrations, see Jacod (85). Insider problems in finance

• Density approach introduced in progressive enlargement of filtrations for credit risk modelling by El Karoui, Jeanblanc, Jiao (09,10) successive defaults without marks:

- More general setting than intensity approach: one can express the intensity of each default time in terms of the density. Semimartingale invariance property **(H')** holds and Immersion hypothesis **(H)** (martingale invariance property) is not required.

Auxiliary survival density

• Let us define α_T^k, $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, $k = 0, \ldots, n-1$, by recursive induction from $\alpha_T^n = \alpha_T$,

$$\alpha_T^k(\boldsymbol{\theta}_k, \mathbf{e}_k) = \int_T^\infty \int_E \alpha_T^{k+1}(\boldsymbol{\theta}_k, \theta, \mathbf{e}_k, e) d\theta \eta_{k+1}(\mathbf{e}_k, de),$$

so that

$$\mathbb{P}\big[\tau_{k+1} > T | \mathcal{F}_T\big] = \int_{\Delta_k \times E^k} \alpha_T^k(\boldsymbol{\theta}_k, \mathbf{e}_k) d\boldsymbol{\theta}_k \eta(d\mathbf{e}_k),$$

where $d\boldsymbol{\theta}_k = d\theta_1 \ldots d\theta_k$, $\eta(d\mathbf{e}_k) = \eta_1(de_1) \ldots \eta_k(\mathbf{e}_{k-1}, de_k)$.

Decomposition result

The value function V_0 is obtained by backward induction from the optimization problems in the $\mathbb{F}$-filtration:

$$V_n(x, \boldsymbol{\theta}, \mathbf{e}) = \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}^n_{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{n,x}\big) \alpha_T(\boldsymbol{\theta}, \mathbf{e}) \Big| \mathcal{F}_{\theta_n} \Big]$$

$$V_k(x, \boldsymbol{\theta}_k, \mathbf{e}_k) = \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k_{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{k,x}\big) \alpha_T^k(\boldsymbol{\theta}_k, \mathbf{e}_k) + \int_{\theta_k}^T \int_E V_{k+1}\big(X^{k,x}_{\theta_{k+1}} + \pi^k_{\theta_{k+1}}.\gamma^k_{\theta_{k+1}}(e_{k+1}), \boldsymbol{\theta}_{k+1}, \mathbf{e}_{k+1}\big) \eta_{k+1}(\mathbf{e}_k, de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big].$$

Comments

- This $\mathbb{F}$-decomposition of the $\mathbb{G}$-control problem can be viewed as a **nonlinear extension of Dellacherie-Meyer and Jeulin-Yor formula**, which relates linear expectation under $\mathbb{G}$ in terms of linear expectation under $\mathbb{F}$, and is used in option pricing for credit derivatives.

- Each step in the backward induction $\longleftrightarrow$ stochastic control problem in the $\mathbb{F}$-filtration (solved e.g. by dynamic programming and BSDE)

BSDEs formulation

- Consider an utility function:

$$U(x) \quad = \quad -\exp(-px), \quad p > 0, \quad x \in \mathbb{R}.$$

and assume that $\mathbb{F} = \mathbb{F}^W$ Brownian filtration generated by W.

▶ Then, the value functions V_k, $k = 0, \ldots, n$, are given by

$$V_k(x, \theta_k, \mathbf{e}_k) \quad = \quad U\big(x - Y^k_{\theta_k}(\theta_k, \mathbf{e}_k)\big),$$

where Y^k, $k = 0, \ldots, n$, are characterized by means of a recursive system of (indexed) BSDEs, derived from dynamic programming arguments in the $\mathbb{F}$-filtration.

BSDE after n defaults

$$
\begin{aligned}
Y_t^n(\boldsymbol{\theta}, \mathbf{e}) \quad = \quad & \frac{1}{p} \ln \alpha_T(\boldsymbol{\theta}, \mathbf{e}) + \int_t^T f^n(r, Z_r^n, \boldsymbol{\theta}, \mathbf{e}) dr \\
& - \int_t^T Z_r^n . dW_r, \quad t \geq \theta_n,
\end{aligned}
$$

with a (quadratic) generator f^n:

$$
f^n(t, z, \boldsymbol{\theta}, \mathbf{e}) \quad = \quad \inf_{\pi \in A^n} \Big\{ \frac{p}{2} \big| z - \sigma_t^n(\boldsymbol{\theta}, \mathbf{e})' \pi \big|^2 - b^n(\boldsymbol{\theta}, \mathbf{e}) . \pi \Big\}.
$$

Remark. Similar BSDE as in El Karoui, Rouge (00), Hu, Imkeller, Müller (04), Sekine (06), for default-free market

BSDE after k defaults, $k = 0, \ldots, n-1$

$$
\begin{aligned}
Y_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) &= \frac{1}{p} \ln \alpha_T^k(\boldsymbol{\theta}_k, \mathbf{e}_k) \\
&\quad + \int_t^T f^k(r, Y_r^k, Z_r^k, \boldsymbol{\theta}_k, \mathbf{e}_k) dr - \int_t^T Z_r^k . dW_r, \quad t \geq \theta_k
\end{aligned}
$$

with a generator

$$
\begin{aligned}
f^k(t, y, z, \boldsymbol{\theta}_k, \mathbf{e}_k) &= \inf_{\pi \in A^k} \Big\{ \frac{p}{2} \big| z - \sigma_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k)' \pi \big|^2 - b_t^k(\boldsymbol{\theta}_k, \mathbf{e}_k) . \pi \\
&\quad + \frac{1}{p} U(y) \int_E U\big(\pi . \gamma_t^k(e_{k+1}) - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \mathbf{e}_k, e_{k+1})\big) \\
&\qquad \eta_{k+1}(de_{k+1}) \Big\}.
\end{aligned}
$$

BSDE characterization of the optimal investment problem

Theorem. Under standard boundedness conditions on the coefficients of the model (b, σ, γ, α), there exists a unique solution $(\mathbf{Y}, \mathbf{Z}) = (Y^0, \ldots, Y^n, Z^0, \ldots, Z^n) \in \mathbf{S}^\infty \times \mathbf{L}^2$ to the recursive system of quadratic BSDEs. The initial value function is

$$V_0(x) = U\big(x - Y_0^0\big),$$

and the optimal strategies between τ_k and τ_{k+1} by

$$\pi_t^k \in \arg\min_{\pi \in A^k} \Big\{ \frac{p}{2}\big|Z_t^k - (\sigma_t^k)'\pi\big|^2 - b_t^k.\pi + \frac{1}{p} U(Y_t^k) \int_E U\big(\pi.\gamma_t^k(e) - Y_t^{k+1}(t,e)\big)\eta_{k+1}(\mathbf{e}_k, de)\Big\}.$$

Technical remarks

• Existence for the system of recursive BSDEs: quadratic term in z + exponential term in y:

- Kobylanski techniques + approximating sequence + convergence

• Uniqueness: verification arguments + BMO techniques

• We don't need to assume boundedness condition on the portfolio control set

Default times density

- Two defaultable assets with default times $(\tau_1, \tau_2) \perp \mathbb{F}$.
  - $\tau_i \rightsquigarrow \mathcal{E}(a_i)$, and dependence of (τ_1, τ_2) via a copula function:

$$\begin{aligned} \mathbb{P}[\tau_1 \geq \theta_1, \tau_2 \geq \theta_2] &= C(\mathbb{P}[\tau_1 \geq \theta_1], \mathbb{P}[\tau_2 \geq \theta_2]) \\ \text{(Gumbel example)} &= \exp\big(-((a_1\theta_1)^\beta + (a_2\theta_2)^\beta)^{1/\beta}\big), \end{aligned}$$

  $\beta \geq 1 \leftrightarrow$ nonnegative correlation between τ_1 and τ_2.
  - Density of (τ_1, τ_2):

$$\alpha^\tau(\theta_1, \theta_2) = a_1 a_2 e^{-a_1\theta_1 - a_2\theta_2} \frac{\partial^2 C}{\partial u_1 \partial u_2}(e^{-a_1\theta_1}, e^{-a_2\theta_2})$$

  - Density of ranked default times and index marks $(\hat{\tau}_1, \hat{\tau}_2, \iota_1, \iota_2)$:

$$\alpha(\hat{\theta}_1, \hat{\theta}_2, i, j) = 1_{\{i=1, j=2\}}\alpha^\tau(\hat{\theta}_1, \hat{\theta}_2) + 1_{\{i=2, j=1\}}\alpha^\tau(\hat{\theta}_2, \hat{\theta}_1).$$

Defaultable assets

- Before any default: BS model for the two assets with drift b^0, volatility σ^0, correlation ρ.
- At default τ_i of asset $i = 1, 2$:
  - Asset i drops to zero (no more traded)
  - Asset j jumps by relative size $\gamma \in (-1, \infty)$: $\gamma < 0 \leftrightarrow$ loss, and $\gamma > 0 \leftrightarrow$ gain, and then follows a BS model with coefficients $b^1 = 0.01$, $\sigma^1 = 0.2$, until its default.
- Investment horizon $T = 1$.

BSDEs as ODEs (I)

$$
\begin{aligned}
Y^2(\boldsymbol{\theta}, i, j) &= \frac{1}{p} \ln \alpha(\boldsymbol{\theta}, i, j), \quad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \Delta_2, \; i, j \in \{1, 2\}, i \neq j \\
Y_t^{1,i}(\theta_1) &= \frac{1}{p} \Big[\beta \ln a_i + (\beta - 1) \ln \theta_1 + \frac{1}{\beta} \ln((a_i \theta_1)^\beta + (a_j t)^\beta) \\
&\quad - ((a_i \theta_1)^\beta + (a_j t)^\beta)^{1/\beta} \Big] + \int_t^T f^{1,i}(s, Y_s^{1,i}, \theta_1) ds,
\end{aligned}
$$

where

$$
f^{1,i}(t, y, \theta_1) = \inf_{\pi \in \mathbb{R}} \Big\{ \frac{p}{2} |\sigma^1 \pi|^2 - b^1 \pi + \frac{1}{p} e^{-p(y - \pi)} \alpha(\theta_1, t, i, j) \Big\},
$$

BSDEs as ODEs (II)

$$Y_t^0 = -\frac{T}{p}(a_1^\beta + a_2^\beta)^{1/\beta} + \int_t^T f^0(s, Y_s^0) ds,$$

where

$$f^0(t,y) = \inf_{\pi=(\pi^1,\pi^2)\in\mathbb{R}^2} \Big\{ \frac{p}{2}\big|(\sigma^0)'\pi\big|^2 - b^0.\pi + \frac{1}{p} e^{-py}\Big[e^{-p(-\pi^1+\pi^2\gamma - Y_t^{1,1}(t))} + e^{-p(\pi^1\gamma-\pi^2-Y_t^{1,2}(t))}\Big]\Big\}.$$

Value function $V^0(t)$ for different jump sizes

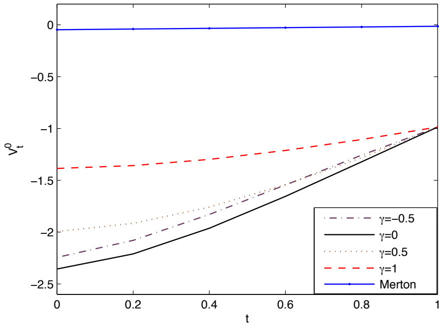

Figure: Value function $V^0(t)$: $a_1 = a_2 = 0.01$, $\beta = 2$

Optimal strategy in function of jump size for various default intensities

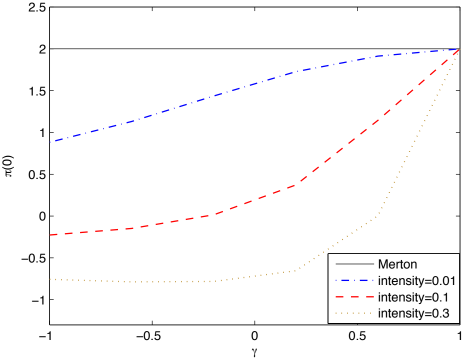

Figure: optimal strategy by varying intensity $a_1 = a_2$, and fixed $\beta = 2$

Optimal strategies in both assets by varying jump sizes and default intensities

Table: Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ before any defaults with various γ and default intensities.

γ	-0.5	-0.1	0	0.5	1	Merton
$a_1 = 0.01,\ a_2 = 0.1,\ \beta = 2$						
$\hat{\pi}^1$	0.462	1.659	1.892	2.621	2.832	2
$\hat{\pi}^2$	-1.047	-0.709	-0.498	0.623	1.168	2
$a_1 = 0.1,\ a_2 = 0.1,\ \beta = 2$						
$\hat{\pi}^1$	-0.353	-0.210	-0.147	0.556	2	2
$\hat{\pi}^2$	-0.353	-0.210	-0.147	0.556	2	2
$a_1 = 0.3,\ a_2 = 0.1,\ \beta = 2$						
$\hat{\pi}^1$	-1.723	-1.719	-1.647	-0.697	1.293	2
$\hat{\pi}^2$	-0.132	0.453	0.521	1.121	2.707	2

Optimal strategies in both assets by varying correlation parameters

Table: Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ before any defaults with various ρ and β. $a_1 = 0.01$, $a_2 = 0.1$

γ	-0.5	-0.1	0	0.5	1	Merton
$\rho = 0,\ \beta = 1$						
$\hat{\pi}^1$	0.228	0.942	1.099	1.966	2.459	2
$\hat{\pi}^2$	-0.867	-0.452	-0.278	0.856	1.541	2
$\rho = 0,\ \beta = 2$						
$\hat{\pi}^1$	0.462	1.659	1.892	2.621	2.832	2
$\hat{\pi}^2$	-1.047	-0.709	-0.498	0.623	1.168	2
$\rho = 0.3,\ \beta = 1$						
$\hat{\pi}^1$	0.492	1.081	1.188	1.715	2.025	1.539
$\hat{\pi}^2$	-0.959	-0.504	-0.348	0.519	1.052	1.539
$\rho = 0.3,\ \beta = 2$						
$\hat{\pi}^1$	0.863	1.939	2.077	2.399	2.450	1.539
$\hat{\pi}^2$	-1.235	-0.817	-0.626	0.216	0.627	1.539

Concluding remarks (I)

• Beyond the optimal investment problem considered here, we provide a general formulation of stochastic control under progressive enlargement of filtration with multiple random times and marks:

- Change of regimes in the state process, control set and gain functional after each random time
- Includes in particular the formulation via jump-diffusion controlled processes

• Recursive decomposition on each default scenario of the $\mathbb{G}$-control problem into $\mathbb{F}$-stochastic control problems by relying on the density hypothesis

Concluding remarks (II)

• $\mathbb{F}$-decomposition method $\rightarrow$ another perspective for the study of controlled diffusion processes with (finite number of) jumps, (quadratic) BSDEs with (finite number of) jumps

$\rightarrow$ Get rid of the jump terms

- obtain comparison theorems under weaker conditions
- Alternative approach for numerical schemes of BSDEs with jumps

$\rightarrow$ Recent works by Kharroubi and Lim (11 a,b).