A stochastic dynamic Dorfman-Steiner model

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Outline

I Topic: Revenue Management, Pricing

dynamic **pricing**

– there is an extensive literature on „dynamic pricing"

– structural properties for continuous/discrete time, determ./ stoch. models

– continuous time: Gallego, van Ryzin 1994 (time-homogeneous)
and: McAfee, te Velde 2008 (time-inhomogeneous)

dynamic **pricing and advertising**

– dynamic models:

carry-over-effects, deterministic (Arrow & Nerlove 1962; Sethi)
innovation- und imitation effects; diffusion processes (Bass 1961)

– MacDonald, Rasmussen 2009: Gallego, van Ryzin - model incl. advertising

– today: McAfee, te Velde – model, including advertising

Literature

„Optimal marketing-mix"

Dorfman, R., P. O. Steiner (1954). *Optimal Advertising and Optimal Quality,* American Economic Review, Vol. 44, 826-836.

„Dynamic pricing"

Gallego, G., G. van Ryzin (1994). *Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons*. Management Science, Vol. 40, 999-1020.

(*) McAfee, R. P., V. te Velde (2008). *Dynamic Pricing with Constant Demand Elasticity*. Production and Operations Management, Vol. 17, No. 4, 432-438.

„Dynamic pricing and advertising"

MacDonald, L., H. Rasmussen (2009). *Revenue Management with Dynamic Pricing and Advertising*. Production Journal of Revenue and Pricing Management, Vol. 9, 126-136.

The Dorfman-Steiner Theorem

Theorem 1 (Dorfman-Steiner)

Assumptions: – one product, one market

– marketing instruments: price a and advertising w

– rate of sales $\lambda(a,w)$ and cost function are known

Problem (1 period): (quantity • unit profit – advertising costs)

$$\max_{a,w>0} \{\lambda(a,w) \cdot (a - cost) - w\}$$

Claim:

For the optimal pricing and advertising actions

$$\frac{w^*(b)}{\lambda\left(a^*(b), w^*(b)\right) \cdot a^*(b)} \equiv \frac{advertising\ elasticity}{price\ elasticity}.$$

II Dynamic stochastic pricing and advertising models

Problem: b_0 identical items

finite time horizon, continuous time, **monopoly situation**

Dynamic: rate of sales is known; the intensity of the Poisson process decreases in price and increases in advertising expenses

Actions: price asked and amount money spent on advertising

Objective: maximize expected **profit**;

find an **optimal non-anticipating policy**; it will depend on the (present) inventory level (state) and time

Notation

retail price:	a	$\in A = \mathbb{R}^+$; the set of possible prices
advertising expenses:	w	$\in W = \mathbb{R}^+$; the set of poss. adv. levels
time:	t	$t \in [0, T]$, (actual time, not time-to-go)
rate of sales:	$\lambda_t(a, w)$	(average number of sales in t per unit of time)
state space:	$\{0, 1, 2, ..., b_0\}$	number of items b left
random inventory:	B_t	at time t
policy:	$h_t(b) = \left(a_t(b), w_t(b)\right)$,	$t \in [0, T]$

Problem formulation

We maximize the expected profit:

$$\max_h E\left[\int_0^T a_t(B_t)\cdot\lambda_t\big(a_t(B_t),w_t(B_t)\big)\ dt-\int_0^T w_t(B_t)\ dt\ \Big|B_0=b_0\right],$$

where h is a non-anticipating Markov policy.

III A special class of P&A models

rate of sales $\lambda_t(a,w) = u(t)\cdot g(w)\cdot(1-F(a))$ (Poisson arrival process)

→ special case $\lambda_t(a,w) := u(t)\cdot w^{\delta}\cdot a^{-\varepsilon}$, time-inhomogeneous, **constant** price- and advertising elasticities $\varepsilon > 1$, $0 < \delta < 1$.

prob. of sale $(1-F(a)) := a^{-\varepsilon}$ (Pareto distributed; $F(a)$ CDF)

price elasticity $$\frac{a}{1-F(a)}\cdot\frac{\partial(1-F(a))}{\partial a} = \varepsilon$$

advertising $g(w) := w^{\delta}$, $0 < \delta < 1$ (concave increasing in *w*)

advert. elasticity $$\frac{w}{g(w)}\cdot\frac{\partial g(w)}{\partial w} = \delta$$

Explicit solutions of a particular Dorfman-Steiner model

Theorem 2 For $\lambda_t(a,w) = u(t)\cdot a^{-\varepsilon}\cdot w^{\delta}$, $\gamma := \dfrac{\varepsilon-\delta}{1-\delta} > 1$ we obtain:

Value function $$V_t(b) = \alpha_t \cdot \beta_b = \frac{\varepsilon-\delta}{\varepsilon}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot U_t^{\gamma^{-1}}\cdot\beta_b$$

Optimal prices $$a_t(b) := \frac{\varepsilon}{\varepsilon-1}\cdot\underbrace{\alpha_t\cdot(\beta_b-\beta_{b-1})}_{\Delta V_t(b)} = \left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot U_t^{\gamma^{-1}}\cdot\beta_b^{\frac{1}{1-\gamma}}$$

Optimal adv. levels $$w_t(b) := \left(\frac{\varepsilon}{u(t)\cdot\delta}\cdot a_t^{\varepsilon-1}\right)^{\frac{1}{\delta-1}} = u(t)^{\frac{1}{1-\delta}}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot U_t^{\frac{1-\gamma}{\gamma}}\cdot\beta_b$$

For all t and b: $$\frac{w_t^*(b)}{\lambda_t\left(a_t^*(b),w_t^*(b)\right)\cdot a_t^*(b)} \equiv \frac{\delta}{\varepsilon}$$ (Dorfman-Steiner property)

Sketch of the proof: Step (1) Optimality conditions

The Bellman-equation for this case: (boundary conditions: $V_T(b) = V_t(0) = 0$)

$$\max_{a>0,w>0} \Big\{ \underbrace{V_t'(b) + u(t)\cdot a^{-\varepsilon}\cdot w^{\delta}\cdot\big(a - \Delta V_t(b)\big) - w}_{=:K(a,w)} \Big\} = 0 .$$

Optimality conditions (for price and advertising): ($\Delta V_t(b) := V_t(b) - V_t(b-1)$)

(price) $$\frac{\partial K(a,w)}{\partial a} \overset{!}{=} 0 \quad \Leftrightarrow \quad a_t^*(b) = \frac{\varepsilon}{\varepsilon - 1}\cdot \Delta V_t(b)$$

(advertising) $$\frac{\partial K(a,w)}{\partial w} \overset{!}{=} 0 \quad \Leftrightarrow \quad w_t^*(b) = \left(\frac{\varepsilon}{u(t)\cdot\delta}\cdot a_t^*(b)^{\varepsilon-1} \right)^{\frac{1}{\delta-1}}$$

The Hessian matrix of K is negative definite, and the point $\big(a_t^*(b), w_t^*(b)\big)$ is a (local) maximum; for this class it is actually a global maximum.

Step (2): Solving the Bellman-Equation

The formulas of the optimal price and advertising policies yield the following nonlinear difference (**in *b***) – differential (**in *t***)-equation; $c = \left(\frac{\varepsilon}{\delta}\right)^{\frac{\delta}{\delta-1}} \cdot \left(\frac{\varepsilon-\delta}{\varepsilon}\right)^{\gamma}$ and $\gamma := \frac{\varepsilon-\delta}{1-\delta} > 1$:

$$0 = V_t'(b) + u(t)^{\frac{1}{1-\delta}} \cdot \frac{c}{\gamma} \cdot \left(\frac{\gamma}{\gamma-1}\right)^{1-\gamma} \cdot \Delta V_t(b)^{1-\gamma}, \qquad (*)$$

Idea: **Separate the effect of time and inventory, i.e. assume** $V_t(b) \stackrel{!}{=} \alpha_t \cdot \beta_b$ with a (time-) function α_t and an (inventory-) sequence β_b.

We thus obtain the following **conditions** for an unique solution of (*) (cf. McAfee, te Velde (2008) for the case without advertising).

Step (3): Separation of time and inventory

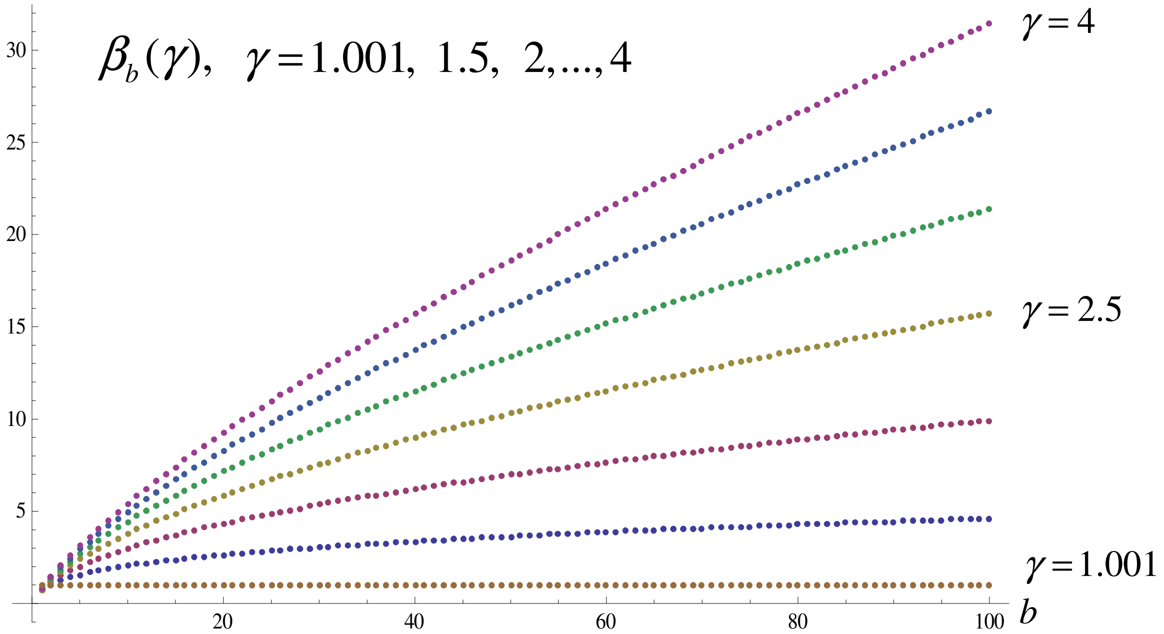
(1) effect of time: The time-dependence is given by the function

$$\alpha_t = (c \cdot U_t)^{\frac{1}{\gamma}}, \quad \text{where} \quad U_t := \int_t^T u(s)^{\frac{1}{1-\delta}} ds \overset{time\ hom.}{=} u^{\frac{1}{1-\delta}} \cdot (T-t)$$

(2) effect of inventory: The sequence $(\beta_b)_{b\geq 0}$ satisfies the (implicit) recursion

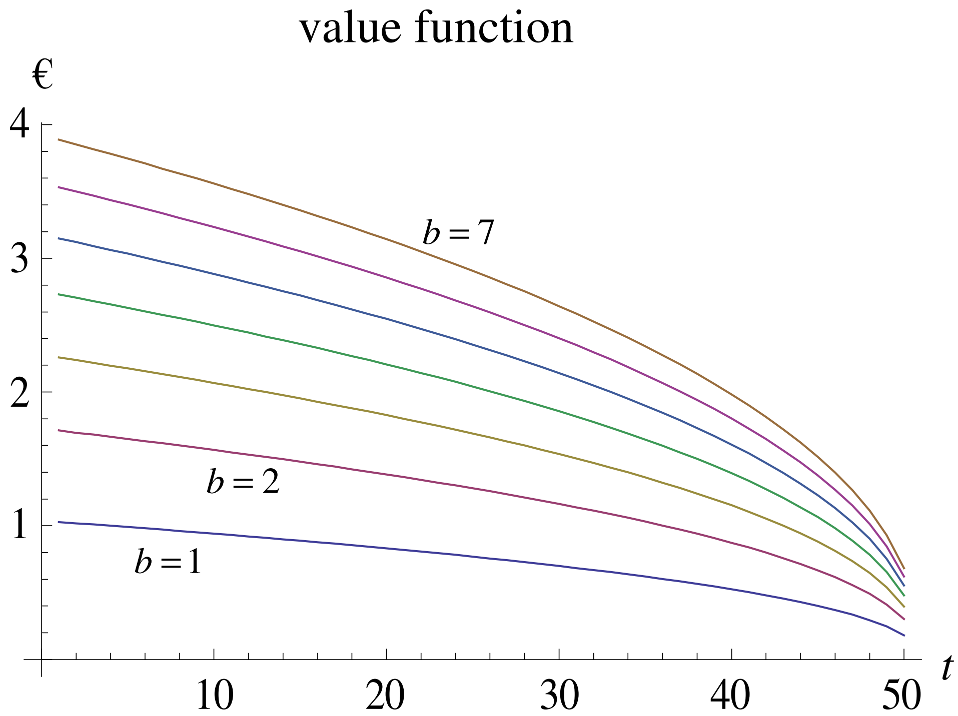
$$\beta_0 = 0 \quad \text{and} \quad \beta_b \cdot (\beta_b - \beta_{b-1})^{\gamma-1} = \left(\frac{\gamma}{\gamma-1}\right)^{1-\gamma}, \ b = 1, 2, 3, \ldots$$

Illustrations: (i) The (inventory-) sequence $(\beta_b)_{b\geq 0}$



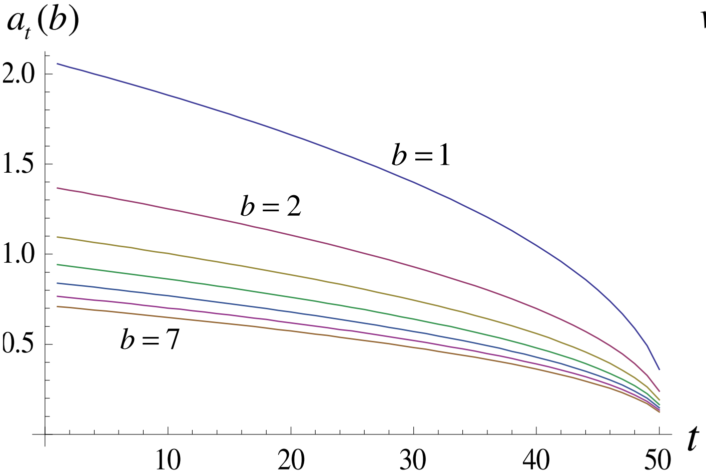
The sequence is nonnegative, concave and increasing in b for all $\gamma > 1$.

Illustrations: (ii) Solution of the Bellman-equation

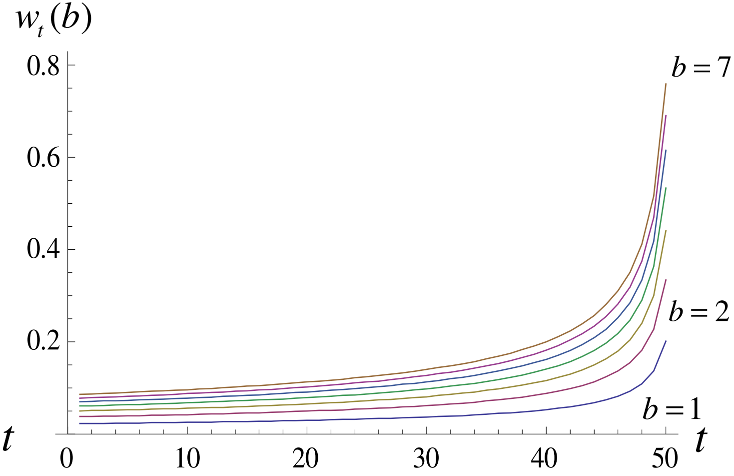


Illustrations: (iii) The optimal policy

optimal price policy



optimal advertising policy



Observations: the (optimal) price is monotone decreasing price in time;
the (optimal) advertising policy is increasing in time.

Structural properties and sensitivity analysis

Theorem 3 (Structural properties **for the solved model**)

(A1) The **value function** is a concave *decreasing* function **in *t***, and it is a concave and *increasing* function **in *b***.

(A2) The optimal **price** *decreases* **in *b***, **advertising** rate *increases* **in *b***.

(S1) The optimal **pricing policy** *decreases* **in *t***.

(S2) If $u(t)$ is decreasing in t, the optimal **advertising** *increases* **in *t***.

Additional results

Based on the explicit solution formulas we can derive several other results:

(for example)

- formulas for the expected evolution of the inventory-level
- state probabilities
- the distribution of the (time) points of sale
- formulas for the expected sales prices
- ideas for efficiently simulating these and other quantities
- asymptotic properties

IV Extensions and generalizations

(1) A generalized 2-dimensional model

(2) A multi-dimensional model

(3) The discounted price and advertising model

(4) A particular deterministic inventory problem on $\mathbb{R}^+$

IV.1 A generalized 2-dimensional model

the intensity function: $\lambda_t(a,w) = u(t)\cdot a^{-\varepsilon}\cdot w^{\delta}$, where $u(t) > 0$ and constant price resp. advertise elasticities $\varepsilon > 1,\ 0 < \delta < 1$.

gen. payment function: $\bar{R}_t(a,w) = \lambda_t(a,w)\cdot \underline{v\cdot a} - \underline{k\cdot w},\qquad v,k>0$

Solution formulas:

$$V_t(b) = \underline{k^{\frac{\delta}{\delta-\varepsilon}}\cdot v^{\frac{\varepsilon}{\varepsilon-\delta}}}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot\left(\frac{\varepsilon-\delta}{\varepsilon}\right)\cdot U_t^{\gamma^{-1}}\cdot\beta_b \qquad =: \left(\frac{v^{\varepsilon}}{k^{\delta}}\right)^{\frac{1}{\varepsilon-\delta}}\cdot V_t^{(1,1)}(b)$$

$$a_t(b) = \underline{k^{\frac{\delta}{\delta-\varepsilon}}\cdot v^{\frac{\delta}{\varepsilon-\delta}}}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot U_t^{\gamma^{-1}}\cdot\beta_b^{\frac{1}{1-\gamma}} \qquad =: \left(\frac{v^{\delta}}{k^{\delta}}\right)^{\frac{1}{\varepsilon-\delta}}\cdot a_t^{(1,1)}(b)$$

$$w_t(b) = \underline{k^{\frac{\varepsilon}{\delta-\varepsilon}}\cdot v^{\frac{\varepsilon}{\varepsilon-\delta}}}\cdot u(t)^{\frac{1}{1-\delta}}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot U_t^{\frac{1-\gamma}{\gamma}}\cdot\beta_b \qquad =: \left(\frac{v^{\varepsilon}}{k^{\varepsilon}}\right)^{\frac{1}{\varepsilon-\delta}}\cdot w_t^{(1,1)}(b)$$

IV.2 A multi-dimensional model

sales rate: $$\lambda_t(a,\vec{w}) = \lambda_t(a, w_1, \ldots, w_J) = u(t) \cdot a^{-\varepsilon} \cdot \prod_j w_j^{\delta_j}, \qquad \delta_j \geq 0$$

payment function: $$\bar{R}_t(a,\vec{w}) := \bar{R}_t(a, w_1, \ldots, w_J) := \lambda_t(a,\vec{w}) \cdot v \cdot a - \sum_{j \in J} k_j \cdot w_j, \qquad k_j \geq 0$$

Bellman-Equ.: $$\max_{a>0, \vec{w}>0} \left\{ V_t'(b) + u(t) \cdot a^{-\varepsilon} \cdot \prod_j w_j^{\delta_j} \cdot \left(v \cdot a - \Delta V_t(b) \right) - \sum_j k_j \cdot w_j \right\} = 0$$

FOC **(i)** $$\frac{v \cdot a}{\varepsilon} = v \cdot a - \Delta V_t(b) \qquad \text{(unchanged)}$$

(ii) $$\frac{\delta_1}{k_1} \cdot w_1^{-1} \cdot \prod_j w_j^{\delta_j} = \frac{u(t)^{-1} \cdot a^{\varepsilon}}{v \cdot a - \Delta V_t(b)} \quad \Rightarrow \quad \boxed{\frac{w_j^*(b)}{w_k^*(b)} = \frac{\delta_j \cdot k_k}{\delta_k \cdot k_j} \quad \forall j,k \in J}$$

Hence $$\lambda_t(a,\vec{w}) = \underbrace{u(t)\cdot\prod_j\left(\frac{\delta_j\cdot k_1}{\delta_1\cdot k_j}\right)^{\delta_j}}_{=:\ \tilde{u}(t)}\cdot a^{-\varepsilon}\cdot w_1^{\overbrace{\sum_j \delta_j}^{=:\ \tilde{\delta}}} = \tilde{u}(t)\cdot a^{-\varepsilon}\cdot w_1^{\tilde{\delta}} =: \tilde{\lambda}_t(a,w_1).$$

$$\bar{R}(a,\vec{w}) = \tilde{\lambda}_t(a,w_1)\cdot v\cdot a - \underbrace{\sum_{j\in J} k_j\cdot\frac{\delta_j\cdot k_1}{\delta_1\cdot k_j}}_{=:\ \tilde{k}}\cdot w_1.$$ If $0<\tilde{\delta}=\sum_j \delta_j<1<\varepsilon$, we can reduce

the multi-dimensional model to the generalized **2-dimensional** model.

1+1-dim.	$\tilde{\delta}$	$\tilde{\varepsilon}$	$\tilde{k}$	$\tilde{v}$	$\tilde{u}(t)$
|J|+1-dim. model	$\sum_j \delta_j$	ε	$\frac{k_1}{\delta_1}\cdot\sum_{j\in J}\delta_j = \frac{k_1}{\delta_1}\cdot\tilde{\delta}$	v	$u(t)\cdot\left(\frac{k_1}{\delta_1}\right)^{\tilde{\delta}}\cdot\prod_j\left(\frac{\delta_j}{k_j}\right)^{\delta_j}$

IV.3 The discounted price and advertising model

problem:

$$\max_{a,w} E\left[\int_0^T \underline{e^{-r\cdot t}} \cdot a_t \cdot \lambda_t(a,w)\ dt - \int_0^T \underline{e^{-r\cdot t}} \cdot w_t\ dt \ \middle|B_0 = b_0\right]$$

Diff.-DE

$$0 = V_t'(b) + u(t)^{\frac{1}{1-\delta}} \cdot \frac{c}{\gamma} \cdot \left(\frac{\gamma}{\gamma-1}\right)^{1-\gamma} \cdot \Delta V_t(b)^{1-\gamma} - \underline{r \cdot V_t(b)}$$

time effect: For $r > 0$ we obtain

$$\tilde{U}_t := -e^{-\gamma\cdot r\cdot t} \cdot \int_t^T e^{\gamma\cdot r\cdot s} \cdot u(s)^{\frac{1}{1-\delta}}\ ds \overset{time\ hom.}{=} \frac{u^{\frac{1}{1-\delta}} \cdot c}{\gamma \cdot r} \cdot \left(1 - e^{-\gamma\cdot r\cdot(T-t)}\right) \begin{cases} \xrightarrow{T\to\infty} \dfrac{u^{\frac{1}{1-\delta}} \cdot c}{\gamma \cdot r} \cdot \left(1 - e^{\gamma\cdot r\cdot t}\right) \\ \xrightarrow{r\to 0} u^{\frac{1}{1-\delta}} \cdot c \cdot (T-t) \end{cases}$$

$$\alpha_t = \left(-c \cdot e^{-\gamma\cdot r\cdot t} \cdot \int_t^T e^{\gamma\cdot r\cdot s} \cdot u(s)\ ds\right)^{\frac{1}{\gamma}} = (c \cdot \tilde{U}_t)^{\gamma^{-1}} \xrightarrow{r\to 0} (c \cdot U_t)^{\gamma^{-1}}$$

IV.4 The Dorfman-Steiner model with a continuous state space as a deterministic inventory problem

Let $b(t) \in \mathbb{R},\ b(0) = b_0$ and $b'(t) = -\lambda_t(a,w) = -u(t) \cdot a^{-\varepsilon} \cdot w^{\delta},$ if $b(t) > 0$

det. inventory problem: $$\max_{a,w} \left[\int_0^T a_t(b(t)) \cdot \lambda_t\big(a_t(b(t)), w_t(b(t))\big) - w_t(b(t))\ dt \right]$$

Bellman-Equ. (PDE) $$0 = V_t'(b) + u(t)^{\frac{1}{1-\delta}} \cdot \frac{c}{\gamma} \cdot \left(\frac{\gamma}{\gamma - 1}\right)^{1-\gamma} \cdot \frac{\partial V_t(b)^{1-\gamma}}{\partial b} \ \big(-r \cdot V_t'(b)\big)$$

(boundary conditions: $V_T(b) = V_t(0) = 0$)

Again (as before): $V_t(b) = \alpha(t) \cdot \beta(b)$

time effect: $\alpha_t = (c \cdot U_t)^{\frac{1}{\gamma}}$, where $U_t := \int_t^T u(s)^{\frac{1}{1-\delta}} ds \overset{time\ hom.}{=} u^{\frac{1}{1-\delta}} \cdot (T - t)$

Inventory effect (fct.): $\beta(0)=0$ and $\beta(b)^{\frac{1}{\gamma-1}}\cdot\beta'(b)=\frac{\gamma-1}{\gamma},\ b\in\mathbb{R}^+$

Solution (Bernoulli-DE): $\beta(b)=b^{\frac{\gamma-1}{\gamma}} \Leftrightarrow b=\beta(b)^{\frac{\gamma}{\gamma-1}}$

$\Rightarrow$ value function: $V_t(b)=k^{\frac{\delta}{\delta-\varepsilon}}\cdot v^{\frac{\varepsilon}{\varepsilon-\delta}}\cdot\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\cdot\left(\frac{\varepsilon-\delta}{\varepsilon}\right)\cdot U_t^{\gamma^{-1}}\cdot\beta(b)$

opt. rate of sales $b'(t)=-\lambda_t\left(a_t^*(b(t)),w_t^*(b(t))\right)=-\frac{u(t)^{\frac{1}{1-\delta}}}{U_t}\cdot\beta(b(t))^{\frac{\gamma}{\gamma-1}}\overset{time\ hom.}{=}-\frac{b(t)}{(T-t)}$

opt. inventory evolution $\Rightarrow$ $b(t)=b(0)\cdot\frac{U_t}{U_0}\overset{time\ hom.}{=}b(0)\cdot\frac{T-t}{T}=b_0-\frac{b_0}{T}\cdot t$

and the inventory decreases at a **constant rate** from b_0 at $t=0$ to 0 at $t=T$;

cf. an almost linear decreasing expected inventory level in the discrete case.

Thank you for your attention!