Stopping problems with discontinuous cost functionals

Lukasz Stettner
stettner@impan.gov.pl

Workshop on Stochastic Models and Control, Black Forest, March 29th - April 1st

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with values in a locally compact separable metric space E;

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with values in a locally compact separable metric space E; transition operator P_t such that $P_t C_0 \subset C_0$

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$
with values in a locally compact separable metric space E;
transition operator P_t such that $P_t C_0 \subset C_0$
where C_0 is the space of continuous functions on E vanishing at infinity

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with values in a locally compact separable metric space E; transition operator P_t such that $P_t C_0 \subset C_0$ where C_0 is the space of continuous functions on E vanishing at infinity

stopping problems with cost functionals: a. with time discontinuity

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$
with values in a locally compact separable metric space E;
transition operator P_t such that $P_t C_0 \subset C_0$
where C_0 is the space of continuous functions on E vanishing at infinity
stopping problems with cost functionals: a. with time discontinuity

$$J^1(s,x,\tau) = E_{sx}\left\{\int_0^\tau e^{-\alpha u} f(s+u,x(u))du \right.$$
$$+1_{\tau<T-s}e^{-\alpha\tau}G(s+\tau,x(\tau)) +$$
$$\left. 1_{\tau=T-s}e^{-\alpha(T-s)}H(T,x(T-s))\right\}$$

$(x(s))$ is a right continuous Markov on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$
with values in a locally compact separable metric space E;
transition operator P_t such that $P_t C_0 \subset C_0$
where C_0 is the space of continuous functions on E vanishing at infinity
stopping problems with cost functionals: a. with time discontinuity

$$J^1(s,x,\tau) = E_{sx}\Big\{\int_0^\tau e^{-\alpha u} f(s+u,x(u))du$$
$$+1_{\tau<T-s} e^{-\alpha\tau} G(s+\tau,x(\tau)) +$$
$$1_{\tau=T-s} e^{-\alpha(T-s)} H(T,x(T-s))\Big\}$$

$$w^1(s,x) = \sup_{\tau \leqslant T-s} J^1(s,x,\tau)$$

b. with exit from an open set discontinuity

b. with exit from an open set discontinuity

$$\tau_{\mathcal{O}} = \{t : x(t) \notin \mathcal{O}\}$$

b. with exit from an open set discontinuity

$\tau_{\mathcal{O}} = \{t : x(t) \notin \mathcal{O}\}$

$$J^2(s,x,\tau) = E_{sx}\Big\{ \int_0^{\tau\wedge\tau_{\mathcal{O}}} e^{-\alpha u} f(s+u, x(u))\, du$$
$$+ 1_{\tau<\tau_{\mathcal{O}}} e^{-\alpha\tau} G(s+\tau, x(\tau))$$
$$+ 1_{\tau\geqslant\tau_{\mathcal{O}}} e^{-\alpha\tau_{\mathcal{O}}} H(s+\tau_{\mathcal{O}}, x(\tau_{\mathcal{O}}))\Big\},$$

b. with exit from an open set discontinuity

$\tau_{\mathcal{O}} = \{t : x(t) \notin \mathcal{O}\}$

$$J^2(s,x,\tau) = E_{sx}\Big\{ \int_0^{\tau \wedge \tau_{\mathcal{O}}} e^{-\alpha u} f(s+u, x(u))\, du$$
$$+1_{\tau < \tau_{\mathcal{O}}} e^{-\alpha \tau} G(s+\tau, x(\tau))$$
$$+1_{\tau \geqslant \tau_{\mathcal{O}}} e^{-\alpha \tau_{\mathcal{O}}} H(s+\tau_{\mathcal{O}}, x(\tau_{\mathcal{O}}))\Big\},$$

$$w^2(s,x) = \sup_{\tau} J^2(s,x,\tau)$$

c. with other discontinuities

$$J^3(s,x,\tau) = E_{sx}\Big\{ \int_0^\tau e^{-\alpha u} f\big(s+u, x(u)\big)\, du + e^{-\alpha\tau} F\big(s+\tau, x(\tau)\big)\Big\}.$$

c. with other discontinuities

$$J^3(s,x,\tau) = E_{sx}\Big\{ \int_0^\tau e^{-\alpha u} f(s+u, x(u))\,du + e^{-\alpha\tau} F(s+\tau, x(\tau))\Big\}.$$

$F(s,x) = G(s,x)$ for $x \in \mathcal{O}$ and $F(s,x) = H(s,x)$ for $x \in \mathcal{O}^c$,

c. with other discontinuities

$$J^3(s,x,\tau) = E_{sx}\Big\{ \int_0^\tau e^{-\alpha u} f(s+u,x(u))du + e^{-\alpha\tau} F(s+\tau, x(\tau))\Big\}.$$

$F(s,x) = G(s,x)$ for $x \in \mathcal{O}$ and $F(s,x) = H(s,x)$ for $x \in \mathcal{O}^c$,

G, H and also f are bounded continuous functions

c. with other discontinuities

$$J^3(s,x,\tau) = E_{sx}\Big\{ \int_0^\tau e^{-\alpha u} f(s+u, x(u))\, du + e^{-\alpha\tau} F(s+\tau, x(\tau))\Big\}.$$

$F(s,x) = G(s,x)$ for $x \in \mathcal{O}$ and $F(s,x) = H(s,x)$ for $x \in \mathcal{O}^c$,

G, H and also f are bounded continuous functions

$$w^3(s,x) = \sup_\tau J^3(s,x,\tau).$$

$$J^r(s,T,x,b,\tau) = E_x\left\{1_{\tau<T-s}G(s+\tau,x(\tau),b)+1_{\tau\geqslant T-s}H(T,x(T-s),b)\right\}$$
$$w^r(s,T,x,b) = \sup_\tau J^r(s,T,x,b,\tau)$$

Theorem

w^r is continuous with possible discontinuity at $s=T$,

$$\tau_s^\varepsilon = \inf\left\{t\geqslant 0 : w^r(t+s,T,x(t),b)\leqslant F(t+s,x(t),b)+\varepsilon\right\},$$

where $F(u,x,b)=\begin{cases} G(u,x,b), & u<T,\\ H(T,x,b), & u=T,\end{cases}$

is ε-optimal for $\varepsilon>0$;
whenever $G\leqslant H$ for $s=T$, then w^r is continuous and τ^0 is an optimal stopping time.

time discretization: V. Mackevicius 1973, J. Palczewski, L. Stettner (SICON 2010)

$$w^l(s,T,x,b) = \sup_{\tau \leqslant T-s} E_x\Big\{1_{\tau\leqslant t_1(b)-s} G(s+\tau, x(\tau), b)$$
$$+ 1_{\tau > t_1(b)-s} H(t_1(b)\vee s, T, x((t_1(b)-s)\vee 0), b)\Big\},$$

where $t_1 : B \mapsto R$ is continuous

Theorem

If $G(t_1(b), x, b) \geqslant H(t_1(b), t_1(b), x, b)$ then w^l is continuous bounded and $\tau_s = 0$ for $s > t_1(b)$ and $\tau_s = \inf\{t \in [0, t_1(b)-s] : w^l(s+t, T, x(t), b) \leqslant G(s+t, x(t), b)\} \wedge (T-s)$, for $s \leqslant t_1(b)$ with convention $\inf\emptyset = \infty$ is an optimal stopping time.

Impulse control with decision lag and execution delay

Impulse strategy $\Pi = (\tau_i, \xi_i)$, where (τ_i) are stopping times with respect to the history (F_t) and variables ξ_i are F_{τ_i}-measurable. At the moment $\tau_i + \Delta$ the process $x(t)$ is shifted to the state given by $\Gamma(x_-(\tau_i + \Delta), \xi_i)$, where $x_-(\tau_i + \Delta)$ represents the state of the process strictly before the exercise of the impulse;

τ_i, called the *ordering time*, a decision is made upon the action ξ_i. It is then *executed* at time $\tau_i + \Delta$; Δ is execution delay

$\tau_{i+1} \geqslant \tau_i + h$;

Value $h \geqslant 0$ has the meaning of a decision lag - the minimal time gap separating ordering times.

Main results

$$J(x,\Pi,T) = E_x\Big\{ \int_0^T e^{-\alpha s} f(t, X^\Pi(t))dt + e^{-\alpha T} g(X^\Pi(T)) + \sum_{i=1}^{\infty} 1_{\tau_i+\Delta\leqslant T} e^{-\alpha(\tau_i+\Delta)} c(X^\Pi_{i-1}(\tau_i+\Delta),\xi_i) \Big\},$$

$v(x) = \sup_\Pi J(x,\Pi,T)$

Theorem

v is continuous and bounded and there exists an optimal strategy.

E. Bayraktar and M. Egami (SPTA 2007), B. Bruder and H. Pham (SPTA 2009), B. Oksendal and A. Sulem (AMO 2008) J. Palczewski and L. Stettner (SICON 2010) (time discretization)

For a fixed $\beta > 0$ find solutions $w^{\star,\beta}$ and then let $\beta \to \infty$

For a fixed $\beta > 0$ find solutions $w^{\star,\beta}$ and then let $\beta \to \infty$

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-\alpha u}\left(f(s+u,x(u)) + \beta(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^+\right) du + e^{-\alpha(T-s)}H(T,x(T-s))\right\},$$

For a fixed $\beta > 0$ find solutions $w^{\star,\beta}$ and then let $\beta \to \infty$

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-\alpha u}\left(f(s+u,x(u))+\right.\right.$$
$$\left.\beta(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^+\right) du +$$
$$\left. e^{-\alpha(T-s)}H(T,x(T-s))\right\},$$

$$w^{2,\beta}(s,x) = E_{sx}\left\{\int_0^{\tau_{\mathcal{O}}} e^{-\alpha u}\left[f(s+u,x(u)) +\right.\right.$$
$$\left.\beta\Big(G(s+u,x(u)) - w^{2,\beta}(s+u,x(u))\Big)^+\right]du +$$

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big] du \Big\}.$$

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big]du\Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ whenerever $\beta \to \infty$?

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F-w^{3,\beta})^+(s+u,x(u))\Big]du\Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ whenerever $\beta \to \infty$?

is $w^\star$ continuous?

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big]du\Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ whenerever $\beta \to \infty$?

is $w^\star$ continuous?

do we have an optimal stopping time τ^*?

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big]du\Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ wheneerever $\beta \to \infty$?

is $w^\star$ continuous?

do we have an optimal stopping time τ^*?

can we approximate an optimal stopping time

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big]du\Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ whenerever $\beta \to \infty$?

is $w^\star$ continuous?

do we have an optimal stopping time τ^*?

can we approximate an optimal stopping time

using penalty functions $\tau^\beta \uparrow \tau^*$

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^\infty e^{-\alpha u}\Big[f(s+u,x(u)) + \beta(F - w^{3,\beta})^+(s+u,x(u))\Big] du \Big\}.$$

when $w^{\star,\beta}$ converges to $w^\star$ whenerever $\beta \to \infty$?

is $w^\star$ continuous?

do we have an optimal stopping time τ^*?

can we approximate an optimal stopping time

using penalty functions $\tau^\beta \uparrow \tau^*$

References: penalty method: M. Robin 1976, L. Stettner, Zabczyk 1981, L. Stettner 2009, L. Stettner, J. Palczewski 2010

Lemma: the following equations are equivalent

$$z(s,x)=E_{sx}\left\{\int_0^{T-s}e^{\int_0^u d(s+t,x(t))dt}g(s+u,x(u))du+ e^{\int_0^{T-s}d(s+t,x(t))dt}h(T,x(T-s))\right\},$$

$$z(s,x)=E_{sx}\left\{\int_0^{T-s}e^{\int_0^u (d(s+t,x(t))-b(s+t))dt}\left[g(s+u,x(u))+ b(s+u)z(s+u,x(u))\right]du+ e^{\int_0^{T-s}(d(s+t,x(t))-b(s+t))dt}h(T,x(T-s))\right\}$$

equivalent form for $w^{1,\beta}$

equivalent form for $w^{1,\beta}$

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-(\alpha+\beta)u}\left(f(s+u,x(u))+\right.\right.$$
$$\left.\beta\left[(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^{+} + w^{1,\beta}(s+u,x(u))\right]\right) du+$$
$$\left. e^{-(\alpha+\beta)(T-s)} H(T,x(T-s))\right\}$$

equivalent form for $w^{1,\beta}$

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-(\alpha+\beta)u}\left(f(s+u,x(u))+ \beta\left[(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^+ + w^{1,\beta}(s+u,x(u))\right]\right) du + e^{-(\alpha+\beta)(T-s)}H(T,x(T-s))\right\}$$

for w^2 and w^3 we assume: (A1) The stopped semigroup

$$P_t^{\tau_{\mathcal{O}}} h(x) = E_x\{1_{t<\tau_{\mathcal{O}}} h(x(t))\}$$

maps the space of continuous bounded functions h into itself

(A2) $\lim_{\eta\to 0} P_x\{\tau_{\mathcal{O}} < \eta\} = 0$ uniformly in x from compact subsets of $\mathcal{O}$.

(A3) $(x(t))$ is strongly Feller, i.e., the mapping $x \mapsto E_x\{h(x(t))\}$ is continuous for any measurable bounded function h and $t > 0$.

(A4) $\lim_{x\to\partial\mathcal{O},\ x\in\mathcal{O}} h_\eta(x) = 0$. where $h_\eta(x) = P_x\{\tau_{\mathcal{O}} > \eta\}$

(A2') for any $\epsilon > 0$

$$\lim_{t\to 0} P_x\{\sup_{s\in[0,t]} \rho(x, X(s)) \geqslant \epsilon\} = 0$$

uniformly in x from compact sets.

Proposition

(A2)-(A4) imply (A1), furthermore (A1) implies (A4) and (A2') implies (A2)

Lemma: If M_β is the set of progressively measurable processes b with values from $[0,\beta]$ then

$$w^{2,\beta}(s,x) = \sup_{b\in M_\beta} E_x\Big\{ \int_0^{\tau_{\mathcal{O}}} e^{-\alpha u - \int_0^u b(t)dt}\Big[f(s+u,X(u)) + b(u)G(s+u,X(u))\Big]du + e^{-\alpha\tau_{\mathcal{O}} - \int_0^{\tau_{\mathcal{O}}} b(t)dt} H(\tau_{\mathcal{O}}, X(\tau_{\mathcal{O}}))\Big\},$$

Proposition

The functions $w^{1,\beta}(s,x)$ increase pointwise to $w^1(s,x)$ and under (A1) also the functions $w^{2,\beta}(s,x)$ increase pointwise to $w^2(s,x)$, as $\beta\to\infty$.

Lemma 1

If G has a decomposition
$G(s,x) =$
$E_x\Big\{ \int_0^{\tau_{\mathcal{O}}} e^{-\alpha u} g_1(s+u, X(u))du + e^{-\alpha\tau_{\mathcal{O}}} g_2(s+\tau_{\mathcal{O}}, X(\tau_{\mathcal{O}}))\Big\}$
for $g_1, g_2 \in C_0([0,\infty)\times E)$
then

$$w^{2,\beta}(s,x) - G(s,x) \geqslant -\frac{\|f-g_1\|}{\alpha+\beta} - E_x\{e^{-(\alpha+\beta)\tau_{\mathcal{O}}}\|H-g_2\|\}.$$

If, moreover, $G \leqslant H$ then

$$w^{2,\beta}(s,x) - G(s,x) \geqslant -\frac{\|f-g_1\|}{\alpha+\beta}.$$

$$\gamma_T(x,R) = P_x\left\{\exists_{s\in[0,T]}\ \rho(x,x(s)) \geqslant R\right\}.$$

$$\gamma_T(x,R) = P_x\left\{\exists_{s\in[0,T]}\ \rho(x,x(s)) \geqslant R\right\}.$$

Lemma 2

For any compact set $K \subset E$

$$\sup_{x\in K} \gamma_T(x,R) \to 0$$

as $R \to \infty$.

Theorem

If $H \geqslant G$ then $w^{1,\beta}(s,x)$ converges uniformly on compact subsets to $w^1(s,x)$, which is a continuous function. Furthermore there is an optimal stopping time $\hat{\tau}^1$ for $w^1(s,x)$ of the form

$$\hat{\tau}^1(s) = \inf\left\{t \geqslant 0 : w^1(s+t, x(t)) = G(s+t, x(t))\right\} \wedge (T-s).$$

Under (A1) and $H \geqslant G$ we have that $w^{2,\beta}(s,x)$ converges uniformly on compact subsets to $w^2(s,x)$, and

$$\hat{\tau}^2(s) = \inf\left\{t \geqslant 0 : w^2(s+t, x(t)) = G(s+t, x(t)) \text{ or } x(t) \notin \mathcal{O}\right\}$$

is an optimal stopping time for $w^2(s,x)$.

Theorem

If $H \geqslant G$ then for

$$\hat{\tau}^{1,\beta}(s) = \inf\left\{t \geqslant 0 : w^{1,\beta}(s+t, x(t)) \geqslant G(s+t, x(t))\right\} \wedge (T-s).$$

we have $\hat{\tau}^{1,\beta}(s) \uparrow \hat{\tau}^{1}(s)$ *as* $\beta \to \infty$.
Under (A1) and $H \geqslant G$ *for*

$$\hat{\tau}^{2,\beta}(s) = \inf\left\{t \geqslant 0 : w^{2,\beta}(s+t, x(t)) \geqslant G(s+t, x(t)) \text{ or } x(t) \notin \mathcal{O}\right\}$$

we have $\hat{\tau}^{2,\beta}(s) \uparrow \hat{\tau}^{2}(s)$, *as* $\beta \to \infty$.

(A5) For any $x \in \partial\mathcal{O}$ we have
$\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, P_x-a.s., and
$\lim_{\varepsilon\to 0} \sigma_\varepsilon^c = 0$, P_x-a.s. where

$$\sigma_\varepsilon = \inf\{u \geqslant 0 : x(u) \in E \setminus (\mathcal{O} \cup \Gamma_\varepsilon)\},$$

$$\sigma_\varepsilon^c = \inf\{u \geqslant 0 : x(u) \in E \setminus (\mathcal{O}^c \cup \Gamma_\varepsilon)\}$$

and Γ_ε is the ε-neighbourhood of $\partial\mathcal{O}$.

(A6) $P_x\{x(T) \in \partial\mathcal{O}\} = 0$ for any $x \in E$ and $T > 0$.

Theorem

Under (A2') and (A3) the function w^3 is continuous on $[0,\infty) \times (E \setminus \partial\mathcal{O})$. Furthermore,

- *if F is l.s.c. (lower semicontinuous) then functions w^3, $w^{3,\infty}(s,x) := \lim_{\beta\to\infty} w^{3,\beta}(s,x)$ are l.s.c. and $w^{3,\infty} \geqslant w^3$*
- *if assumptions (A5) and (A6) are satisfied and $F \leqslant G \vee H$ on $\partial\mathcal{O}$, then $w^{3,\beta}$ converges to w^3 uniformly on compact subsets of $[0,\infty) \times (E \setminus \partial\mathcal{O})$.*

Lemma

Assume (A2'), (A3) and let $A \subset E$ be an open set. Then

$$\lim_{\beta \to \infty} E_x \Big\{ \int_0^T e^{-\alpha u - \beta \int_0^u 1_{x(t) \in A} dt} du \Big\} = 0$$

uniformly in x in compact subsets of A.

Discrete time optimal stopping problem

$$w(s,x) = \sup_{\tau \leqslant T-s} E_{sx} \left\{ \sum_{i=0}^{\tau-1} \gamma^i f(s+i, x(i)) + \chi_{\tau<T-s}\gamma^\tau G(s+\tau, x(\tau)) + \chi_{\tau=T-s}\gamma^{T-s} H(T, x(T-s)) \right\}$$

Discrete time penalty equation; for $b \in (0,1)$ find w^b s.t.

$$w^b(s,x) = f(s,x) + \frac{b}{1-b}(G(s,x) - w^b(s,x))^+ + \gamma \int_E w^b(s+1,y)P(x,dy) \tag{1}$$

with $w^b(T,x) = H(T,x)$.

Theorem: For $b \in (0,1)$ and bounded measurable functions f, g, h there is a unique bounded solution w^b to (1). Moreover w^b for $s < T$ is also a solution to

$$w^b(s,x) = \max_{0 \leqslant b(s,x) \leqslant b} (1 - b(s,x)) f(s,x) + b(s,x) G(s,x) + (1 - b(s,x)) \gamma \int_E w^b(s+1,y) P(x,dy)$$

where $b(s,x)$ is takes values from the interval $[0,b]$ and maximum is attained for $b(s,x)$ with values from $\{0,b\}$. Furthermore, we also have that

$$w^b(s,x) = \sup_{\tau \leqslant T-s} E_{sx} \left\{ \sum_{i=0}^{\tau-1} \gamma^i f(s+i, x(i)) + \chi_{\tau < T-s} \gamma^\tau \left(G(s+\tau, x(\tau)) - (G(s+\tau, x(\tau)) - w^b(s+\tau, x(\tau)))^+ \right) + \chi_{\tau = T-s} \gamma^{T-s} H(T, x(T-s)) \right\}$$

Moreover w^b converges uniformly to w as $b \to 1$.

$$w_t = essup_{\tau \leqslant T-s} E\left\{\sum_{i=t}^{\tau-1} \gamma^{i-t} f_i + \chi_{\tau<T}\gamma^{\tau-t}G_\tau + \chi_{\tau=T}\gamma^{T-t}H_T | F_t\right\}$$

For $b \in (0,1)$ find a right process (w_t^b) such that

$$w_t^b = E\left\{\sum_{i=t}^{T-1} \gamma^{i-t}\left[f_i + \frac{b}{1-b}(G_i - w_i^b)^+\right] + e^{-\alpha(T-t)}H_T | F_t\right\}$$

Lemma. If $z_t = E\left\{\sum_{i=t}^{T-1} \gamma^{i-t} f_i + \gamma^{T-t} H_T | F_t\right\}$ then for any adapted process (b_j) taking values from $[0,b]$, $b<1$ we have

$$z_t = E\left\{\sum_{i=t}^{T-1} \gamma^{i-t} \prod_{j=t}^{i-1}(1-b_j)\left[(1-b_i)f_i + b_i z_i\right] + \gamma^{T-t}\prod_{j=t}^{T-1}(1-b_j)H_T | F_t\right\}$$

with inverse implication for constant processes only.

Proposition. $w_t^b \to w_t$ as $b \to 1$.

$$w_t = essup_{\tau \leqslant T-s} E\left\{\int_0^\tau e^{-\alpha(s-t)} f_s ds + \chi_{\tau<T} G_\tau + \chi_{\tau=T} e^{-\alpha(T-t)} H_T | F_t\right\}$$

$$w_t^\beta = E\left\{\int_t^T e^{-(\alpha+\beta)(s-t)} \left[f_s + \beta(G_s - w_s^\beta)^+\right] ds + e^{-(\alpha+\beta)(T-t)} H_T | F_t\right\}$$

Lemma. If $z_t = E\left\{\int_t^T e^{-\alpha(s-t)} f_s ds + e^{-\alpha(T-t)} H_T | F_t\right\}$ then for any progressively measurable process (b_s) taking values from $[0, b]$ we have (with inverse implication)

$$z_t = E\left\{\int_t^T e^{-\alpha(s-t)} e^{-\int_t^T b_u du} [(f_s + b_s z_s] + e^{-\alpha(T-t)} e^{-\int_t^T b_u du} H_T | F_t\right\}$$

Proposition. $w_t^\beta \to w_t$ as $\beta \to \infty$.