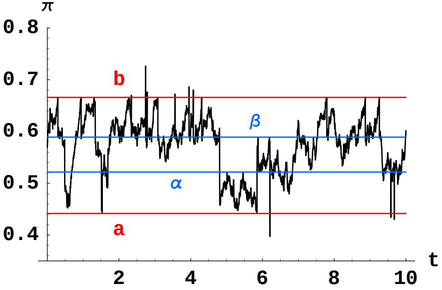
# Maximizing the asymptotic growth rate under fixed and proportional transaction costs in a financial market with jumps

Jörn Sass, sass@mathematik.uni-kl.de

joint work with Alexandra Kochendörfer

Department of Mathematics, University of Kaiserslautern




Bad Herrenalb, March 29, 2011

# Outline

- Market model and trading without transaction costs
- The asymptotic growth rate without transaction costs
- Types of transaction costs
- Fixed and proportional costs
- Quasi-variational inequalities
- Optimality of constant-boundary strategies
- Numerical examples

# Financial market and trading without costs

**Financial market:**

- Bank account with interest rate $r = 0$.

# Financial market and trading without costs

**Financial market:**

- Bank account with interest rate $r = 0$.
- Stock: $dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \int_0^t \int_E z\widetilde{J}(dz, ds)\right)$, i.e.

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma + \lambda \mathrm{E}[\log(1+Z) - Z]\right)t + \sigma W_t + \int_0^t \int_E \log(1+z)\widetilde{J}(ds, dz)\right\}$$

# Financial market and trading without costs

**Financial market:**

- Bank account with interest rate $r = 0$.
- Stock: $dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \int_0^t \int_E z\widetilde{J}(dz, ds)\right)$, i.e.

$$S_t = S_0 \exp\left\{\left(\mu - \tfrac{1}{2}\sigma + \lambda \mathrm{E}[\log(1+Z) - Z]\right)t + \sigma W_t + \int_0^t\int_E \log(1+z)\widetilde{J}(ds, dz)\right\}$$

- $J$ Poisson random measure of a compound Poisson process $\sum_{i=1}^{N_t} Z_i$
- Jumps i.i.d. like $Z$ with values in $E \subseteq (-1, \infty)$, intensity $\lambda > 0$
- $\widetilde{J}$ compensated PRM, i.e. $\widetilde{J}(dt, dz) = J(dt, dz) - \lambda dt P^Z(dz)$

# Financial market and trading without costs

**Financial market:**

- Bank account with interest rate $r = 0$.
- Stock: $dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \int_0^t \int_E z\widetilde{J}(dz, ds)\right)$, i.e.

$$S_t = S_0 \exp\left\{\left(\mu - \tfrac{1}{2}\sigma + \lambda \mathrm{E}[\log(1+Z)-Z]\right)t + \sigma W_t + \int_0^t\int_E \log(1+z)\widetilde{J}(ds, dz)\right\}$$

- $J$ Poisson random measure of a compound Poisson process $\sum_{i=1}^{N_t} Z_i$
- Jumps i.i.d. like $Z$ with values in $E \subseteq (-1, \infty)$, intensity $\lambda > 0$
- $\widetilde{J}$ compensated PRM, i.e. $\widetilde{J}(dt, dz) = J(dt, dz) - \lambda dt P^Z(dz)$

**Trading without costs**

- $\pi_t$ is the fraction of wealth $X_t$ which is invested in the stock
- Without costs $dX_t = \pi_{t-} X_{t-} \left(\mu\, dt + \sigma\, dW_t + \int_E z\widetilde{J}(dt, dz)\right)$.
- Without costs we may look at $X$ as controlled by $K = \pi$.

# Maximizing the growth rate without transaction costs

- We want to maximize the **asymptotic growth rate**

$$\hat{R} = \sup_K R^K, \quad R^K = \liminf_{T\to\infty} \frac{1}{T}\,\mathrm{E}[\log(X_T^K)]\ .$$

# Maximizing the growth rate without transaction costs

- We want to maximize the **asymptotic growth rate**

$$\hat{R} = \sup_K R^K, \quad R^K = \liminf_{T\to\infty} \frac{1}{T}\,\mathrm{E}[\log(X_T^K)]\,.$$

- The wealth for given $K = (\pi_t)_{t\geq 0}$ is

$$X_T^K = x_0 \exp\left\{ \int_0^T R(\pi_t)\,dt + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz) \right\}.$$

# Maximizing the growth rate without transaction costs

- We want to maximize the **asymptotic growth rate**

$$\hat{R} = \sup_K R^K, \quad R^K = \liminf_{T\to\infty} \frac{1}{T}\,\mathrm{E}[\log(X_T^K)]\,.$$

- The wealth for given $K = (\pi_t)_{t\geq 0}$ is

$$X_T^K = x_0 \exp\left\{ \int_0^T R(\pi_t)\,dt + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz) \right\}.$$

- Here,

$$R(x) = \mu x - \tfrac{1}{2}\sigma^2 x^2 + \lambda \mathrm{E}[\log(1+xZ) - xZ]$$

is the asymptotic growth rate for constant $\pi_t = x$. In particular,

$$\begin{aligned} R_0 := R(0) &= 0 \quad \text{pure bond strategy} \\ R_1 := R(1) &= \mu - \tfrac{1}{2}s^2 + \lambda\mathrm{E}[\log(1+Z) - Z] \quad \text{pure stock strategy} \\ \hat{R} := R(\hat{\eta}) &= \max\{R(x) : x\in[0,1]\} \quad \text{optimal strategy (Merton).} \end{aligned}$$

# Maximizing the growth rate without transaction costs

- We want to maximize the **asymptotic growth rate**

$$\hat{R} = \sup_K R^K, \quad R^K = \liminf_{T\to\infty} \frac{1}{T} \mathrm{E}[\log(X_T^K)] .$$

- The wealth for given $K = (\pi_t)_{t\geq 0}$ is

$$X_T^K = x_0 \exp\left\{ \int_0^T R(\pi_t)\, dt + \int_0^T \sigma\, \pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz) \right\} .$$

- Here,

$$R(x) = \mu x - \tfrac{1}{2}\sigma^2 x^2 + \lambda \mathrm{E}[\log(1+xZ) - xZ]$$

is the asymptotic growth rate for constant $\pi_t = x$. In particular,

$$\begin{aligned} R_0 := R(0) &= 0 \quad \text{pure bond strategy} \\ R_1 := R(1) &= \mu - \tfrac{1}{2}s^2 + \lambda\mathrm{E}[\log(1+Z) - Z] \quad \text{pure stock strategy} \\ \hat{R} := R(\hat{\eta}) &= \max\{R(x) : x \in [0,1]\} \quad \text{optimal strategy (Merton).} \end{aligned}$$

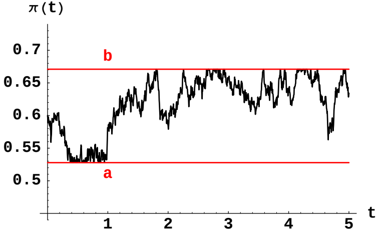
- We assume $\hat{\eta} \in (0,1)$ ($\Leftrightarrow \mu > 0, \mu - \sigma^2 - \lambda \mathrm{E}[Z^2/(1+Z)] < 0$).

# Types of transaction costs

An investor who has wealth $x$ and makes a transaction $\Delta$ may face fees of the form $\gamma\,|\Delta|$ (proportional), $C > 0$ (constant), $\delta\,x$ (fixed).

$\gamma\,|\Delta|$, $\gamma \in (0,1)$

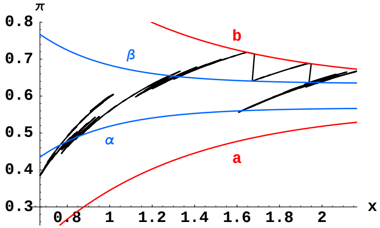
proportional costs




Davis und Norman (1990)
Shreve and Soner (1994)
Akian, Sulem and Taksar (2001)

$C + \gamma\,|\Delta|$, $C > 0$

constant plus proport. costs



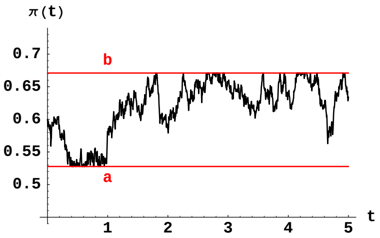

Eastham and Hastings (1988)
Korn (1998)
Øksendal and Sulem (2001)

# Types of transaction costs 2

An investor who has wealth $x$ and makes a transaction $\Delta$ may face fees of the form $\gamma\,|\Delta|$ (proportional), $C > 0$ (constant), $\delta\,x$ (fixed).

$\gamma\,|\Delta|$, $\gamma \in (0,1)$

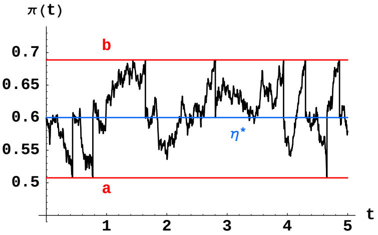
proportional costs




Davis and Norman (1990)
Shreve and Soner (1994)
Akian, Sulem and Taksar (2001)

$\delta\,x$, $\delta \in (0,1)$

fixed costs




Morton and Pliska (1995)
Bielecki and Pliska (2000)

When trading, $X_\tau = (1-\delta)X_{\tau-}$.

# Fixed and proportional costs for the growth rate

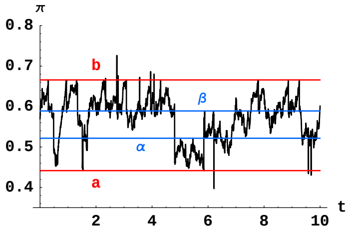
- Fees $\delta x + \gamma |\Delta|$, where $\delta > 0$, $\gamma \in [0, 1 - \delta)$.
- The wealth $(X_t)_{t \geq 0}$ and the risky fraction process $(\pi_t)_{t \geq 0}$, are both controlled by an impulse control strategy $K = (\tau_n, \Delta_n)_{n \in N_0}$.

# Fixed and proportional costs for the growth rate

- Fees $\delta x + \gamma |\Delta|$, where $\delta > 0$, $\gamma \in [0, 1 - \delta)$.
- The wealth $(X_t)_{t \geq 0}$ and the risky fraction process $(\pi_t)_{t \geq 0}$, are both controlled by an impulse control strategy $K = (\tau_n, \Delta_n)_{n \in \mathbf{N}_0}$.

We expect that an impulse control strategy with constant boundaries (CB) is optimal, given by

- $a, b$ for stopping boundaries, $0 < a < b < 1$,
- $\alpha, \beta \in (a, b)$ for new risky fractions after trading $a, b$.

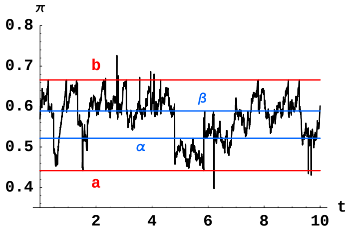


# Fixed and proportional costs for the growth rate

- Fees $\delta x + \gamma |\Delta|$, where $\delta > 0$, $\gamma \in [0, 1 - \delta)$.
- The wealth $(X_t)_{t \geq 0}$ and the risky fraction process $(\pi_t)_{t \geq 0}$, are both controlled by an impulse control strategy $K = (\tau_n, \Delta_n)_{n \in N_0}$.

We expect that an impulse control strategy with constant boundaries (CB) is optimal, given by

- $a, b$ for stopping boundaries, $0 < a < b < 1$,
- $\alpha, \beta \in (a, b)$ for new risky fractions after trading $a, b$.




Existence of optimal impulse-control strategies:

- Irle/S. (2005): BS-model: Renewal theory, best CB strategy
- Irle/S. (2006): BS-model: CB strategy is optimal
- Tamura (2006): BS-model: Ex. of optimal impulse control strategy
- Duncan/Pasik Duncan/Stettner (2010): With jumps. Existence of of optimal impulse control strategy under diversification

# Trading for fixed and proportional transaction costs

For a combination of fixed and proportional costs

$$\delta\, x + \gamma\, |\Delta|\,, \quad \delta > 0\,, \quad \gamma \in [0, 1-\delta)\,,$$

the wealth process $(X_t)_{t\geq 0}$ and the risky fraction process $(\pi_t)_{t\geq 0}$, both controlled by $K = (\tau_n, \Delta_n)_{n\in N_0}$, satisfy

$$\overline{X}_n = (1-\delta) X_{\tau_n -} - \gamma |\Delta_n|\,, \quad \text{new wealth,}$$

$$\overline{\pi}_n = \frac{\pi_{\tau_n -} X_{\tau_n -} + \Delta_n}{\overline{X}_n}\,, \quad \text{new fraction,}$$

at trading time $\tau_n$;

# Trading for fixed and proportional transaction costs

For a combination of fixed and proportional costs

$$\delta\, x + \gamma\, |\Delta|\,, \quad \delta > 0\,, \quad \gamma \in [0, 1-\delta)\,,$$

the wealth process $(X_t)_{t\geq 0}$ and the risky fraction process $(\pi_t)_{t\geq 0}$, both controlled by $K = (\tau_n, \Delta_n)_{n\in N_0}$, satisfy

$$\overline{X}_n = (1-\delta)X_{\tau_n -} - \gamma|\Delta_n|\,, \quad \text{new wealth,}$$

$$\overline{\pi}_n = \frac{\pi_{\tau_n -}X_{\tau_n -} + \Delta_n}{\overline{X}_n}\,, \quad \text{new fraction,}$$

at trading time $\tau_n$; In between we have for $t \in (\tau_n, \tau_{n+1}]$

$$X_t = (1-\overline{\pi}_n)\overline{X}_n + \overline{\pi}_n\overline{X}_n\, S_t/S_{\tau_n}\,, \quad \text{wealth,}$$

$$\pi_t = \frac{\overline{\pi}_n\overline{X}_n S_t/S_{\tau_n}}{X_t}\,, \quad \text{risky fraction,}$$

# Trading for fixed and proportional transaction costs

For a combination of fixed and proportional costs

$$\delta\, x + \gamma\, |\Delta|\,, \quad \delta > 0\,, \quad \gamma \in [0, 1-\delta)\,,$$

the wealth process $(X_t)_{t\geq 0}$ and the risky fraction process $(\pi_t)_{t\geq 0}$, both controlled by $K = (\tau_n, \Delta_n)_{n\in N_0}$, satisfy

$$\overline{X}_n = (1-\delta)X_{\tau_n-} - \gamma|\Delta_n|\,, \quad \text{new wealth,}$$

$$\overline{\pi}_n = \frac{\pi_{\tau_n-}X_{\tau_n-} + \Delta_n}{\overline{X}_n}\,, \quad \text{new fraction,}$$

at trading time $\tau_n$; In between we have for $t \in (\tau_n, \tau_{n+1}]$

$$X_t = (1-\overline{\pi}_n)\overline{X}_n + \overline{\pi}_n\overline{X}_n\, S_t/S_{\tau_n}\,, \quad \text{wealth,}$$

$$\pi_t = \frac{\overline{\pi}_n\overline{X}_n S_t/S_{\tau_n}}{X_t}\,, \quad \text{risky fraction,}$$

We compute for type of trading $A_n = \mathrm{sign}((1-\delta)\overline{\pi}_n - \pi_{\tau_n})$

$$\Delta_n = \frac{(1-\delta)\overline{\pi}_n - \pi_{\tau_n}}{1 + A_n\gamma\,\overline{\pi}_n}\, X_{\tau_n-}\,.$$

# Reduction to a one-dimensional control problem

We look at $(\pi_t)_{t\geq 0}$ as controlled by $K = (\tau_n, \eta_n)_{n\in N_0}$, $\eta_n := \overline{\pi}_n$, where $\tau_0 < \tau_1 < \dots$ are stopping times, $\eta_n$ new risky fractions after $\tau_n$.

# Reduction to a one-dimensional control problem

We look at $(\pi_t)_{t\geq 0}$ as controlled by $K=(\tau_n,\eta_n)_{n\in N_0}$, $\eta_n := \overline{\pi}_n$, where $\tau_0<\tau_1<\dots$ are stopping times, $\eta_n$ new risky fractions after $\tau_n$. Then,

$$R^K = \liminf_{T\to\infty}\frac{1}{T}E\Big[\underbrace{\int_0^T R(\pi_t)dt}_{\text{growth rate without costs}} + \sum_{n\in N_0}\underbrace{\Gamma(\pi_{\tau_n-},\eta_n)}_{\text{costs}}\mathbf{1}_{\{\tau_n<T\}}+\dots\Big].$$

where

$$\Gamma(\pi,\eta)=\begin{cases}\log\frac{1-\delta-\gamma\pi}{1-\gamma\,\eta}, & \eta<\frac{\pi}{1-\delta},\\ \log\frac{1-\delta+\gamma\pi}{1+\gamma\,\eta}, & \eta\geq\frac{\pi}{1-\delta},\end{cases}$$

$$\pi_t = \int_0^t \pi_s(1-\pi_s)(\mu-\sigma^2\pi_s)ds + \int_0^t \pi_s(1-\pi_s)\sigma\, dW_s$$
$$+\sum_{n\in N_0}(\eta_n-\pi_{\tau_n-})\mathbf{1}_{\{\tau_n<t\}} + \int_0^t\int_E \frac{z}{1+\pi_{s-}z}\widetilde{J}(ds,dz).$$

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n\in N_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n < T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[G(T) + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz)\right].$$

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n\in \mathbf{N}_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n<T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[G(T) + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz)\right].$$

Find $v : (0,1) \to \mathbf{R}$ and $\rho \in \mathbf{R}$ such that

$$\begin{aligned} (I) &\quad \mathcal{L}v(x) + R(x) - \rho \le 0 \quad \text{for all} \quad x \in (0,1), \\ (II) &\quad v(y) - v(x) + \Gamma(x,y) \le 0 \quad \text{for all} \quad x,y \in (0,1), \end{aligned}$$

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n \in N_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n < T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[ G(T) + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz) \right].$$

Find $v : (0,1) \to \boldsymbol{R}$ and $\rho \in \boldsymbol{R}$ such that

$$\begin{aligned} (I) \quad & \mathcal{L}v(x) + R(x) - \rho \le 0 \quad \text{for all} \quad x \in (0,1), \\ (II) \quad & v(y) - v(x) + \Gamma(x,y) \le 0 \quad \text{for all} \quad x, y \in (0,1), \end{aligned}$$

since for $\widetilde{\sigma}(x) = \sigma x(1-x)$, $\widetilde{v}(x,z) = v\left(\frac{(1+z)x}{1+xz}\right) - v(x)$

$$G(T) = \int_0^T R(\pi_t)dt + \sum_{n,\tau_n \le T} \Gamma(\pi_{\tau_n -}, \eta_n) + \sum_{n,\tau_n \le T} (v(\eta_n) - v(\pi_{\tau_n -})) + \rho T - \rho T$$

$$- v(\pi_T) + v(\pi_0) + \int_0^T \mathcal{L}v(\pi_t)dt + \int_0^T \widetilde{\sigma}(\pi_t)v'(\pi_t)dW_t + \int_0^T \int_E \widetilde{v}(\pi_{t-}, z)\widetilde{J}(dt,dz).$$

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n\in \mathbf{N}_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n<T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[G(T) + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz)\right].$$

Find $v : (0,1) \to \mathbf{R}$ and $\rho \in \mathbf{R}$ such that

$$\begin{aligned}(I) \quad & \mathcal{L}v(x) + R(x) - \rho \le 0 \quad \text{for all} \quad x \in (0,1),\\ (II) \quad & v(y) - v(x) + \Gamma(x,y) \le 0 \quad \text{for all} \quad x, y \in (0,1),\end{aligned}$$

since for $\widetilde{\sigma}(x) = \sigma x(1-x)$, $\widetilde{v}(x,z) = v\left(\frac{(1+z)x}{1+xz}\right) - v(x)$

$$G(T) = \rho T + v(\pi_0) - v(\pi_T) + \int_0^T \widetilde{\sigma}(\pi_t)v'(\pi_t)dW_t + \int_0^T\int_E \widetilde{v}(\pi_{t-},z)\widetilde{J}(dt,dz)$$
$$+ \int_0^T (\mathcal{L}v(\pi_t) + R(\pi_t) - \rho)dt + \sum_{n,\tau_n\le T}(v(\eta_n) - v(\pi_{\tau_n -}) + \Gamma(\pi_{\tau_n -},\eta_n)).$$

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n\in N_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n<T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[G(T) + \int_0^T \sigma\, \pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz)\right].$$

Find $v : (0,1) \to \boldsymbol{R}$ and $\rho \in \boldsymbol{R}$ such that

$$\begin{aligned} (I) \quad & \mathcal{L}v(x) + R(x) - \rho \le 0 \quad \text{for all} \quad x \in (0,1), \\ (II) \quad & v(y) - v(x) + \Gamma(x,y) \le 0 \quad \text{for all} \quad x, y \in (0,1), \end{aligned}$$

since for $\widetilde{\sigma}(x) = \sigma x(1-x)$, $\widetilde{v}(x,z) = v\left(\frac{(1+z)x}{1+xz}\right) - v(x)$

$$\begin{aligned} G(T) = \rho T + v(\pi_0) - v(\pi_T) + \int_0^T \widetilde{\sigma}(\pi_t)v'(\pi_t)dW_t + \int_0^T \int_E \widetilde{v}(\pi_{t-}, z)\widetilde{J}(dt,dz) \\ + \int_0^T (\mathcal{L}v(\pi_t) + R(\pi_t) - \rho)dt + \sum_{n,\tau_n\le T} (v(\eta_n) - v(\pi_{\tau_n -}) + \Gamma(\pi_{\tau_n -}, \eta_n)). \end{aligned}$$

- If $v$, $\lambda$ satisfy $(I)$, $(II)$ and ... for all $\pi = \pi^K$ then $R^* \le \rho$.

# Quasi-variational inequalities

For $G(T) = \int_0^T R(\pi_t)dt + \sum_{n\in N_0} \Gamma(\pi_{\tau_n}, \eta_n)\mathbf{1}_{\{\tau_n<T\}}$ we have

$$R^K = \liminf_{T\to\infty} \frac{1}{T} E\left[G(T) + \int_0^T \sigma\,\pi_t dW_t + \int_0^T \int_E \log(1+\pi_{t-}z)\widetilde{J}(dt,dz)\right].$$

Find $v : (0,1) \to R$ and $\rho \in R$ such that

$$\begin{aligned} (I) \quad & \mathcal{L}v(x) + R(x) - \rho \le 0 \quad \text{for all} \quad x \in (0,1), \\ (II) \quad & v(y) - v(x) + \Gamma(x,y) \le 0 \quad \text{for all} \quad x,y \in (0,1), \end{aligned}$$

since for $\widetilde{\sigma}(x) = \sigma x(1-x)$, $\widetilde{v}(x,z) = v\left(\frac{(1+z)x}{1+xz}\right) - v(x)$

$$\begin{aligned} G(T) = {} & \rho T + v(\pi_0) - v(\pi_T) + \int_0^T \widetilde{\sigma}(\pi_t)v'(\pi_t)dW_t + \int_0^T\int_E \widetilde{v}(\pi_{t-},z)\widetilde{J}(dt,dz) \\ & + \int_0^T (\mathcal{L}v(\pi_t) + R(\pi_t) - \rho)dt + \sum_{n,\tau_n\le T} (v(\eta_n) - v(\pi_{\tau_n-}) + \Gamma(\pi_{\tau_n-},\eta_n)). \end{aligned}$$

- If $v$, $\lambda$ satisfy $(I)$, $(II)$ and ... for all $\pi = \pi^K$ then $R^* \le \rho$.
- If '=' for some $\pi^* = \pi^{K^*}$, then $K^*$ is optimal and $R^* = \rho$.

# Solving the first qvi

- The generator $\mathcal{L}$ of the uncontrolled $(\pi_t)_{t\in[0,T]}$ is given by

$$\mathcal{L}h(x) = x(1-x)\left(\mu - \sigma^2 x - \lambda \mathrm{E}[Z]\right) h'(x) + \frac{1}{2}x(1-x)^2\sigma^2 h''(x) + \lambda \mathrm{E}\left[h\left(\frac{x(1+Z)}{1+xZ}\right) - h(x)\right]$$

# Solving the first qvi

- The generator $\mathcal{L}$ of the uncontrolled $(\pi_t)_{t\in[0,T]}$ is given by

$$\begin{aligned}\mathcal{L}h(x) =& x(1-x)\left(\mu-\sigma^2 x-\lambda\mathrm{E}[Z]\right)h'(x)\\ &+\tfrac{1}{2}x(1-x)^2\sigma^2 h''(x)+\lambda\mathrm{E}\left[h\left(\frac{x(1+Z)}{1+xZ}\right)-h(x)\right]\end{aligned}$$

- A solution of $\mathcal{L}v(x)+R(x)-\rho=0$ with $v'(c)=0$ is given by

$$v_0(x)=h(x)+\rho h_1(x)+\psi(c)h_0(x)$$

where $h, h_1, h_0$ are solutions of

$$\mathcal{L}h=-R,\quad \mathcal{L}_\pi h_1=1,\quad \mathcal{L}_\pi h_0=0.$$

# Solving the first qvi

- The generator $\mathcal{L}$ of the uncontrolled $(\pi_t)_{t\in[0,T]}$ is given by

$$\mathcal{L}h(x) = x(1-x)\left(\mu - \sigma^2 x - \lambda \mathrm{E}[Z]\right) h'(x) + \tfrac{1}{2}x(1-x)^2\sigma^2 h''(x) + \lambda \mathrm{E}\left[h\left(\frac{x(1+Z)}{1+xZ}\right) - h(x)\right]$$

- A solution of $\mathcal{L}v(x) + R(x) - \rho = 0$ with $v'(c) = 0$ is given by

$$v_0(x) = h(x) + \rho h_1(x) + \psi(c) h_0(x)$$

where $h, h_1, h_0$ are solutions of

$$\mathcal{L}h = -R, \quad \mathcal{L}_\pi h_1 = 1, \quad \mathcal{L}_\pi h_0 = 0.$$

- Explicit solutions are

$$h(x) = \log(1-x), \quad h_1(x) = \frac{1}{R_1}\log\frac{x}{1-x}, \quad h_0(x) = -\left(\frac{1-x}{x}\right)^p$$

where $p$ is unique solution in $(0,1)$ of (ass. $\mu - \frac{1}{2}\sigma^2 - \lambda \mathrm{E}[Z] > 0$)

$$\mu - \tfrac{1}{2}(1+p)\sigma^2 + \tfrac{\lambda}{p} - \tfrac{\lambda}{p}\mathrm{E}[pZ + (1+Z)^{-p}].$$

# Necessary conditions for optimality of CB-strategies

- We consider *CB*-strategies $(a, b, \alpha, \beta)$, $a < \alpha < \beta < b$. So we want

$$v_0(\beta) - v_0(b) + \Gamma(b, \beta) = 0 \ , \quad v_0(\alpha) - v_0(a) + \Gamma(a, \alpha) = 0 \ .$$

# Necessary conditions for optimality of CB-strategies

- We consider $CB$-strategies $(a, b, \alpha, \beta)$, $a < \alpha < \beta < b$. So we want

$$v_0(\beta) - v_0(b) + \Gamma(b, \beta) = 0 \ , \quad v_0(\alpha) - v_0(a) + \Gamma(a, \alpha) = 0 \ .$$

- We shall determine $v_0$ (i.e. $c$), $\rho$, $a$, $\alpha$, $\beta$, $b$, using the conditions

$$v_0'(b) = -\frac{\gamma}{1-\delta-\gamma b}, \quad v_0'(\beta) = -\frac{\gamma}{1-\gamma\beta}, \quad \int_\beta^b v_0'(x)\,dx = \Gamma\,(b, \beta),$$

$$v_0'(a) = \frac{\gamma}{1-\delta+\gamma a}, \quad v_0'(\alpha) = \frac{\gamma}{1+\gamma\alpha}, \quad \int_a^\alpha v_0'(x)\,dx = -\Gamma\,(a, \alpha).$$

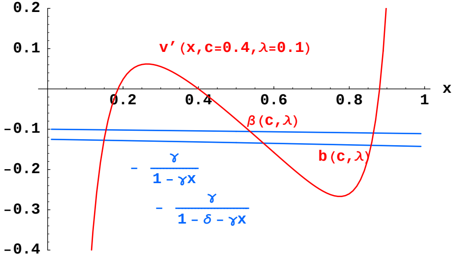
# Necessary conditions for optimality of CB-strategies

- We consider $CB$-strategies $(a, b, \alpha, \beta)$, $a < \alpha < \beta < b$. So we want
$$v_0(\beta) - v_0(b) + \Gamma(b, \beta) = 0 \ , \quad v_0(\alpha) - v_0(a) + \Gamma(a, \alpha) = 0 \ .$$

- We shall determine $v_0$ (i.e. $c$), $\rho$, $a$, $\alpha$, $\beta$, $b$, using the conditions
$$v_0'(b) = -\frac{\gamma}{1 - \delta - \gamma b}, \quad v_0'(\beta) = -\frac{\gamma}{1 - \gamma\beta}, \quad \int_\beta^b v_0'(x)\, dx = \Gamma\left(b, \beta\right),$$
$$v_0'(a) = \frac{\gamma}{1 - \delta + \gamma a}, \quad v_0'(\alpha) = \frac{\gamma}{1 + \gamma\alpha}, \quad \int_a^\alpha v_0'(x)\, dx = -\Gamma\left(a, \alpha\right).$$

- **Theorem (sufficiency) :** For a solution $c, \rho, a, \alpha, \beta, b$ of the necessary conditions,
$$v(x) := \begin{cases} v_0(\alpha) + \Gamma(x, \alpha), & x \leq a, \\ v_0(x), & x \in (a, b), \\ v_0(\beta) + \Gamma(x, \beta), & x \geq b, \end{cases}$$
satisfies the qvis.

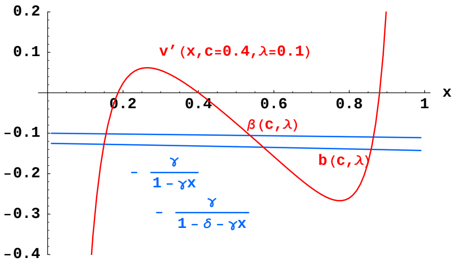
# Existence of an optimal solution




We determine $\rho_1 = \rho_1(c)$, $b = b(c, \rho_1)$ and $\beta = \beta(c, \rho_1)$ for a fixed $c$ such that the conditions for $b$ and $\beta$ are fulfilled. Similarly we find $\rho_2 = \rho_2(c)$, $a = a(c, \rho_2)$, $\alpha = \alpha(c, \rho_2)$ and show, that there exists $c^*$, such that

$$\rho_1(c^*) = \rho_2(c^*).$$

# Existence of an optimal solution




We determine $\rho_1 = \rho_1(c), b = b(c, \rho_1)$ and $\beta = \beta(c, \rho_1)$ for a fixed $c$ such that the conditions for $b$ and $\beta$ are fulfilled. Similarly we find $\rho_2 = \rho_2(c), a = a(c, \rho_2), \alpha = \alpha(c, \rho_2)$ and show, that there exists $c^*$, such that

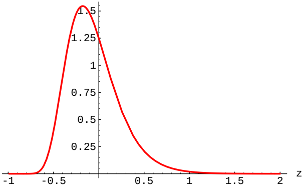
$$\rho_1(c^*) = \rho_2(c^*).$$

## Theorem

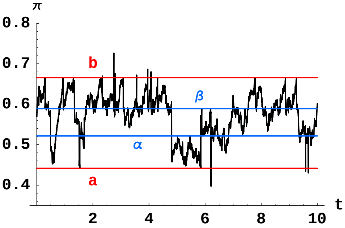
*Kochendörfer/S.: Under weak conditions on the costs and on the model parameters, unique parameters $\rho, a, \alpha, b, \beta$ exist such that the CB-strategy defined by $(a, \alpha, b, \beta)$ is optimal with growth rate $\lambda$.*

# Example

To obtain the following numerical results we use realistic transaction costs $\gamma = 0.003, \delta = 0.0001$, parameters of the stock price $\sigma = 0.4, \mu = 0.19$ and $\log(1+Z) \sim \mathcal{N}(\log(0.9), 0.09)$.



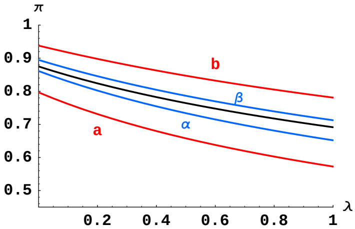

Shifted lognormal distribution




Optimally controlled risky fraction $(\pi_t)_{t\geq 0}$ for $\lambda = 2$.

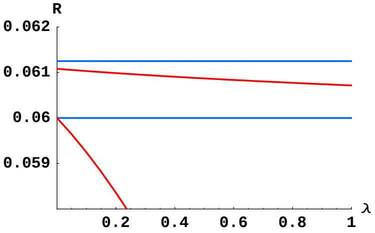
# Dependency of the boundaries on the intensity

For the following graphs we start with the model without jumps and then increase the rate $\lambda$ to show the dependency of the boundaries, the costs and the growth rate on the expected number of jumps. For each intensity $\lambda$ the drift $\mu$ is fixed such that we get without costs the same optimal growth rate as in the original model with no jumps.

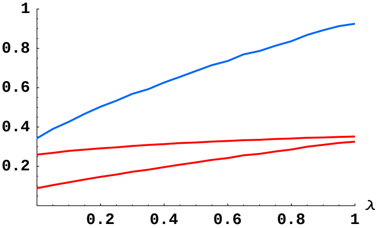



Constant boundary strategy against jump intensity

# Growth rates and costs of trading

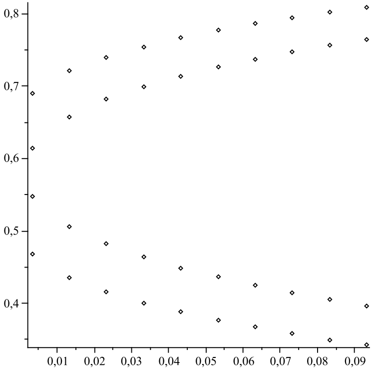



Upper curves: Optimal growth rate without costs, with costs.
Lower curves: Buy-and-hold growth rate for one stock $R_1$ without jumps, with jumps.

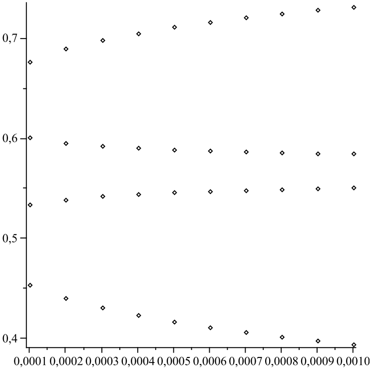



Number of trades per year,
upper red line: relative costs per trade$\times 1000$,
lower line: relative costs per year

# Dependency of the boundaries on the costs




Boundaries depending on $\gamma$




Boundaries depending on $\delta$

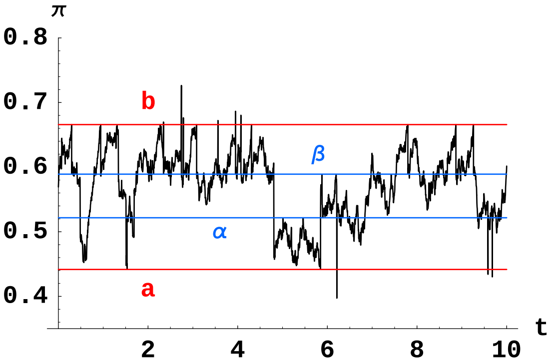
# Numerical examples without jumps

| $\mu$ | $\delta$ | $\gamma$ | $a^*$ | $\alpha^*$ | $\beta^*$ | $b^*$ | $R^*$ | $\hat{\eta}$ | $\hat{R}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.096 | 0.01 | 0.3 | 0.49 | 0.57 | 0.63 | 0.71 | 0.03 | 0.60 | 0.03 |
| 0.01 | 0.01 | 0.3 | 0.02 | 0.05 | 0.07 | 0.12 | $> 0$ | 0.06 | $> 0$ |
| 0.159 | 0.01 | 0.3 | 0.96 | 0.99 | $< 1$ | $< 1$ | 0.08 | 0.99 | 0.08 |
| 0.096 | 0.01 | 99 | 0.11 | 0.14 | $< 1$ | $< 1$ | 0.02 | 0.60 | 0.03 |
| 0.096 | 65 | 0.3 | 0.01 | 0.67 | 0.67 | $< 1$ | 0.02 | 0.60 | 0.03 |
| -0.10 | 0.01 | 0.3 | -0.90 | -0.71 | -0.51 | -0.40 | 0.03 | -0.63 | 0.03 |
| 0.20 | 0.01 | 0.3 | 1.14 | 1.20 | 1.28 | 1.39 | 0.12 | 1.25 | 0.13 |
| 26.0 | 0.01 | 0.3 | 56.4 | 60.6 | 145 | 156 | 1096 | 162 | 2113 |

Cost parameters $\delta$, $\gamma$ in %.

# Extensions and future work

- More dimensions, more jumps, more costs
- Approximation, convergence, non-constant parameters




- **Thanks for your attention**