DIVERSE BELIEFS AND MARKET SELECTION

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Overview

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- Equilibrium pricing under diverse beliefs
- Market selection and starvation
- Going broke
- Hungry misers and happy bankrupts
- Asymptotically equivalent pricing
- Conclusions

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Preprint downloadable from `http://www.statslab.cam.ac.uk/~chris/MSH.pdf`

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Central planner equilibrium: Form the combined objective

$$\begin{aligned} F(\bar{c}) &= \sup \Big\{ \sum_{j=1}^J \nu_j^{-1} F_j(c^j) : \sum_j c^j = \bar{c} \Big\} \\ &= \sup \Big\{ \mathbb{E}^0 \Big[\int_0^\infty \sum_j \nu_j^{-1} \Lambda_t^j\, U_j(t, c_t^j)\, dt \Big] : \sum_j c_t^j = \bar{c}_t \Big\} \end{aligned}$$

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$$w_t = \zeta_t^{-1} \, \mathbb{E}^0 \left[\int_t^\infty \zeta_u c_u \, du \,\middle|\, \mathcal{F}_t \right]$$

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Different assumptions from earlier work of Blume & Easley, Kogan et al, Cvitanic & Malamud, Yan, Sandroni, Cvitanic-Jouini-Malamud-Napp, ...

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Then if $\nu_2 = p$, $\nu_1 = 1$, and $e^{\rho t}\zeta_t \downarrow 0$ through the values (x_n) we see that agent 1 starves . . . Need for some (all) $0 < p < 1$ that

$$\sup_{x>0} \frac{I(px)}{I(x)} < \infty.$$

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(ii) If (1) holds, then Agent 2 starves $\Rightarrow$ Agent 2 has inferior beliefs.

Corollary. If R is bounded and bounded away from 0, then

Agent 2 starves $\Leftrightarrow$ Agent 2 has inferior beliefs.

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See the preprint with Nishide.

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Will take $R = 1/2$, and make $\Lambda_t \to 0$, yet $\liminf_{t\to\infty} w_t^{(2)} \geq a > 0$ with positive probability.

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If $\tau > n$, then

$$\begin{aligned}
w_n^{(2)} &\geq \mathbb{E}_n \left[\int_n^{n+1} e^{-\rho(s-n)} \Lambda_s^{1/R} \, ds \right] \geq \mathbb{E}_n \left[1_{\{\tau \leq n+1\}} \int_n^{n+1} e^{-\rho(s-n)} \Lambda_s^{1/R} \, ds \right] \\
&\geq \mathbb{E}_n \left[1_{\{\tau \leq n+1\}} \int_\tau^{n+1} e^{-\rho(s-n)} (a_{n+1} + \xi_n)^{1/R} \, ds \right] \\
&= (a_{n+1} + \xi_n)^{1/R} \int_0^1 \nu_n e^{-\nu_n t} \left(\int_t^1 e^{-\rho s} \, ds \right) \, dt \\
&= \frac{(a_{n+1} + \xi_n)^{1/R}}{\rho(\rho + \nu_n)} \, e^{-\rho} \left[\nu_n (e^\rho - 1) - \rho(1 - e^{-\nu_n}) \right].
\end{aligned}$$

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Easy to see that $P(\tau = \infty) > 0$.

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- Agent 1 goes broke.

YES!! Agent 1 gets all the consumption, Agent 2 gets all the wealth!!

Price impact.

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In a benchmark economy where all agents have the same beliefs, $\Lambda^j \equiv 1$ for all j,

$$\tilde{\zeta}_t(\alpha) = \alpha_j e^{-\rho t} u'(c_t^j)$$

for some $\alpha_j > 0$.

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$$\lim_{t \to \infty} \frac{\zeta_{t+T}/\zeta_t}{\tilde{\zeta}_{t+T}(\alpha)/\tilde{\zeta}_t(\alpha)} = 1$$

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We change agent j in such a way that he has 'no price impact', and then the ratio of the prices of the asset in the two economies tends to infinity?!

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A non-negative future cashflow $(c_s)_{s \geq t}$ is priced as

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Definition. Families $(\pi_t)_{t\geq 0}$ and $(\tilde{\pi}_t)_{t\geq 0}$ are asymptotically equivalent if there exists some stopping time t_0 such that for all $t \geq t_0$,

$$\mathcal{A}_t \equiv \{c \geq 0; \pi_t(c) < \infty\} = \tilde{\mathcal{A}}_t \equiv \{c \geq 0; \tilde{\pi}_t(c) < \infty\},$$

and

$$\sup_{c\in\mathcal{A}_t, c\leq 1} \frac{\pi_t(c)}{\tilde{\pi}_t(c)} \to 1, \qquad \sup_{c\in\mathcal{A}_t, c\leq 1} \frac{\tilde{\pi}_t(c)}{\pi_t(c)} \to 1 \quad (t \to 1).$$

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Theorem. Families $(\pi_t)_{t \geq 0}$ and $(\tilde{\pi}_t)_{t \geq 0}$ are asymptotically equivalent if and only if there exist positive α and β and a stopping time t_0 such that

(i) for all $t \geq t_0$,

$$\alpha_t \leq \frac{\zeta_{t,t+T}}{\tilde{\zeta}_{t,t+T}} \equiv \frac{\zeta_{t+T}/\zeta_t}{\tilde{\zeta}_{t+T}/\tilde{\zeta}_t} \leq \beta_t \quad \forall t, T \geq 0; \tag{7}$$

(ii)

$$\frac{\alpha_t}{\beta_t} \to 1 \quad (t \to \infty). \tag{8}$$

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