Existence of Strict Optimal Controls for Constrained, Controlled Markov Processes in Continuous Time

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Outline

- Stochastic control problems
  - constraints
  - technical assumptions
  - types of controls
- Equivalent linear programming reformulations
  (essentially from Kurtz & S (1998, 2001), Cho & S (2001))
- Sufficient conditions for strict optimal control
  (ideas from Haussmann & Lepeltier (1990), Warga (1972))
- Examples

Long-term Average Control Problem

$$\begin{cases} \inf & \overline{\lim_{t\to\infty}}\, \mathbb{E}\left[t^{-1}\int_0^t \int_U c(X(s),u)\,\Lambda_s(du)\,ds\right] \\ \text{S.t.} & f(X(t)) - f(X(0)) - \int_0^t Af(X(s),u)\,\Lambda_s(du)\,ds \\ & \qquad\qquad \text{is a martingale } \forall f \in \mathcal{D}, \\ & \overline{\lim_{t\to\infty}}\, \mathbb{E}\left[t^{-1}\int_0^t \int_U F_i(X(s),u)\,\Lambda_s(du)\,ds\right] \le \lambda_i, \quad i \in \{1,\ldots,m\} \end{cases}$$

X is the state process and Λ is a relaxed control process
E denotes the state space and U denotes the space of controls

Discounted Control Problem

$$
\begin{cases}
\inf & \mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} c(X(s),u)\,\Lambda_s(du)\,ds\right] \\
\text{S.t.} & f(X(t)) - f(X(0)) - \int_0^t Af(X(s),u)\,\Lambda_s(du)\,ds \\
& \qquad\qquad\qquad \text{is a martingale } \forall f \in \mathcal{D}, \\
& X(0) \sim \nu_0, \\
& \mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} F_i(X(s),u)\,\Lambda_s(du)\,ds\right] \le \lambda_i, \quad i \in \{1,\ldots,m\}
\end{cases}
$$

$\alpha > 0$ is a discount factor

Finite Horizon Control Problem

$$\begin{cases} \inf & \mathbb{E}\left[\int_0^T \int_U c(s, X(s), u)\,\Lambda_s(du)\,ds + g(X(T))\right] \\ \text{S.t.} & f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u)\,\Lambda_s(du)\,ds \\ & \qquad \text{is a martingale } \forall f \in \mathcal{D}, \\ & X(0) \sim \nu_0, \\ & \mathbb{E}\left[\int_0^T F_i(s, X(s), u)\,\Lambda_s(du)\,ds \; + H_i(X(T))\right] \leq \lambda_i,\ i \in \{1, \ldots, m\} \\ & \mathbb{E}\left[G_j(X(T))\right] \leq \tilde{\lambda}_j, \qquad j \in \{1, \ldots, n\} \end{cases}$$

First Exit Control Problem and Control Problem with Discretionary Stopping

$$
\begin{cases}
\inf & \mathbb{E}\left[\int_0^\tau \int_U c(s, X(s), u)\,\Lambda_s(du)\,ds + g(\tau, X(\tau))\right] \\
\text{S.t.} & f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u)\,\Lambda_s(du)\,ds \text{ is a martingale} \\
& \qquad \text{is a martingale } \forall f \in \mathcal{D}, \\
& X(0) \sim \nu_0, \\
& \mathbb{E}\left[\int_0^\tau F_i(s, X(s), u)\,\Lambda_s(du)\,ds + H_i(\tau, X(\tau))\right] \leq \lambda_i, \; i \in \{1, \ldots, m\} \\
& \mathbb{E}\left[G_j(\tau, X(\tau))\right] \leq \tilde{\lambda}_i, \qquad j \in \{1, \ldots, n\}
\end{cases}
$$

Technical Assumptions (Kurtz & S (2001))

- E and U are complete, separable metric spaces
- A a linear operator satisfying
  (i) $A : \mathcal{D} \subset \overline{C}(E) \to C(E \times U)$, $1 \in \mathcal{D}$, and $A1 = 0$.
  (ii) There exist $\psi_A \in C(E \times U)$, $\psi_A \geq 1$, and constants a_f, $f \in \mathcal{D}$, such that
  $$|Af(x,u)| \leq a_f \psi_A(x,u), \quad \forall (x,u) \in \mathcal{U}.$$
  (iii) *Defining $A_0 = \{(f, \psi_A^{-1} Af) : f \in \mathcal{D}\}$, A_0 is separable in the sense that there exists a countable collection $\{f_k : k \in \mathbb{N}\} \subset \mathcal{D}$ such that A_0 is contained in the bounded, pointwise closure of the linear span of $\{(f_k, A_0 f_k) = (f_k, \psi_A^{-1} Af_k) : k \in \mathbb{N}\}$.*
  (iv) For each $u \in U$, the operator A_u defined by $A_u f(x) = Af(x,u)$ is a pre-generator (a generator of a Markov process is also a pregenerator).
  (v) $\mathcal{D}$ is closed under multiplication and separates points.

Conditions on the Cost Rate Function

(i) c is lower-semicontinuous and bounded below;

(ii) for each $a > 0$, $\{(x,u) : c(x,u) \leq a\}$ is compact; and

(iii) there exists a nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \frac{\Phi(r)}{r} = \infty$ such that

$$\Phi(\psi_A(x,u)) \leq a + b\, c(x,u).$$

The function Φ is a Young function.

Types of Controls

There exists a filtration $\{\mathcal{F}_t\}$ with respect to which:

- Relaxed controls
  - Λ is a $\mathcal{P}(U)$-valued process such that (X, Λ) is $\{\mathcal{F}_t\}$-progressively measurable
- Strict controls
  - u is a U-valued process such that (X, u) is $\{\mathcal{F}_t\}$-progressively measurable
  - $f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u(s))\, ds$ is an $\{\mathcal{F}_t\}$-martingale for every $f \in \mathcal{D}(A)$
    - $\Lambda_s(\cdot) = \delta_{\{u(s)\}}(\cdot)$

Key Idea: Expected Occupation Measures (Time-Homogeneous)

- Long-term Average Problem

$$\mu_0(G) = \overline{\lim_{t\to\infty}}\, t^{-1}\mathbb{E}\left[\int_0^t \int_U I_G(X(s),u)\Lambda_s(du)\,ds\right], \qquad G \in \mathcal{B}(E\times U)$$

- Discounted Problem

$$\mu_0(G) = \mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} I_G(X(s),u)\Lambda_s(du)\,ds\right], \qquad G \in \mathcal{B}(E\times U)$$

Equivalent Linear Programming Formulations

Theorem: (LP Equivalence for Long-term Average Cost)

Suppose there exists a feasible pair (X, Λ) for which the long-term average cost is finite. Then the long-term average stochastic control problem is equivalent to the linear program

$$\begin{cases} \textit{Min.} & \int c(x,u)\,\mu_0(dx \times du) & & \\ \textit{S.t.} & \int Af(x,u)\,\mu_0(dx \times du) & = & 0, \quad f \in \mathcal{D}, \\ & \int F_i(x,u)\,\mu_0(dx \times du) & \leq & \lambda_i, \quad i \in \{1,\ldots,m\}, \\ & \mu_0 \in \mathcal{P}(E \times U). & & \end{cases}$$

Moreover, an optimal measure μ_0^* and an optimal stationary relaxed solution (X^*, Λ^*) exist with the property that $X^*(s)$ has distribution μ_E^* and $\Lambda_s^*(\cdot) = \eta^*(X^*(s), \cdot)$ for $s \geq 0$, where μ_E^* is the state marginal of μ_0^* and η^* is the regular conditional distribution of μ_0^* on U given x.

Theorem: (LP Equivalence for Discounted Cost)

$$
\begin{cases}
\textit{Min.} & \int c(x,u)\,\mu_0(dx \times du) & & \\
\textit{S.t.} & \int A^\alpha f(x,u)\,\mu_0(dx \times du) & = & 0, & f \in \mathcal{D}, \\
& \int F_i(x,u)\,\mu_0(dx \times du) & \leq & \alpha\lambda_i, & i \in \{1,\ldots,m\}, \\
& \mu_0 \in \mathcal{P}(E \times U). & & &
\end{cases}
$$

where $A^\alpha f(x,u) = Af(x,u) + \alpha(\int f\, d\nu_0 - f(x))$

Intuition:

- View the discounting as an exponential lifetime (modula a factor of α)
- restart the process upon death and
- evaluate the long-term average cost of this process.

Unified Formulation

Theorem: (Time-Homogeneous LPs)

$$\left\{\begin{array}{llcll} \text{Min.} & \int c(x,u)\,\mu_0(dx\times du) & & & \\ \text{S.t.} & \int \widehat{A}f(x,u)\,\mu_0(dx\times du) & = & 0, & f\in\mathcal{D}, \\ & \int F_i(x,u)\,\mu_0(dx\times du) & \leq & \widehat{\lambda}_i, & i\in\{1,\ldots,m\}, \\ & \mu_0\in\mathcal{P}(E\times U). & & & \end{array}\right.$$

where

Long-term Average	Discounted
$\widehat{A}f(x,u) = Af(x,u)$	$\widehat{A}f(x,u) = A^{\alpha}f(x,u) = Af(x,u) + \alpha(\int f\,d\nu_0 - f(x))$
$\widehat{\lambda}_i = \lambda_i$	$\widehat{\lambda}_i = \alpha\lambda_i$

Sufficiency Condition (Time-Homogeneous)

For each $x \in E$, define the sets

$$\kappa(x) = \{(z, u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(x, u), z_i \geq F_i(x, u), \quad i = 1, \ldots, m\}$$

and

$$\mathcal{K}(x) = \left\{ \left(z, (Af_k(x, u))_{k \in \mathbb{N}}\right) \in \mathbb{R}^{m+1} \times \mathbb{R}^{\mathbb{N}} : (z, u) \in \kappa(x) \right\}.$$

Condition: (Time-Homogeneous Generator)

For each x, the set $\mathcal{K}(x)$ is closed and convex.

Existence Result

Theorem: (Strict Control for L-t Avg/Disc Problem)

Assume there exists a feasible measure μ_0 for the linear program for which $\int c \, d\mu_0 < \infty$.

If $\mathcal{K}(x)$ is closed and convex for each x, then there exists an optimal strict feedback control u^*.

Idea of Proof

- The LP gives existence of an optimal measure μ_0^* and a process X^* such that $(X^*, \eta^*(X^*, \cdot))$ is a relaxed solution and the optimal cost is given by $\int c\, d\mu_0^*$.

  Here η^* is a regular conditional distribution of μ_0^* given x; μ_E^* denotes the state marginal distribution.

- Define
$$\begin{aligned} \overline{c}_0(x) &= \int c(x,u)\,\eta^*(x,du), & \\ \overline{F}_i(x) &= \int F_i(x,u)\,\eta^*(x,du), & i = 1,\ldots,m, \\ \overline{Af}_k(x) &= \int \widehat{A}f_k(x,u)\,\eta^*(x,du), & k \in \mathbb{N} \end{aligned}$$

- Then $\left(\left(\overline{c}_0(x), \overline{F}_1(x), \ldots, \overline{F}_m(x)\right), \left(\overline{Af}_k(x)\right)_{k\in\mathbb{N}} \right) \in \mathcal{K}(x)$.

Idea of Proof

- Consequently, there exist measurable functions $u^* : E \mapsto U$ and $v : E \mapsto \mathbb{R}_+^{m+1}$ such that for all $x \in E$,

$$\begin{aligned} c(x, u^*(x)) + v_0(x) &= \bar{c}_0(x) \\ F_i(x, u^*(x)) + v_i(x) &= \bar{F}_i(x), \qquad i \in \{1, \ldots, m\}, \text{ and} \\ \widehat{A}f_k(x, u^*(x)) &= \overline{Af}_k(x), \quad k \in \mathbb{N}. \end{aligned}$$

- Integrating with respect to μ_E^* yields

$$\begin{aligned} \int \widehat{A}f_k(x, u^*(x))\, \mu_E^*(dx) &= \int \int \widehat{A}f_k(x, u)\eta^*(x, du)\mu_E^*(dx) \\ &= \int \widehat{A}f_k(x, u)\, \mu_0^*(dx \times du) = 0, \quad k \in \mathbb{N}, \\ \int F_i(x, u^*(x))\, \mu_E^*(dx) &\leq \int F_i(x, u)\, \mu_0^*(dx \times du) \leq \widehat{\lambda}_i, \quad i \leq m \\ \int c(x, u^*(x))\, \mu_E^*(dx) &\leq \int c(x, u)\, \mu_0^*(dx \times du) \end{aligned}$$

Key Idea: Expected Occupation Measures (Time-Inhomogeneous)

- Finite Horizon Problem

$$\begin{aligned} \mu_0(G) &= \mathbb{E}\left[\int_0^T \int_U I_G(s, X(s), u)\Lambda_s(du)\, ds\right], G \in \mathcal{B}([0,T] \times E \times U) \\ \mu_1(\cdot) &\sim X(T) \end{aligned}$$

- Exit/Stopping Problem

$$\begin{aligned} \mu_0(G) &= \mathbb{E}\left[\int_0^\tau \int_U I_G(s, X(s), u)\Lambda_s(du)\, ds\right], \; G \in \mathcal{B}(\mathbb{R}_+ \times E \times U) \\ \mu_1(\cdot) &\sim (\tau, X(\tau)) \end{aligned}$$

Finite Horizon/Exit/Stopping Control Problems

Theorem: (LP Equivalence)

$$
\left\{
\begin{array}{lll}
\textit{Minimize} & \int c\, d\mu_0 + \int g\, d\mu_1 & \\
\textit{Subject to} & \int \tilde{A}f\, d\mu_0 + \int Bf\, d\mu_1 = 0, & f \in \mathcal{D}, \\
 & \int F_i\, d\mu_0 + \int H_i\, d\mu_1 \leq \lambda_i, & i \in \{1, \ldots, m\}, \\
 & \int G_i\, d\mu_1 = \tilde{\lambda}_i, & i \in \{1, \ldots, n\}, \\
 & \mu_0 \in \mathcal{M}_F(E_1 \times U), \mu_1 \in \mathcal{P}(E_2). &
\end{array}
\right.
$$

where $\tilde{A}f(t,x,u) = \frac{\partial f}{\partial t}(t,x) + Af(t,x,u)$

and $Bf(t,x) = \int f(0,y)\, \nu_0(dy) - f(t,x)$

Finite Horizon:	$E_1 = [0,T) \times E$	$E_2 = E$
Exit:	$E_1 \subset \mathbb{R}_+ \times E$ is open	$E_2 = G^c$
Stopping:	$E_1 = \mathbb{R}_+ \times E$	$E_2 = \mathbb{R}_+ \times E$

Time Inhomogeneous Problems

Condition: (Time-Dependent)

For each $x \in E$*, define the sets*
$\kappa(t,x) = \{(z,u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(t,x,u), z_i \geq F_i(t,x,u), i = 1,\ldots,m\}$ *and*
$\mathcal{K}(t,x) = \left\{\left(z, (Af_k(t,x,u))_{k\in\mathbb{N}}\right) \in \mathbb{R}^{m+1} \times \mathbb{R}^{\mathbb{N}} : (z,u) \in \kappa(t,x)\right\}.$
For each (t,x)*, the set* $\mathcal{K}(t,x)$ *is closed and convex.*

Theorem: (Strict Control for FH/Exit/Stop Control Problems)

Assume there exists a feasible measure μ_0 *for the linear program for which* $\int c\, d\mu_0 < \infty$*.*

If $\mathcal{K}(t,x)$ *is closed and convex for each* (t,x)*, then there exists an optimal strict feedback control* u^**.*

Example 1: Controlled Diffusions (Time-Dependent)

Let X satisfy

$$X(t) = x + \int_0^t b(s, X(s), u(s))\, ds + \int_0^t \sigma(s, X(s), u(s))\, dW(s)$$

so the generator of the controlled process is

$$Af(t, x, u) = \frac{1}{2} \sum_{i,j} a_{ij}(t, x, u) f_{x_i x_j}(x) + \sum_i b_i(t, x, u) f_{x_i}(x).$$

Condition: (Haussmann and Lepeltier (1990))

For each $x \in \mathbb{R}^d$, define the set

$$K(t, x) = \{(a(t, x, u), b(t, x, u), z) : u \in U, z \in \mathbb{R}^{m+1}, z_0 \geq c(t, x, u), \\ z_i \geq F_i(t, x, u), i = 0, 1, \ldots, m\}.$$

Assume $K(t, x)$ is closed and convex for almost all $(t, x) \in \mathbb{R}_+ \times E$.

Controlled Diffusions

Proposition:

For the controlled diffusion process X, the Haussmann and Lepeltier Condition on $K(t,x)$ implies the Time-Dependent Condition that $\mathcal{K}(t,x)$ is closed and convex for almost all (t,x).

Remarks

- The joint convexity of the diffusion coefficients implies the joint convexity of $(Af_k(t,x,u))_{k\in\mathbb{N}}$.
- Similarly, the closedness of $(a(t,x,u), b(t,x,u), z)$ implies the closedness of $(z, (Af_k(t,x,u))_{k\in\mathbb{N}})$ in the product topology.

Example 2: Controlled Markov Chains

- State space $E = \{x_\ell : \ell \in \mathcal{I}\}$, where $\mathcal{I}$ denote $\{1, \ldots, n\}$ for some $n < \infty$ or $\mathcal{I} = \mathbb{N}$
- Control space U a complete, separable, metric space
- Generator

$$Af(x_\ell, u) = \sum_{j \in \mathcal{I}} (f(x_j) - f(x_\ell))\, q_{\ell j}(u)$$

with $\mathcal{D} = B(E)$

Controlled Markov Chains (Time-Homogeneous)

Condition: (Rate Matrix)

For each $x_\ell \in E$, define the set

$$k(x_\ell) = \{(z, u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(x_\ell, u), z_i \geq F_i(x_\ell, u), i = 1, \ldots, m\}$$

and then define the set

$$K(x_\ell) = \{(z, (q_{\ell j}(u))_{j \in \mathcal{I}}) \in \mathbb{R}^{m+1} \times \mathbb{R}^{\mathcal{I}} : (z, u) \in k(x_\ell)\}.$$

Assume $K(x_\ell)$ is closed and convex for every $x_\ell \in E$.

Proposition:

Let X be a controlled Markov chain having generator A of the previous slide. Then the Rate Matrix Condition on $K(x_\ell)$ implies the Time-Homogeneous Generator Condition on $\mathcal{K}(x_\ell)$.

Example 3: Simple Markov Jump Process

- Let $\widehat{\lambda} : E \times U \to \mathbb{R}_+$ be bounded.
- Define the generator A by

$$Af(x,u) = \widehat{\lambda}(x,u) \int_E [f(y) - f(x)]\, \mu(x,u,dy), \qquad f \in \widehat{C}(E),$$

  where μ is a control-dependent transition function.
- The dependence of A on the control variable u occurs in the product $\widehat{\lambda}(x,u)\mu(x,u,\cdot)$.
- $\widehat{\lambda}(x,u)\mu(x,u,\cdot)$ represents the rate at which the process jumps from x into measurable sets using control u.

Sufficient Conditions

Condition: (Simple Jump Process)

For each $x \in E$, define the set

$$k(x) = \{(z, u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(x, u), z_i \geq F_i(x, u), i = 1, \ldots, m\}$$

and then define the set

$$K(x) = \{(z, \widehat{\lambda}(x, u)\mu(x, u, \cdot)) \in \mathbb{R}^{m+1} \times \mathcal{M}_F(E) : (z, u) \in k(x)\}.$$

Assume $K(x)$ is closed and convex for every $x \in E$.

The topology of weak convergence is used for $\mathcal{M}_F(E)$.

Sufficiency

Proposition:

For the controlled simple Markov jump process having generator given in on the previous slide, the Time-Homogeneous Generator Condition is implied by the Simple Jump Process Condition.

Key Observation. Suppose $\{(z_n, u_n) : n \in \mathbb{N}\}$ and (z_∞, u_∞) are such that $(z_n, \widehat{\lambda}(x, u_n)\mu(x, u_n, \cdot))$ converges to $(z_\infty, \widehat{\lambda}(x, u_\infty)\mu(x, u_\infty, \cdot))$, where

$$\widehat{\lambda}(x, u_n)\mu(x, u_n, \cdot) \Rightarrow \widehat{\lambda}(x, u_\infty)\mu(x, u_\infty, \cdot).$$

Then

$$\widehat{\lambda}(x, u_n) \int_E [f(y) - f(x)]\, \mu(x, u_n, dy) \to \widehat{\lambda}(x, u_\infty) \int_E [f(y) - f(x)]\, \mu(x, u_\infty, dy).$$

Example 4: Solutions of controlled Lévy SDEs

Process X (formulation from Øksendal and Sulem (2004))

$$\begin{aligned} X(t) &= X(0) + \int_0^t b(X(s),u(s))\,ds + \int_0^t \sigma(X(s),u(s))\,dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} \phi(X(s-),u(s-),y)\,\tilde{N}(ds \times dy). \end{aligned}$$

Generator

$$\begin{aligned} Af(x,u) &= b(x,u)f'(x) + \tfrac{1}{2}a(x,u)f''(x) \\ &\quad + \int_{\mathbb{R}} [f(x+\phi(x,u,y)) - f(x) - f'(x)\cdot\phi(x,u,y)]\,\nu(dy) \end{aligned}$$

for $f \in C_c^2(\mathbb{R})$.
ν is the Lévy measure associated with N.

Transition Kernel

For each $x \in E$ and $u \in U$, define a kernel Q on $\mathbb{R}$ given $\mathbb{R} \times U$ by

$$Q(x,u,G) = \int_{\mathbb{R}} I_G(\phi(x,u,y))\,\nu(dy), \qquad \forall G \in \mathcal{B}(\mathbb{R}).$$

For $f \in C_c^2(\mathbb{R})$, define the function

$$\bar{f}(x,z) = f(x+z) - f(x) - f'(x)\cdot z.$$

It then follows that

$$\int [f(x+\phi(x,u,y)) - f(x) - f'(x)\cdot\phi(x,u,y)]\nu(dy) = \int \bar{f}(x,z)\,Q(x,u,dz)$$

Sufficiency Condition

Condition: (Lévy Process)

For each $x \in E$, *define*

$$k(x) = \{(z,u) \in \mathbb{R}^{m+1} \times U : z_0 \geq c(x,u), z_i \geq F_i(x,u), i = 1, \ldots, m\}$$

and

$$K(x) = \{(z, a(x,u), b(x,u), Q(x,u,\cdot)) \in \mathbb{R}^{m+1} \times \mathbb{R}_+ \times \mathbb{R} \times \mathcal{M}_{\sigma F}(\mathbb{R}) : (z,u) \in k(x)\}.$$

Assume $K(x)$ *is closed and convex for every* $x \in E$.

Proposition:

The Lévy Process Condition implies the Generator Condition.

Concluding Remarks

- The closedness and convexity condition on $(z, (Af_k)_{k \in \mathbb{N}})$ is sufficient to prove the existence of an optimal control in the class of strict controls for the relaxed stochastic control problem.

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Concluding Remarks

- The closedness and convexity condition on $(z, (Af_k)_{k\in\mathbb{N}})$ is sufficient to prove the existence of an optimal control in the class of strict controls for the relaxed stochastic control problem.
- Simpler closedness and convexity conditions on the model coefficients imply the more general condition.
- The equivalent LP formulation provides a single unified approach to prove existence of optimal strict controls for the long-term average, infinite-horizon discounted, finite-horizon, and first exit control problems, as well as control problems with discretionary stopping.

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- The equivalent LP formulation provides a single unified approach to prove existence of optimal strict controls for the long-term average, infinite-horizon discounted, finite-horizon, and first exit control problems, as well as control problems with discretionary stopping.
- The measurable selection approach of Haussmann and Lepeltier for diffusion processes adapts to this more general model.
- A significance of this work is the broad applicability of the results.