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# Cramér–von Mises distance: Probabilistic interpretation, confidence intervals, and neighborhood-of-model validation

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We give a probabilistic interpretation of the Cramér–von Mises distance $\Delta(F, F_0) = \int (F - F_0)^2 \mathrm{d}F_0$ between continuous distribution functions $F$ and $F_0$. If $F$ is unknown, we construct an asymptotic confidence interval for $\Delta(F, F_0)$ based on a random sample from $F$. Moreover, for given $F_0$ and some value $\Delta_0 > 0$, we propose an asymptotic equivalence test of the hypothesis that $\Delta(F, F_0) \geq \Delta_0$ against the alternative $\Delta(F, F_0) < \Delta_0$. If such a 'neighborhood-of-$F_0$ validation test', carried out at a small asymptotic level, rejects the hypothesis, there is evidence that $F$ is within a distance $\Delta_0$ of $F_0$. As a neighborhood-of-exponentiality test shows, the method may be extended to the case that $H_0$ is composite.

**Keywords:** representation of Cramér–von Mises distance; confidence interval for the Cramér–von Mises distance; equivalence testing; neighborhood-of-exponentiality test; model validation

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## 1. Introduction

The Cramér–von Mises distance

$$\Delta(F, F_0) = \int_{-\infty}^{\infty} (F(x) - F_0(x))^2 \, \mathrm{d}F_0(x) \tag{1.1}$$

between continuous distribution functions is one of the distinguished measures of deviation between distributions. At first sight, $\Delta(F, F_0)$ seems to be difficult to interpret, but we will show that $\frac{2}{3} + \Delta(F, F_0)$ can be regarded a probability

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involving four independent random variables, two of which having distribution function (df) $F$, and the other two following df $F_0$.

For testing the hypothesis

$$H_0 : F = F_0$$

that an unknown continuous df $F$ equals some given continuous df $F_0$, based on an independent and identically distributed sample $X_1, X_2, \ldots, X_n$ from $F$, there is the time-honored Cramér–von Mises statistic

$$\omega_n^2 \;=\; n\,\Delta(F_n, F_0) \;=\; n \int_{-\infty}^{\infty} (F_n(x) - F_0(x))^2 \,\mathrm{d}F_0(x).$$

Here, $F_n(x) = n^{-1} \sum_{j=1}^{n} \mathbf{1}\{X_j \le x\}$ is the empirical distribution function (edf) of $X_1, \ldots, X_n$, and $\mathbf{1}\{A\}$ denotes the indicator function of an event $A$ (see, e.g. Csörgő and Faraway (1996) for a short historical account, especially with respect to the contributions of Smirnov (1936) and Smirnov (1937)).

The Cramér–von Mises test which rejects $H_0$ for large values of $\omega_n^2$ is one of the most prominent goodness-of-fit tests. The test statistic has the computationally simple form

$$\omega_n^2 = \frac{1}{12n} + \sum_{j=1}^{n} \left( F_0(X_{(j)}) - \frac{2j-1}{2n} \right)^2,$$

where $X_{(1)} \le \ldots \le X_{(n)}$ are the order statistics of $X_1, \ldots, X_n$. The $H_0$-limit distribution of $\omega_n^2$ is well-known (see, e.g. Anderson and Darling (1952)), and the finite-sample distribution of $\omega_n^2$ approaches this limit very quickly (see, e.g. Csörgő and Faraway (1996), p. 229, for extensive tables of quantiles of $\omega_n^2$).

The limit distribution of $\omega_n^2$ under fixed alternatives is also known, see, e.g., Shorack and Wellner (1986). For an elementary derivation we refer to Angus (1983). To state the result, consider any continuous df $F \ne F_0$, and rewrite (1.1) as $\Delta(F) := \Delta(F, F_0)$, thus suppressing the dependence on $F_0$.

Under the alternative $F$,

$$\sqrt{n}\left( \frac{\omega_n^2}{n} - \Delta(F) \right) \;\xrightarrow{\mathcal{D}}\; \mathrm{N}(0, \sigma^2(F)), \tag{1.2}$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution,

$$\sigma^2(F) = 4 \iint (F(x) - F_0(x))\,(F(y) - F_0(y))\,[F(x \wedge y) - F(x)F(y)]\;\mathrm{d}F_0(x)\mathrm{d}F_0(y) \tag{1.3}$$

and $x \wedge y$ is shorthand for $\min(x, y)$. Here and in what follows, an unspecified integral is over $\mathbb{R}$.

If $\sigma^2(F) > 0$ and $\widehat{\sigma}_n^2$ denotes a consistent estimator of $\sigma^2(F)$, Slutsky's Lemma yields

$$\frac{\sqrt{n}}{\widehat{\sigma}_n}\left(\frac{\omega_n^2}{n} - \Delta(F)\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1). \tag{1.4}$$

The paper is organized as follows. In Section 2 we give an interpretation of $\frac{2}{3} + \Delta(F, F_0)$ in terms of a probability involving four independent random variables, of which two follow the df $F$ and the other the df $F_0$. In Section 3, we use (1.2) to approximate the power function of the Cramér–von Mises test. Moreover, (1.4) yields asymptotic confidence intervals for $\Delta(F)$. Even more important is the fact that we may construct an asymptotic equivalence test of the hypothesis $H_{\Delta_0} : \Delta(F) \geq \Delta_0$ versus the alternative $K_{\Delta_0} : \Delta(F) < \Delta_0$. Here, $\Delta_0$ is a given value that defines a neighborhood $\mathcal{N}_0 := \{F : \Delta(F) < \Delta_0\}$ of $F_0$. If $\alpha$ is a small positive number, an asymptotic level $\alpha$-test of $H_{\Delta_0}$ versus $K_{\Delta_0}$ leads to the rejection of $H_{\Delta_0}$, there is much evidence that the unknown underlying distribution belongs to the above neighborhood $\mathcal{N}_0$ and in this sense is 'sufficiently close to $F_0$'. For more information on equivalence testing, especially bioequivalence testing, we refer to Wellek (2010) and Romano (2005). The latter paper derives asymptotically optimal equivalence tests for a real-valued functional within a regular parametric family of distributions. Moreover, Munk and Czado (1998) and Freitag and Munk (2005) study equivalence testing for equality of two distributions and for structural relationship in a semiparametric two-sample context, respectively, based on trimmed versions of the Mallows distance.

Section 4 shows that the approach also covers the more general case of a composite hypothesis $H_0$ provided that the limit distribution of the Cramér–von Mises statistic under fixed alternatives to $H_0$ is available. Section 5 considers a real-data example, and Section 6 contains some concluding remarks. For the sake of readability, lengthy proofs are deferred to Section 7.

## 2. A probabilistic representation of the Cramér–von Mises distance

It is sometimes argued that $\Delta(F, F_0)$, being a weighted $L^2$-distance between distribution functions, is difficult to interpret. The following result shows that $\frac{2}{3} + \Delta(F, F_0)$ is nothing but a probability involving four independent random variables; this probability attains its minimum value $\frac{2}{3}$ if, and only if, the random variables are identically distributed.

THEOREM 1 *Let $X_1, X_2, Z_1, Z_2$ be independent random variables, where $X_1, X_2$ have continuous df $F$ and $Z_1, Z_2$ have continuous df $F_0$. Then*

$$\Delta(F, F_0) = \mathbb{P}(X_1 \vee X_2 < Z_1) + \mathbb{P}(Z_1 \vee Z_2 < X_1) - \frac{2}{3},$$

*where $x \vee y$ is shorthand for* $\max(x, y)$.

*Proof.* Notice that Fubini's theorem yields

$$\begin{aligned}
\Delta(F, F_0) &= \int (F(x) - F_0(x))^2 \, \mathrm{d}F_0(x) \\
&= \int \mathbb{E}\left[(\mathbf{1}\{X_1 \le x\} - \mathbf{1}\{Z_1 \le x\})\,(\mathbf{1}\{X_2 \le x\} - \mathbf{1}\{Z_2 \le x\})\right] \mathrm{d}F_0(x) \\
&= \mathbb{E}\left[\int (\mathbf{1}\{X_1 \le x\} - \mathbf{1}\{Z_1 \le x\})\,(\mathbf{1}\{X_2 \le x\} - \mathbf{1}\{Z_2 \le x\}) \, \mathrm{d}F_0(x)\right] \\
&= \mathbb{E}\Big[\int (\mathbf{1}\{X_1 \vee X_2 \le x\} + \mathbf{1}\{Z_1 \vee Z_2 \le x\} \\
&\qquad\qquad - \mathbf{1}\{Z_1 \vee X_2 \le x\} - \mathbf{1}\{X_1 \vee Z_2 \le x\}) \, \mathrm{d}F_0(x)\Big].
\end{aligned}$$

Since, by the continuity of $F_0$, we have that $\int \mathbf{1}\{X_1 \vee X_2 \le x\} \, \mathrm{d}F_0(x) = 1 - F_0(X_1 \vee X_2)$ and the other integrals lead to similar expressions, it follows that

$$\Delta(F, F_0) = \mathbb{E}(W),$$

where

$$W = F_0(X_1 \vee Z_2) + F_0(X_2 \vee Z_1) - F_0(X_1 \vee X_2) - F_0(Z_1 \vee Z_2). \qquad (2.1)$$

In what follows, let $Z_0$ be independent of $X_1, X_2, Z_1, Z_2$ and having the distribution function $F_0$. By conditioning on $X_1, Z_2$ and using the fact that the joint distribution of $(Z_0, Z_2)$ is that of $(Z_1, Z_2)$, we have

$$\begin{aligned}
\mathbb{E}[F_0(X_1 \vee Z_2)] &= \mathbb{P}(Z_0 < X_1 \vee Z_2) \\
&= \mathbb{P}(Z_0 < X_1 < Z_2) + \mathbb{P}(Z_0 < Z_2 < X_1) \\
&\qquad + \mathbb{P}(X_1 < Z_0 < Z_2) + \mathbb{P}(Z_2 < Z_0 < X_1) \\
&= \mathbb{P}(Z_1 < X_1 < Z_2) + \mathbb{P}(Z_1 < Z_2 < X_1) \\
&\qquad + \mathbb{P}(X_1 < Z_1 < Z_2) + \mathbb{P}(Z_2 < Z_1 < X_1).
\end{aligned}$$

By symmetry, it follows that $\mathbb{E}[F_0(X_2 \vee Z_1)] = \mathbb{E}[F_0(X_1 \vee Z_2)]$. Moreover,

$$\begin{aligned}
\mathbb{E}[F_0(X_1 \vee X_2)] &= \mathbb{P}(Z_0 < X_1 \vee X_2) \\
&= \mathbb{P}(X_1 < Z_0 < X_2) + \mathbb{P}(X_2 < Z_0 < X_1) \\
&\qquad + \mathbb{P}(Z_0 < X_1 < X_2) + \mathbb{P}(Z_0 < X_2 < X_1) \\
&= \mathbb{P}(X_1 < Z_1 < X_2) + \mathbb{P}(X_2 < Z_1 < X_1) \\
&\qquad + \mathbb{P}(Z_1 < X_1 < X_2) + \mathbb{P}(Z_1 < X_2 < X_1).
\end{aligned}$$

Since $\mathbb{E}[F_0(Z_1 \vee Z_2)] = \mathbb{P}(Z_0 < Z_1 \vee Z_2) = 2/3$, (2.1) and symmetry give

$$\begin{aligned}\mathbb{E}(W) &= \mathbb{P}(Z_1 \wedge Z_2 < X_1 < Z_1 \vee Z_2) + 2\mathbb{P}(Z_1 \vee Z_2 < X_1) + \mathbb{P}(X_1 < Z_1 \wedge Z_2) \\ &\quad -\mathbb{P}(X_1 \wedge X_2 < Z_1 < X_1 \vee X_2) - \mathbb{P}(Z_1 < X_1 \wedge X_2) - \frac{2}{3}.\end{aligned}$$

Since $1 = \mathbb{P}(Z_1 \wedge Z_2 < X_1 < Z_1 \vee Z_2) + \mathbb{P}(Z_1 \vee Z_2 < X_1) + \mathbb{P}(X_1 < Z_1 \wedge Z_2)$ and $\mathbb{P}(X_1 \wedge X_2 < Z_1 < X_1 \vee X_2) + \mathbb{P}(Z_1 < X_1 \wedge X_2) = 1 - \mathbb{P}(X_1 \vee X_2 < Z_1)$, the result follows. $\square$

*Remark* 1 (i) Theorem 1 shows that

$$\Delta(F, F_0) = \int (F - F_0)^2 \,\mathrm{d}F_0 = \int (F_0 - F)^2 \,\mathrm{d}F = \Delta(F_0, F).$$

At first glance, this symmetry looks somewhat surprising. It follows, however, also from

$$0 = \frac{1}{3}\int \mathrm{d}(F-F_0)^3 = \int (F-F_0)^2 \,\mathrm{d}(F-F_0) = \int (F-F_0)^2 \,\mathrm{d}F - \int (F-F_0)^2 \,\mathrm{d}F_0.$$

(ii) Using integration by parts, we have

$$\frac{8}{3} = \frac{1}{3}\int \mathrm{d}(F + F_0)^3 = \int (F + F_0)^2 \,\mathrm{d}(F + F_0),$$

whence

$$\begin{aligned}\frac{8}{3} + 2\Delta(F, F_0) &= \int (F + F_0)^2 \,\mathrm{d}(F + F_0) + \int (F - F_0)^2 \,\mathrm{d}(F + F_0) \\ &= 2\int F^2 \,\mathrm{d}(F + F_0) + 2\int F_0^2 \,\mathrm{d}(F + F_0) \\ &= \frac{4}{3} + 2\int F^2 \,\mathrm{d}F_0 + 2\int F_0^2 \,\mathrm{d}F.\end{aligned}$$

Therefore,

$$\Delta(F, F_0) = \int F^2 \,\mathrm{d}F_0 + \int F_0^2 \,\mathrm{d}F - \frac{2}{3}.$$

By noting that

$$\int F^2 \,\mathrm{d}F_0 = \mathbb{P}(X_1 \vee X_2 < Z_1) \quad \text{and} \quad \int F_0^2 \,\mathrm{d}F = \mathbb{P}(Z_1 \vee Z_2 < X_1)$$

we have another proof of Theorem 1.

(iii) In comparison to Theorem 1, it is interesting to note that Lehmann (1951) based a nonparametric two-sample test on the measure of deviation

$$\int (F - F_0)^2 \, \mathrm{d}(F + F_0) = \mathbb{P}(X_1 \vee X_2 < Z_1 \wedge Z_2) + \mathbb{P}(Z_1 \vee Z_2 < X_1 \wedge X_2) - \frac{1}{3}$$

(see also Sundrum (1954)).

## 3. The case of a simple hypothesis

This section exploits statistical applications of (1.2). In what follows, it is indispensable to have a consistent estimator $\widehat{\sigma}_n^2$ of $\sigma^2(F)$ figuring in (1.2). Such an estimator is obtained if $F$ in (1.3) is replaced throughout by the edf $F_n$. Putting $U_j := F_0(X_j)$, $U_{(j)} := F_0(X_{(j)})$, $j = 1, \ldots, n$, and $G_n(t) = n^{-1} \sum_{j=1}^n \mathbf{1}\{U_j \le t\}$, $0 \le t \le 1$, we have

$$\widehat{\sigma}_n^2 = 4 \int_0^1 \int_0^1 (G_n(s) - s)\,(G_n(t) - t)\,(G_n(s \wedge t) - G_n(s)G_n(t)) \, \mathrm{d}s \, \mathrm{d}t. \qquad (3.1)$$

To state an expression for $\widehat{\sigma}_n^2$ in terms of $U_{(1)} \ldots, U_{(n)}$ that is suitable for computations, let

$$\begin{aligned} S \;:=\; & \sum_{i=1}^n \sum_{j=1}^n (1 - U_{(j)})(1 - U_{(i)} \vee U_{(j)}) + \sum_{i=1}^n \sum_{j<k} (1 - U_{(k)})(1 - U_{(i)} \vee U_{(k)}) \\ & + \sum_{i=1}^n \sum_{k<j} (1 - U_{(j)})(1 - U_{(i)} \vee U_{(k)}). \end{aligned}$$

Proposition 1 *Putting*

$$\overline{U^k} := \frac{1}{n} \sum_{i=1}^n U_{(i)}^k, \qquad V_k := \sum_{i=1}^n (i-1) U_{(i)}^k, \quad k \ge 1, \qquad (3.2)$$

*we have*

$$\begin{aligned} \frac{\widehat{\sigma}_n^2}{4} \;=\; & \frac{S}{n^3} - 1 + \frac{1}{n}\left(2\overline{U^1} + \overline{U^1} \cdot \overline{U^2} - \overline{U^3}\right) + \frac{\overline{U^4}}{4} - \frac{\overline{U^2}^2}{4} \\ & + \frac{4V_1 - V_3 - \overline{U^1}^2 + 2V_1 \overline{U^2}}{n^2} - \frac{4\overline{U^1} V_1}{n^3} - \frac{4V_1^2}{n^4} - \frac{1}{n^2} \sum_{j=1}^{n-1} \sum_{i=j+1}^n U_{(i)} U_{(j)}^2. \end{aligned}$$

The proof of Proposition 1 is given in Section 7.

*Remark* 2 Angus (1983) stated his result (1.2) without discussing whether the variance $\sigma^2(F)$ of the limiting normal distribution is positive. Since

$$F(x\wedge y) - F(x)F(y) = \int (\mathbf{1}\{u \le x\} - F(x))(\mathbf{1}\{u \le y\} - F(y))\,\mathrm{d}F(u), \;\; x, y \in \mathbb{R},$$

we obtain by Fubini's theorem that $\sigma^2(F)$ given in (1.3) takes the form

$$\sigma^2(F) = 4 \int \left( \int (F(x) - F_0(x))(\mathbf{1}\{u \le x\} - F(x))\,\mathrm{d}F_0(x) \right)^2 \mathrm{d}F(u). \tag{3.3}$$

Choosing $F = F_n$, the edf of $X_1, \ldots, X_n$, we see that $\widehat{\sigma}_n^2 = \sigma^2(F_n) \ge 0$. If $F$ and $F_0$ have supports that are disjoint closed intervals, for example, it follows from (1.3) that $\sigma^2(F) = 0$. This observation contradicts an assertion of Tiago de Oliveira (1987). Recognizing that $\omega_n^2$ is a $V$-statistic this author uses the standard asymptotic theory of $V$-statistics, see, e.g., Serfling (1980), to obtain the limit distribution of $\omega_n^2$ for each fixed alternative. Starting from the alternative formula for the variance given there, the author argues that the variance vanishes if, and only if, $F = F_0$. In fact, if $F$ and $F_0$ are continuous and mutually absolutely continuous with the same (closed) interval as support, then (3.3) readily shows that $\sigma^2(F) = 0$ if, and only if, $F = F_0$. In the remainder of this section, only alternative distributions $F$ with $\sigma^2(F) > 0$ are considered.

From (1.2) we obtain the following approximation of the power of the Cramér–von Mises test against a fixed alternative distribution function $F$.

Corollary 1 *Let $c_n = c_n(\alpha)$ be the critical value of the Cramér–von Mises test, carried out at level $\alpha$. Then, under a fixed alternative distribution function $F$ , the power $\mathbb{P}_F(\omega_n^2 > c_n)$ may be approximated by*

$$\mathbb{P}_F(\omega_n^2 > c_n) \approx 1 - \Phi\left( \frac{\sqrt{n}}{\sigma(F)} \left( \frac{c_n}{n} - \Delta(F) \right) \right), \tag{3.4}$$

*where $\Phi$ denotes the distribution function of the standard normal distribution.*

*Proof.* The result follows immediately from

$$\begin{aligned} \mathbb{P}_F(\omega_n^2 > c_n) &= \mathbb{P}_F\left( \frac{\sqrt{n}}{\sigma(F)} \left( \frac{\omega_n^2}{n} - \Delta(F) \right) > \frac{\sqrt{n}}{\sigma(F)} \left( \frac{c_n}{n} - \Delta(F) \right) \right) \\ &\approx 1 - \Phi\left( \frac{\sqrt{n}}{\sigma(F)} \left( \frac{c_n}{n} - \Delta(F) \right) \right). \quad \square \end{aligned}$$

*Example* 1 To give an impression on the quality of the approximation provided by the right-hand side of (3.4), we consider the case where $F_0(t) = t$, $0 \le t \le 1$, and $H_\gamma(t) := t^\gamma$, $0 \le t \le 1$, is the alternative distribution function parametrized

| | $\gamma$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0.3 | 0.5 | 0.7 | 0.8 | 0.9 | 1.1 | 1.2 | 1.3 | 1.5 | 1.7 | 1.9 | |
| 20 | .99 | .73 | .28 | .14 | .07 | .06 | .10 | .15 | .32 | .52 | .70 | MC |
| | .95 | .65 | .16 | .01 | .00 | .00 | .00 | .03 | .21 | .43 | .61 | App |
| 50 | 1 | .98 | .56 | .27 | .10 | .08 | .18 | .34 | .69 | .91 | .98 | MC |
| | 1 | .93 | .50 | .16 | .00 | .00 | .05 | .24 | .62 | .84 | .94 | App |
| 100 | 1 | 1 | .85 | .48 | .14 | .12 | .33 | .61 | .94 | 1 | 1 | MC |
| | 1 | .99 | .77 | .40 | .02 | .01 | .24 | .54 | .88 | .98 | 1 | App |
| 200 | 1 | 1 | .99 | .76 | .24 | .20 | .60 | .90 | 1 | 1 | 1 | MC |
| | 1 | 1 | .94 | .69 | .12 | .07 | .52 | .82 | .99 | 1 | 1 | App |

Table 1. Empirical and approximated power against several alternatives ($\alpha = 0.05$), rounded to 2 decimal places

by the parameter $\gamma > 0$; see also Shorack and Wellner (1986), Excercise 4.4.3. Then

$$\Delta(H_\gamma) = \int_0^1 (t^\gamma - t)^2 \, \mathrm{d}t = \frac{1}{2\gamma+1} - \frac{2}{\gamma+2} + \frac{1}{3}.$$

Moreover, straightforward calculations give

$$\begin{aligned} \sigma^2(H_\gamma) &= 4\int_0^1\int_0^1 (s^\gamma - s)(t^\gamma - t)(s^\gamma \wedge t^\gamma - s^\gamma t^\gamma)\,\mathrm{d}s\mathrm{d}t \\ &= 4\left[\frac{2}{2\gamma+1}\left(\frac{1}{3\gamma+2} - \frac{1}{2\gamma+3}\right) - \frac{2}{\gamma+2}\left(\frac{1}{2\gamma+3} - \frac{1}{\gamma+4}\right)\right. \\ &\quad \left. - \left(\frac{1}{2\gamma+1} - \frac{1}{\gamma+2}\right)^2\right]. \end{aligned}$$

Table 1 displays the empirical power of the Cramér-von Mises test against alternative df's $H_\gamma$ for several values of $\gamma$ and sample sizes $n = 20$, $n = 50$, $n = 100$ and $n = 200$. The nominal level is 0.05. Each entry in the lines denoted by 'MC' is based on a Monte Carlo simulation with 25 000 replications. The lines denoted by 'App' show the corresponding approximations given by the right-hand side of (3.4). In each case, the approximation seems to be a lower bound for the true power.

From (1.4), we obtain the following asymptotic confidence interval for $\Delta(F)$.

Corollary 2 *Given $\alpha \in (0,1)$, let $u_\alpha = \Phi^{-1}(1-\alpha/2)$ be the $(1-\alpha/2)$-quantile of the standard normal distribution. Then*

$$I_n := \left[\frac{\omega_n^2}{n} - \frac{u_\alpha \widehat{\sigma}_n}{\sqrt{n}}, \frac{\omega_n^2}{n} + \frac{u_\alpha \widehat{\sigma}_n}{\sqrt{n}}\right]$$

*is an asymptotic confidence interval for $\Delta(F)$ at level $1-\alpha$, i.e., we have*

$$\lim_{n\to\infty} \mathbb{P}_F(I_n \ni \Delta(F)) = 1-\alpha.$$

For illustration, we pick up Example 1. Table 2 gives the empirical coverage probability of $I_n$ for $\Delta(H_\gamma)$ in the case $\alpha = 0.05$ and $n \in \{20, 50, 100\}$ for various values of $\gamma$. Each of the entries in Table 2 is based on 10 000 replications.

| | $\gamma$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
| 20 | .89 | .90 | .89 | .99 | .98 | .98 | .93 | .90 | .90 | .90 |
| 50 | .93 | .93 | .91 | .94 | .99 | .97 | .91 | .92 | .93 | .93 |
| 100 | .94 | .94 | .93 | .90 | .99 | .93 | .92 | .93 | .94 | .94 |

Table 2. Empirical coverage probability for $\Delta(H_\gamma)$ (10 000 replications), rounded to 2 decimal places, $\alpha = 0.05$

A further application of (1.4) is an equivalence testing procedure or 'neighborhood-of-$F_0$ validation procedure' that tests, for some given positive value $\Delta_0$, the hypothesis

$$H_{\Delta_0} : \Delta(F) \geq \Delta_0 \quad \text{versus} \quad K_{\Delta_0} : \Delta(F) < \Delta_0.$$

Theorem 2 *Let $\alpha \in (0,1)$. The test that rejects $H_{\Delta_0}$ if*

$$\frac{\omega_n^2}{n} \;\leq\; \Delta_0 - \frac{\widehat{\sigma}_n}{\sqrt{n}}\Phi^{-1}(1-\alpha)$$

*is an asymptotic level-$\alpha$-test of $H_{\Delta_0}$ versus $K_{\Delta_0}$. This test is consistent against each alternative distribution.*

*Proof.* Using (1.4) we have for each $F \in H_{\Delta_0}$

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_F\left(\frac{\omega_n^2}{n} \leq \Delta_0 - \frac{\widehat{\sigma}_n}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right) &= \limsup_{n\to\infty} \mathbb{P}_F\left(\frac{\sqrt{n}}{\widehat{\sigma}_n}\left(\frac{\omega_n^2}{n} - \Delta_0\right) \leq \Phi^{-1}(\alpha)\right) \\ &\leq \alpha.\end{aligned}$$

In particular,

$$\lim_{n\to\infty} \mathbb{P}_F\left(\frac{\omega_n^2}{n} \leq \Delta_0 - \frac{\widehat{\sigma}_n}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right) \;=\; \alpha$$

for each $F$ such that $\Delta(F) = \Delta_0$. Therefore, the test has asymptotic level $\alpha$. It is easy to see that

$$\lim_{n\to\infty} \mathbb{P}_F\left(\frac{\omega_n^2}{n} \le \Delta_0 - \frac{\widehat{\sigma}_n}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right) = 1$$

if $\Delta(F) < \Delta_0$. Thus, this test is consistent against each fixed alternative. □

To exhibit the limiting power of the test against special local alternatives we firstly state a limit theorem for $\omega_n^2$ under a triangular scheme. To this end, let $F$ be a continuous df with $\sigma^2(F) > 0$ and $\Delta(F) > 0$. For each $n \ge 2$, suppose $X_{n1}, \dots, X_{nn}$ are independent and identically distributed random variables with continuous df $G_n$ such that $\lim_{n\to\infty} G_n = F$. We can (and do) assume that the random variables $X_{nk}$ are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, say. Let $F_n$ be the empirical df of $X_{n1}, \dots, X_{nn}$. Define (with this $F_n$) $\omega_n^2$ as before.

Theorem 3 *Under these assumptions, we have, as $n \to \infty$:*

(i) $\sigma^2(G_n) \to \sigma(F)$,

(ii) $\sigma^2(F_n) \to \sigma(F)$ *in probability,*

(iii) $\sqrt{n}\left(\frac{\omega_n^2}{n} - \Delta(G_n)\right) \xrightarrow{\mathcal{D}} N(0, \sigma^2(F))$.

*Proof.* (i) obviously follows from the definition of

$$\sigma^2(H) = 4\int\left(\int (H(x) - F_0(x))(\mathbf{1}\{u \le x) - H(x))\,\mathrm{d}F_0(x)\right)^2 \mathrm{d}H(u)$$

for distribution functions $H$. Using a central limit theorem for processes, see, e.g., Van der Vaart and Wellner (1996), Section 2.11, there is a Brownian bridge $(U(t), 0 \le t \le 1)$ with continuous sample paths such that the empirical process $(\sqrt{n}\,(F_n(x) - G_n(x))\,, x \in \mathbb{R})$ converges in distribution to $(U\,(F(x))\,, x \in \mathbb{R})$. Thus, as $n \to \infty$,

$$\sup_{x\in\mathbb{R}}|F_n(x) - F(x)| \to 0 \text{ in probability.} \tag{3.5}$$

By (3.5) and, again, by the definition of $\sigma^2(H)$, (ii) follows. To prove (iii) we argue as in Angus (1983). As $n \to \infty$,

$$2\int (G_n(x) - F_0(x))\left[\sqrt{n}\,(F_n(x) - G_n(x))\right]\,\mathrm{d}F_0(x) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma^2(F)).$$

Therefore, due to

$$\begin{aligned}&\sqrt{n}\left|\frac{\omega_n^2}{n}-\Delta(G_n)-2\int\left(G_n(x)-F_0(x)\right)\left(F_n(x)-G_n(x)\right)\,\mathrm{d}F_0(x)\right|\\=&\sqrt{n}\int\left(F_n(x)-G_n(x)\right)^2\,\mathrm{d}F_0(x)\\\leq&\frac{1}{\sqrt{n}}\left(\sqrt{n}\sup_{x\in\mathbb{R}}|F_n(x)-G_n(x)|\right)^2\to 0\ \text{ in probability,}\end{aligned}$$

the proof is finished. □

Now, suppose $\Delta_0=\Delta(F)$ and assume that the sequence $(G_n)$ has the additional property that

$$\Delta(G_n)=\Delta_0-\frac{a_n}{\sqrt{n}},\quad n\geq 1,$$

where $(a_n)$ is some sequence of positive real numbers converging to some positive real number $a$. Then as an easy consequence of the theorem we obtain

$$\sqrt{n}\left(\frac{\omega_n^2}{n}-\Delta_0\right)\ \xrightarrow{\mathcal{D}}\ \mathrm{N}(-a,\sigma^2(F)).$$

Thus, for such a sequence $(G_n)$ of local alternatives, the asymptotic power is

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\omega_n^2}{n}\leq\Delta_0-\frac{\sigma(F_n)}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right)=\Phi\left(\frac{a}{\sigma(F)}-\Phi^{-1}(1-\alpha)\right).$$

## 4. Composite hypothesis: neighborhood-of-exponentiality testing

This section shows that, at least in principle, the above reasoning carries over to the case that $H_0$ is a composite hypothesis, e.g., a parametric family of distributions. We confine ourselves to the important special case of 'neighborhood-of-exponentiality testing', but it will become apparent what is needed to tackle the problem of a model validation also in other cases (see, e.g., Dette and Munk (2003) and Czado et al. (2007) in other contexts).

In what follows, write $E_\mu$ for the df of the exponential distribution with density $\mu^{-1}\exp(-x/\mu)$ for $x>0$ and 0 else, and put $E=E_1$. The parameter $\mu\in(0,\infty)$ is the mean of $E_\mu$. Let $\mathcal{E}$ be the family of these exponential distributions. The goodness-of-fit problem of testing the hypothesis

$$H_0: F\in\mathcal{E}$$

of exponentiality against general or special alternatives has received considerable interest in the literature, see. e.g. Baringhaus and Henze (1991) or Henze and Meintanis (2005), where many more references are supplied.

Writing $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ for the mean of $X_1, \ldots, X_n$, the Cramér–von Mises statistic for testing $H_0$ versus general alternatives is

$$\begin{aligned} \widehat{\omega}_n^2 &= n\int_{-\infty}^{\infty} (F_n(x) - E_{\overline{X}_n}(x))^2 \,\mathrm{d}E_{\overline{X}_n}(x) \\ &= \frac{1}{12n} + \sum_{j=1}^{n}\left(E_{\overline{X}_n}(X_{(j)}) - \frac{2j-1}{2n}\right)^2 \end{aligned}$$

(see, e.g. D'Agostino and Stephens (1986), p. 133). Since

$$E_{\overline{X}_n}(X_{(j)}) = 1 - \exp(-X_{(j)}/\overline{X}_n), \; j = 1, \ldots, n,$$

$\widehat{\omega}_n^2$ is a function of the scaled random variables $X_j/\overline{X}_n$, $j = 1 \ldots, n$, and hence is distribution-free under $H_0$, that is, the distribution of $\widehat{\omega}_n^2$ does not depend on the true unknown distribution $E_\mu$. For results on the limit null distribution of $\widehat{\omega}_n^2$ we refer to Stephens (1976). For convenience, we present a small table (Table 3) showing the critical values ($(1-\alpha)$-quantiles) of $\widehat{\omega}_n^2$ for $\alpha \in \{0.1, 0.05, 0.025, 0.01\}$ and samples sizes $n = 20$, $n = 50$ and $n = 100$. The values have been obtained by simulations based on 100 million replications.

| | $\alpha = 0.100$ | $\alpha = 0.050$ | $\alpha = 0.025$ | $\alpha = 0.010$ |
|---|---|---|---|---|
| $n = 20$ | 0.1735 | 0.2191 | 0.2660 | 0.3293 |
| $n = 50$ | 0.1741 | 0.2205 | 0.2687 | 0.3343 |
| $n = 100$ | 0.1743 | 0.2210 | 0.2697 | 0.3360 |

Table 3. Critical values for $\widehat{\omega}_n^2$

In the sequel, we assume that $X_1$ is a positive random variable with df $F$ and finite mean $\mu$. We will construct an equivalence test of the hypothesis $H_{\Delta_0} : \Delta(F, \mathcal{E}) \geq \Delta_0$ against the alternative $K_{\Delta_0} : \Delta(F, \mathcal{E}) < \Delta_0$, where $\Delta_0$ is some given positive number and

$$\Delta(F, \mathcal{E}) := \int_0^\infty (F(x\mu) - E(x))^2 \; e^{-x}\mathrm{d}x$$

is the Cramér–von Mises distance between $F$ and the class of exponential distributions. Notice that $x \mapsto F(x\mu)$ is the df of the random variable $X_1/\mu$, which shows that $\Delta(F, \mathcal{E})$ is scale invariant. In order derive an asymptotic test of $H_{\Delta_0}$ versus $K_{\Delta_0}$, we first give the limit distribution of $\widehat{\omega}_n^2$ for fixed alternatives.

Theorem 4 *Let $X_1$ have the df $F$, where $F$ has finite positive mean $\mu$ and finite positive variance $\sigma^2$. Additionally, let $F$ be differentiable with uniformly* *continuous density $f = F'$. Define*

$$V_n = \sqrt{n}\left(\frac{\widehat{\omega}_n^2}{n} \; - \; \Delta(F, \mathcal{E})\right).$$

*Then $V_n \xrightarrow{\mathcal{D}} \mathrm{N}(0, \tau^2(F))$, where*

$$\begin{aligned} &\tau^2(F) \\ &= 4\Bigg[\int_0^\infty \int_0^\infty (F(x\mu) - E(x))\,(F(y\mu) - E(y))\,(F(\mu(x \wedge y)) - F(x\mu)F(y\mu))\,\mathrm{d}E(x)\mathrm{d}E(y) \\ &\quad + 2\rho \int_0^\infty \left(\int_0^{x\mu} y\,\mathrm{d}F(y) - \mu F(x\mu)\right)(F(x\mu) - E(x))\,\mathrm{d}E(x) + \sigma^2\rho^2\Bigg] \end{aligned}$$

*and*

$$\rho = \int_0^\infty x f(x\mu)(F(x\mu) - E(x))\exp(-x)\,\mathrm{d}x. \tag{4.1}$$

The proof of Theorem 4 is given in Section 7.

To avoide estimation of the density $f$ figuring in (4.1) when estimating the parameter $\rho$, notice that, putting $\overline{F} = 1 - F$, $\rho$ can alternatively be written as

$$\rho = \frac{1}{\mu^2}\int_0^\infty x\left(\exp\left(-\frac{x}{\mu}\right) - \overline{F}(x)\right)\exp\left(-\frac{x}{\mu}\right)\,dF(x).$$

A consistent estimator $\widehat{\tau}_n^2$ of $\tau^2(F)$ is obtained if, in the above expression, we replace throughout $F$ by the edf $F_n$, $\mu$ by $\overline{X}_n$ and $\sigma^2$ by the empirical variance $n^{-1}\sum_{j=1}^n (X_j - \overline{X}_n)^2$ (or its unbiased version $(n-1)^{-1}\sum_{j=1}^n (X_j - \overline{X}_n)^2$). The estimator can be expressed in terms of the scaled variables $Y_j := X_j/\overline{X}_n$, $j = 1, \ldots, n$, and their order statistics $Y_{(1)} \leq \ldots \leq Y_{(n)}$. Putting $T_j := \exp(-Y_{(j)})$, $j = 1, \ldots, n$, we have the following result.

Theorem 5 *A consistent estimate $\widehat{\tau}_n^2$ of $\tau^2(F)$ is given by*

$$\widehat{\tau}_n^2 = 4\left[\widehat{J} + 2\widehat{\kappa}\widehat{\eta} + \widehat{\sigma}_Y^2\widehat{\kappa}^2\right],$$

*where*

$$\widehat{J} = \widehat{J}_1 - 2\widehat{J}_2 + \widehat{J}_3 - \widehat{J}_4^2$$

*with*

$$\widehat{J}_1 = \frac{1}{n^3}\sum_{i,j=1}^{n} T_j(T_i \wedge T_j) + \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j<k} T_k(T_i \wedge T_k) + \frac{1}{n^3}\sum_{i=1}^{n}\sum_{k<j} T_j(T_i \wedge T_k),$$

$$\widehat{J}_2 = \frac{1}{n^2}\sum_{i=1}^{n} T_i^2\left(1-\frac{T_i}{2}\right) + \frac{1}{n^2}\sum_{j<i} T_i^2\left(1-\frac{T_i}{2}\right) + \frac{1}{n^2}\sum_{i<j} T_iT_j\left(1-\frac{T_i}{2}\right),$$

$$\widehat{J}_3 = \frac{1}{n}\sum_{i=1}^{n}\left[T_i\left(1-\frac{T_i}{2}\right)\right]^2,$$

$$\widehat{J}_4 = \frac{1}{n^2}\sum_{i,j=1}^{n} T_i \wedge T_j - \frac{1}{n}\sum_{i=1}^{n} T_i\left(1-\frac{T_i}{2}\right),$$

$$\widehat{\kappa} = \frac{1}{n}\sum_{j=1}^{n} Y_{(j)}\left(T_j - \left(1-\frac{j}{n}\right)\right)T_j,$$

$$\widehat{\eta} = \frac{1}{n^2}\sum_{j,k=1}^{n}(Y_{(j)}-1)T_j \wedge T_k - \frac{1}{n}\sum_{j=1}^{n}(Y_{(j)}-1)\left(1-\frac{T_j}{2}\right)T_j,$$

*and* $\widehat{\sigma}_Y^2 = \frac{1}{n}\sum_{j=1}^{n}(Y_j-1)^2$, *the empirical variance of the scaled variables* $Y_1,\ldots,Y_n$.

The proof of Theorem 5 is given in Section 7.

The next results are immediate consequences of Theorem 4 and Theorem 5. Throughout, only alternative distributions $F$ with $\tau^2(F) > 0$ are considered.

Corollary 3 *With* $\alpha$ *and* $u_\alpha$ *as in* Corollary 2, *an asymptotic confidence interval for* $\Delta(F,\mathcal{E})$ *at level* $1-\alpha$ *is*

$$J_n := \left[\frac{\widehat{\omega}_n^2}{n} - \frac{u_\alpha\widehat{\tau}_n}{\sqrt{n}}, \frac{\widehat{\omega}_n^2}{n} + \frac{u_\alpha\widehat{\tau}_n}{\sqrt{n}}\right],$$

*i.e., we have* $\lim_{n\to\infty}\mathbb{P}_F(J_n \ni \Delta(F,\mathcal{E})) = 1-\alpha$.

Corollary 4 *Let* $d_n = d_n(\alpha)$ *be the critical value of the Cramér–von Mises test for exponentiality, carried out at level* $\alpha$. *Then, under a fixed alternative distribution function* $F$ *satisfying the assumptions of* Theorem 4, *the power* $\mathbb{P}_F(\widehat{\omega}_n^2 > d_n)$ *may be approximated by*

$$\mathbb{P}_F(\widehat{\omega}_n^2 > d_n) \approx 1 - \Phi\left(\frac{\sqrt{n}}{\tau(F)}\left(\frac{d_n}{n} - \Delta(F,\mathcal{E})\right)\right). \tag{4.2}$$

*Example* 2 Let $F$ be the df of the Erlang(1,2)-distribution (Gamma distribution with shape parameter 2 and scale parameter 1), that is, $F(x) = 1-(1+x)e^{-x}$ for $x \geq 0$ and 0 else. The density is $f(x) = xe^{-x}$ for $x \geq 0$ and 0 else, the mean is $\mu = 2$, and the variance is $\sigma^2 = 2$. By straightforward calculation we obtain

$$\Delta(F,\mathcal{E}) = \int_0^\infty \left(\exp(-x) - (1+2x)e^{-2x}\right)^2 e^{-x}\,\mathrm{d}x = \frac{11}{1500} = 0.0073333\ldots.$$

The variance of the limit distribution is $\tau^2(F) = 4\left[K + 2\rho\lambda + 2\rho^2\right]$, where, putting $h(x) = e^{-x} - (1+2x)e^{-2x}$ for $x \geq 0$ and 0 else,

$$\begin{aligned}
K &= \iint h(x)h(y)\left(1-(1+2(x\wedge y))\,e^{-2(x\wedge y)} - \left(1-(1+2x)e^{-2x}\right)\left(1-(1+2y)e^{-2y}\right)\right) \\
&\qquad \cdot e^{-x}e^{-y}\,\mathrm{d}x\mathrm{d}y \\
&= 2\int h(x)(1+2x)e^{-3x}\,\mathrm{d}x \int h(y)e^{-y}\,\mathrm{d}y - \left(\int h(x)(1+2x)e^{-3x}\,\mathrm{d}x\right)^2 \\
&\quad - \iint h(x)h(y)\,(1+2(x\wedge y))\,e^{-2(x\wedge y)}\cdot e^{-x}e^{-y}\,\mathrm{d}x\mathrm{d}y \\
&=: 2K_1\cdot K_2 - K_1^2 - K_3\ \text{, say,} \\
\rho &= \int x(f(x\mu)(F(x\mu) - E(x))e^{-x}\,\mathrm{d}x = 2\int_0^\infty \left(x^2e^{-4x} - x^2e^{-5x} - 2x^3e^{-5x}\right)\mathrm{d}x \\
&= -\frac{79}{10000},
\end{aligned}$$

and

$$\begin{aligned}
\lambda &= \int \left(\int_0^{x\mu} y\,\mathrm{d}F(y) - \mu F(x\mu)\right)(F(x\mu) - E(x))\,\mathrm{d}E(x) \\
&= -4\int_0^\infty \left(x^2e^{-4x} - x^2e^{-5x} - 2x^3e^{-5x}\right)\mathrm{d}x = -2\rho.
\end{aligned}$$

The integrals $K_1, K_2$ and $K_3$ are calculated to be

$$\begin{aligned}
K_1 &= \int_0^\infty \left(e^{-4x} + 2xe^{-4x} - e^{-5x} - 4xe^{-5x} - 4x^2e^{-5x}\right)\mathrm{d}x = -\frac{49}{1000}, \\
K_2 &= \int_0^\infty \left(e^{-2x} - e^{-3x} - 2xe^{-3x}\right)\mathrm{d}x = -\frac{1}{18}, \\
K_3 &= 2\int_0^\infty \left(\int_x^\infty \left(e^{-2y} - e^{-3y} - 2ye^{-3y}\right)\mathrm{d}y\right) \\
&\qquad\qquad \cdot \left(e^{-4x} + 2xe^{-4x} - e^{-5x} - 4xe^{-5x} - 4x^2e^{-5x}\right)\mathrm{d}x \\
&= \frac{361}{13712}.
\end{aligned}$$

Thus,

$$K = \frac{1868351}{6174000000}.$$

Putting pieces together we obtain

$$\tau^2(F) = 4(K - 2\tau^2) = \frac{3430351}{4823437500} = 0.000711183\ldots.$$

Table 4 gives the empirical coverage probability of $J_n$ for $\Delta(F, \mathcal{E})$ in the case $\alpha = 0.05$ and $n \in \{20, 50, 100, 200\}$ for the Erlang(1,2)-distribution. Each of the entries in Table 4 is based on 10 000 replications.

| $n$ | 20 | 50 | 100 | 200 |
|---|---|---|---|---|
| | .93 | .94 | .94 | .95 |

Table 4. Empirical coverage probability for $\Delta(F, \mathcal{E})$ (10 000 replications), rounded to 2 decimal places, $\alpha = 0.05$

To conclude this example, Table 5 displays the empirical power of the Cramér-von Mises test for exponentiality against the Erlang(1,2)-distribution for the sample sizes $n = 20$, $n = 50$ and $n = 100$. The nominal level is 0.05. Each entry in the lines denoted by 'MC' is based on a Monte Carlo simulation with 25 000 replications. The lines denoted by 'App' show the corresponding approximations given by the right-hand side of (4.2). Like in Example 1, the approximation seems to be a lower bound for the true power.

| $n$ | 20 | 50 | 100 |
|---|---|---|---|
| MC | .48 | .90 | 1.0 |
| App | .27 | .78 | .97 |

Table 5. Power of the Cramér–von Mises test for exponentiality against the Erlang(1,2)-distribution ($\alpha = 0.05$), rounded to 2 decimal places

Finally, as an analogue to Theorem 2 treating

$$H_{\Delta_0} : \Delta(F, \mathcal{E}) \geq \Delta_0 \quad \text{versus} \quad K_{\Delta_0} : \Delta(F, \mathcal{E}) < \Delta_0, \tag{4.3}$$

where $\Delta_0$ is a given positive number, we obtain an equivalence test or 'neighborhood-of-exponentiality test'.

THEOREM 6 *Let $\alpha \in (0, 1)$. The test that rejects $H_{\Delta_0}$ if*

$$\frac{\widehat{\omega}_n^2}{n} \leq \Delta_0 - \frac{\widehat{\tau}_n}{\sqrt{n}} \Phi^{-1}(1 - \alpha) \tag{4.4}$$

*is an asymptotic level-$\alpha$-test of $H_{\Delta_0}$ against $K_{\Delta_0}$. This test is consistent against each alternative distribution.*

*Proof.* The proof follows the same lines as the proof of Theorem 2 and is therefore omitted. $\square$

## 5. A real data example

To illustrate a new K-S type goodness-of-fit test for exponentiality based on empirical Hankel transforms, Baringhaus and Taherizadeh (2013) considered the $n = 21$ values 33.5, 20.9, 17.8, 102.8, 110.6, 20.7, 54.2, 69.0, 63.4, 37.8, 37.1, 45.7, 35.3, 55.2, 5.8, 0.6, 54.3, 89.8, 58.6, 39.0, 24.8 (in millimeters) of the monthly amount of rainfall in January during the period 1991 to 2011, taken at the weather station Berlin-Tempelhof of the German Weather Service. For these data, a test of exponentiality was rejected at the 5% level. Treating the testing problem (4.3) we applied the test (4.4) to these data, choosing $\Delta_0 = \frac{11}{1500} = 0.0073333\ldots = \Delta(F, \mathcal{E})$, where $F$ is the df of the Erlang distribution Erlang(1,2), see Example 2. Since the values $\widehat{\tau}_n^2 = 0.001379132$ and $\widehat{\omega}_n^2 = 0.2654209$ were observed, and due to the fact that $\widehat{\omega}_n^2/n = 0.01263909$ exceeds $\Delta_0$, the hypothesis is accepted at each level $\alpha \in (0, 1/2)$. A Q-Q plot of the data against the quantiles $F^{-1}((i-3/8)/(n+0.25))$, $i = 1, \ldots, n$, of the Erlang(1,2) distribution shown in Figure 1 seems to confirm this finding.

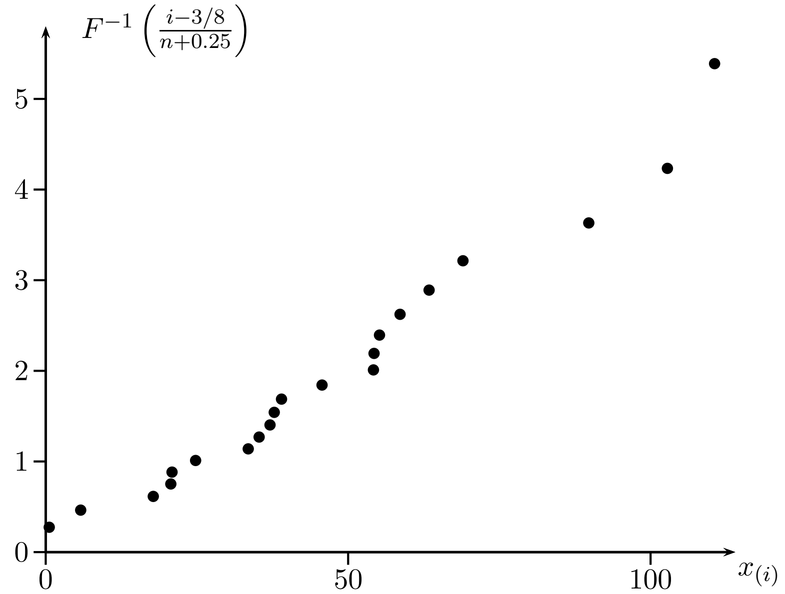


Figure 1. Q-Q plot of rainfall data

## 6. Concluding remarks

We have given a probabilistic representation of the Cramér–von Mises distance between continuous distribution functions. Moreover, we have shown that the time-honored Cramér–von Mises statistic can be used to construct nonparametric confidence intervals for the Cramér–von Mises distance between an unknown continuous df $F$ and a given continuous df $F_0$ when a random sample of $F$ is available. The same holds for a continuous distribution $F$ with support $[0,\infty)$ and the class $\mathcal{E}$ of exponential distributions. Moreover, given a positive number $\Delta_0$, there are asymptotic level-$\alpha$-tests of $H_0 : \Delta(F,F_0) \geq \Delta_0$ versus $K_0 : \Delta(F,F_0) < \Delta_0$ and $H_0 : \Delta(F,\mathcal{E}) \geq \Delta_0$ versus $K_0 : \Delta(F,\mathcal{E}) < \Delta_0$, where $\Delta(F,F_0)$ and $\Delta(F,\mathcal{E})$ denote the Cramér–von Mises distance between $F$ and $F_0$ and $F$ and $\mathcal{E}$, respectively. The method requires an asymptotic normal distribution of the Cramér–von Mises statistic under fixed alternatives and a consistent estimator of the variance of the limit distribution, and will presumably also work in other contexts, for example in the case of testing for normality. In the latter case, Gürtler (2000)) constructed a neighborhood-of-normality test and a pertaining confidence interval based on the BHEP statistic for testing for univariate and multivariate normality (see Baringhaus and Henze (1988) and Epps and Pulley (1983)).

## 7. Proofs

*Proof* of Proposition 1: From (3.1) we have

$$\frac{\widehat{\sigma}_n^2}{4} = I_1 - 2I_2 + I_3 - I_4,$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \int_0^1 G_n(s)G_n(t)G_n(s\wedge t)\,\mathrm{d}s\,\mathrm{d}t,\\
I_2 &= \int_0^1 \int_0^1 s\,G_n(t)G_n(s\wedge t)\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 \int_0^1 t\,G_n(s)G_n(s\wedge t)\,\mathrm{d}s\,\mathrm{d}t,\\
I_3 &= \int_0^1 \int_0^1 st\,G_n(s\wedge t)\,\mathrm{d}s\mathrm{d}t,\\
I_4 &= \int_0^1 \int_0^1 (G_n(s)-s)\,(G_n(t)-t)\,G_n(s)G_n(t)\,\mathrm{d}s\mathrm{d}t = \left[\int_0^1 (G_n(s)-s)\,G_n(s)\,\mathrm{d}s\right]^2.
\end{aligned}$$

Now,

$$I_1 = \frac{1}{n^3}\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \int_0^1 \int_0^1 \mathbf{1}\{U_{(i)} \leq s\}\mathbf{1}\{U_{(j)} \leq t\}\mathbf{1}\{U_{(k)} \leq s\wedge t\}\,\mathrm{d}s\,\mathrm{d}t. \quad (7.1)$$

Since

$$\int_0^1 \mathbf{1}\{U_{(j)} \le t\}\mathbf{1}\{U_{(k)} \le s \wedge t\}\,\mathrm{d}t \;=\; \begin{cases} \mathbf{1}\{U_{(j)} \le s\}(1-U_{(j)}), & \text{if } j=k, \\ \mathbf{1}\{U_{(k)} \le s\}(1-U_{(k)}), & \text{if } j<k, \\ \mathbf{1}\{U_{(k)} \le s\}(1-U_{(j)}), & \text{if } k<j, \end{cases}$$

it follows that the double integral figuring in (7.1) equals $(1-U_{(j)})(1-U_{(i)} \vee U_{(j)})$ if $j=k$, $(1-U_{(k)})(1-U_{(i)} \vee U_{(k)})$ if $j<k$ and $(1-U_{(j)})(1-U_{(i)} \vee U_{(k)})$ if $j>k$. Hence, the triple sum in (7.1) is

$$\begin{aligned} S \;=\; & \sum_{i=1}^n \sum_{j=1}^n (1-U_{(j)})(1-U_{(i)} \vee U_{(j)}) + \sum_{i=1}^n \sum_{j<k} (1-U_{(k)})(1-U_{(i)} \vee U_{(k)}) \\ & + \sum_{i=1}^n \sum_{k<j} (1-U_{(j)})(1-U_{(i)} \vee U_{(k)}). \end{aligned}$$

Likewise,

$$I_2 \;=\; \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \int_0^1 \int_0^1 s\mathbf{1}\{U_{(i)} \le t\}\mathbf{1}\{U_{(j)} \le s \wedge t\}\,\mathrm{d}s\,\mathrm{d}t. \tag{7.2}$$

Since

$$\int_0^1 \mathbf{1}\{U_{(i)} \le t\}\mathbf{1}\{U_{(j)} \le s \wedge t\}\,\mathrm{d}t \;=\; \begin{cases} \mathbf{1}\{U_{(i)} \le s\}(1-U_{(i)}), & \text{if } i=j, \\ \mathbf{1}\{U_{(j)} \le s\}(1-U_{(j)}), & \text{if } i<j, \\ \mathbf{1}\{U_{(j)} \le s\}(1-U_{(i)}), & \text{if } i>j, \end{cases}$$

the double integral figuring in (7.2) is $(1-U_{(i)})(1-U_{(i)}^2)/2$ if $i=j$, $(1-U_{(j)})(1-U_{(j)}^2)/2$ if $i<j$ and $(1-U_{(i)})(1-U_{(j)}^2)/2$ if $i>j$. Using the notation introduced in (3.2), straightforward algebra gives

$$I_2 \;=\; \frac{1}{2} - \frac{\overline{U^1} - \overline{U^3}}{2n} - \frac{V_1}{n^2} - \frac{\overline{U^2}}{2} + \frac{V_3}{2n^2} + \frac{1}{2n^2} \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} U_{(i)} U_{(j)}^2.$$

Next, we have

$$I_3 \;=\; \frac{1}{n} \sum_{j=1}^n \int_0^1 \int_0^1 st\,\mathbf{1}\{U_{(j)} \le s \wedge t\}\,\mathrm{d}s\mathrm{d}t \;=\; \frac{1}{4n} \sum_{j=1}^n \left(1 - U_{(j)}^2\right)^2.$$

Finally,

$$I_4 \;=\; \left[\int_0^1 (G_n(s) - s)\, G_n(s)\,\mathrm{d}s\right]^2.$$

Here, the term within squared brackets equals

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1 \mathbf{1}\{U_{(i)} \le s\}\mathbf{1}\{U_{(j)} \le s\}\,\mathrm{d}s - \frac{1}{n}\sum_{i=1}^{n}\int_0^1 s\mathbf{1}\{U_{(i)} \le s\}\,\mathrm{d}s \\
= &\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1 - U_{(i)} \vee U_{(j)}\right) - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\left(1 - U_{(i)}^2\right) \\
= &\frac{1}{2} - \frac{\overline{U^1}}{n} - \frac{2V_1}{n^2} + \frac{\overline{U^2}}{2}.
\end{aligned}$$

Putting the results together, the assertion follows. □

*Proof* of Theorem 4: Throughout the proof, an unspecified integral is over $[0,\infty)$. Since $\Delta(F,\mathcal{E}) = \int (F(x) - E_\mu(x))^2 \mathrm{d}E_\mu(x)$, we have $V_n = A_n + B_n$, where

$$\begin{aligned}
A_n &= \sqrt{n}\left(\int \left(F_n(x) - E_{\overline{X}_n}(x)\right)^2 \mathrm{d}E_{\overline{X}_n}(x) - \int \left(F(x) - E_{\overline{X}_n}(x)\right)^2 \mathrm{d}E_{\overline{X}_n}(x)\right), \\
B_n &= \sqrt{n}\left(\int \left(F(x) - E_{\overline{X}_n}(x)\right)^2 \mathrm{d}E_{\overline{X}_n}(x) - \int \left(F(x) - E_\mu(x)\right)^2 \mathrm{d}E_\mu(x)\right).
\end{aligned}$$

Since

$$\sqrt{n}\sup_{x\in\mathbb{R}}|F_n(x) - F(x)|^2 = o_{\mathbb{P}}(1) \quad \text{as } n \to \infty, \tag{7.3}$$

we have

$$\begin{aligned}
A_n &= 2\sqrt{n}\left(\int \left(F_n(x) - F(x)\right)\left(F(x) - E_{\overline{X}_n}(x)\right) \mathrm{d}E_{\overline{X}_n}(x)\right) \\
&\quad + \sqrt{n}\left(\int (F_n(x) - F(x))^2 \,\mathrm{d}E_{\overline{X}_n}(x)\right) \\
&= 2\sqrt{n}\left(\int \left(F_n(x) - F(x)\right)\left(F(x) - E_{\overline{X}_n}(x)\right) \mathrm{d}E_{\overline{X}_n}(x)\right) + o_{\mathbb{P}}(1).
\end{aligned}$$

Again using (7.3) and the fact that

$$\lim_{n\to\infty}\int |E_\mu(x) - E_{\overline{X}_n}(x)|\,\mathrm{d}E_{\overline{X}_n}(x) = 0$$

$\mathbb{P}$-almost surely, we obtain

$$A_n = 2\sqrt{n}\left(\int \left(F_n(x) - F(x)\right)\left(F(x) - E_\mu(x)\right) \mathrm{d}E_{\overline{X}_n}(x)\right) + o_{\mathbb{P}}(1).$$

Since $\overline{X}_n \to \mu$ $\mathbb{P}$-a.s. we can apply Scheffé's theorem to obtain

$$\begin{aligned} A_n &= 2\sqrt{n}\left(\int (F_n(x) - F(x))\,(F(x) - E_\mu(x))\;\mathrm{d}E_\mu(x)\right) + o_{\mathbb{P}}(1) \\ &= 2\sqrt{n}\left(\int (F_n(x\mu) - F(x\mu))\,(F(x\mu) - E(x))\;\mathrm{d}E(x)\right) + o_{\mathbb{P}}(1). \end{aligned}$$

The treatment of $B_n$ is similar. We have $B_n = B_{1,n} + B_{2,n}$, where

$$\begin{aligned} B_{1,n} &= \sqrt{n}\int\left(F(x) - E_{\overline{X}_n}(x)\right)^2\,\mathrm{d}E_{\overline{X}_n}(x) - \sqrt{n}\int\left(F(x) - E_\mu(x)\right)^2\,\mathrm{d}E_{\overline{X}_n}(x) \\ &= 2\sqrt{n}\int\left(E_\mu(x) - E_{\overline{X}_n}(x)\right)\left(F(x) - E_\mu(x)\right)\,\mathrm{d}E_{\overline{X}_n}(x) + o_{\mathbb{P}}(1) \\ &= 2\sqrt{n}\int\left(E_\mu(x\overline{X}_n) - E(x)\right)\left(F(x\overline{X}_n) - E_\mu(x\overline{X}_n)\right)\,\mathrm{d}E(x) + o_{\mathbb{P}}(1) \\ &= 2\sqrt{n}\int\left(E_\mu(x\overline{X}_n) - E(x)\right)\left(F(x\mu) - E(x)\right)\,\mathrm{d}E(x) + o_{\mathbb{P}}(1) \end{aligned}$$

and

$$\begin{aligned} B_{2,n} &= \sqrt{n}\int\left(F(x) - E_\mu(x)\right)^2\,\mathrm{d}E_{\overline{X}_n}(x) - \sqrt{n}\int\left(F(x) - E_\mu(x)\right)^2\,\mathrm{d}E_\mu(x) \\ &= \sqrt{n}\int\left(\left(F(x\overline{X}_n) - E_\mu(x\overline{X}_n)\right)^2 - \left(F(x\mu) - E(x)\right)^2\right)\,\mathrm{d}E(x) \\ &= 2\sqrt{n}\int\left(\left(F(x\overline{X}_n) - F(x\mu) - \left(E_\mu(x\overline{X}_n) - E(x)\right)\right)\left(F(x\mu) - E(x)\right)\right)\mathrm{d}E(x) + o_{\mathbb{P}}(1). \end{aligned}$$

Thus,

$$\begin{aligned} B_n &= \sqrt{n}\int\left(F(x\overline{X}_n) - F(x\mu)\right)\left(F(x\mu) - E(x)\right)\,\mathrm{d}E(x) + o_{\mathbb{P}}(1) \\ &= 2\sqrt{n}(\overline{X}_n - \mu)\int_0^\infty xF'(x\mu)(F(x\mu) - E(x))e^{-x}\,\mathrm{d}x + o_{\mathbb{P}}(1). \end{aligned}$$

Remembering the definition of $\rho$ in (4.1) we recognize that $V_n \xrightarrow{\mathcal{D}} V$, where

$$V = 2\left(\int U(F(x\mu))(F(x\mu) - E(x))\exp(-x)\,\mathrm{d}x + Z\rho\right).$$

Here, $(U(t), 0 \le t \le 1, Z)$ is a Gaussian process with $(U(t), 0 \le t \le 1)$ being a Brownian bridge with continuous sample paths, and $Z$ is a centered normal variable with variance $\sigma^2$. Moreover, the covariance of $U(F(x\mu))$ and $Z$ is

$$\mathrm{Cov}(U(F(x\mu)), Z) = \int_0^{x\mu} y\,\mathrm{d}F(y) - \mu F(x\mu),\;\; x \ge 0.$$

The distribution of $V$ is the centered normal distribution stated in the theorem.

□

*Proof* of Theorem 5: Putting $\kappa := \mu\rho$ and

$$\eta := \frac{1}{\mu}\int_0^\infty \left( \int_0^{x\mu} y \,\mathrm{d}F(y) - \mu F(x\mu) \right) (F(x\mu) - E(x)) \,\mathrm{d}E(x),$$

$\tau^2(F)$ figuring in Theorem 4 can be written as

$$\tau^2(F) = 4\left[ J + 2\kappa\eta + \frac{\sigma^2}{\mu^2}\kappa^2 \right],$$

where

$$\begin{aligned}
J \;&:=\; \iint (F(x\mu) - E(x))\,(F(y\mu) - E(y))\,(F(\mu(x\wedge y)) - F(x\mu)F(y\mu))\,\mathrm{d}E(x)\mathrm{d}E(y) \\
&= \iint F(x\mu)F(y\mu)F(\mu(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y) \\
&= -2\iint E(x)F(y\mu)F(\mu(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y) \\
&= +\iint E(x)E(y)F(\mu(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y) \\
&= -\left[ \int (F(x\mu) - E(x))F(x\mu)\,\mathrm{d}E(x) \right]^2 \\
&=: J_1 - 2J_2 + J_3 - J_4^2 \quad \text{(say)}.
\end{aligned}$$

Replacing $F$ by the edf $F_n$ and $\mu$ by $\overline{X}_n$, an estimator of $J_1$ is

$$\begin{aligned}
\widehat{J}_1 &= \iint F_n(x\overline{X}_n)F_n(y\overline{X}_n)F_n(\overline{X}_n(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y) \\
&= \frac{1}{n^3}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \iint \mathbf{1}\{Y_{(i)} \le x\}\mathbf{1}\{Y_{(j)} \le y\}\mathbf{1}\{Y_{(k)} \le x\wedge y\}e^{-(x+y)}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Here, $Y_{(1)} \le \ldots \le Y_{(n)}$ are the order statistics of $Y_1, \ldots, Y_n$. By analogy with the reasoning in the proof of Proposition 1, we have

$$\begin{aligned}
\widehat{J}_1 &= \frac{1}{n^3}\sum_{i,j=1}^n \exp\left(-(Y_{(j)} + Y_{(i)} \vee Y_{(j)})\right) + \frac{1}{n^3}\sum_{i=1}^n\sum_{j<k} \exp\left(-(Y_{(k)} + Y_{(i)} \vee Y_{(k)})\right) \\
&= +\frac{1}{n^3}\sum_{i=1}^n\sum_{k<j} \exp\left(-(Y_{(j)} + Y_{(i)} \vee Y_{(k)})\right).
\end{aligned}$$

Likewise, it follows that

$$\begin{aligned}\widehat{J}_2 &= \iint E(x)F_n(y\overline{X}_n)F_n(\overline{X}_n(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y)\\ &= \frac{1}{n^2}\sum_{i=1}^n e^{-2Y_{(i)}}\left(1-\frac{1}{2}e^{-Y_{(i)}}\right)+\frac{1}{n^2}\sum_{j<i} e^{-2Y_{(i)}}\left(1-\frac{1}{2}e^{-Y_{(i)}}\right)\\ &\quad +\frac{1}{n^2}\sum_{i<j} e^{-(Y_{(i)}+Y_{(j)})}\left(1-\frac{1}{2}e^{-Y_{(i)}}\right).\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}\widehat{J}_3 &= \iint E(x)E(y)F_n(\overline{X}_n(x\wedge y))\,\mathrm{d}E(x)\mathrm{d}E(y)\\ &= \frac{1}{n}\sum_{i=1}^n\left[\int \mathbf{1}\{Y_{(i)}\le x\}(1-e^{-x})e^{-x}\,\mathrm{d}x\right]^2\\ &= \frac{1}{n}\sum_{i=1}^n\left[e^{-Y_{(i)}}\left(1-\frac{1}{2}e^{-Y_{(i)}}\right)\right]^2.\end{aligned}$$

Finally, an estimator of $J_4^2$ is

$$\widehat{J_4^2} = \left[\int (F_n(x\overline{X}_n)-E(x))F_n(x\overline{X}_n)\,\mathrm{d}E(x)\right]^2,$$

where

$$\widehat{J}_4 = \frac{1}{n^2}\sum_{i,j=1}^n e^{-Y_{(i)}\vee Y_{(j)}}-\frac{1}{n}\sum_{i=1}^n e^{-Y_{(i)}}\left(1-\frac{1}{2}e^{-Y_{(i)}}\right).$$

The estimator of $\kappa$ is

$$\begin{aligned}\widehat{\kappa} &= \frac{1}{\overline{X}_n}\int x\left(\exp\left(-\frac{x}{\overline{X}_n}\right)-(1-F_n(x))\right)\exp\left(-\frac{x}{\overline{X}_n}\right)\,\mathrm{d}F_n(x)\\ &= \frac{1}{n}\sum_{j=1}^n Y_{(j)}\left(\exp\left(-Y_{(j)}\right)-\left(1-\frac{j}{n}\right)\right)\exp\left(-Y_{(j)}\right),\end{aligned}$$

and the estimator of $\eta$ takes the form

$$\begin{aligned}\widehat{\eta} &= \frac{1}{\overline{X}_n}\int_0^\infty \left(\int_0^{x\overline{X}_n} y\mathrm{d}F_n(y) - \overline{X}_n F_n(x\overline{X}_n)\right)(F_n(x\overline{X}_n) - E(x))\,\mathrm{d}E(x)\\ &= \frac{1}{n}\sum_{j=1}^n \int_0^\infty \left(Y_{(j)} - 1\right)\mathbf{1}\{Y_{(j)} \le x\}\left(\frac{1}{n}\sum_{k=1}^n \mathbf{1}\{Y_{(k)} \le x\} - \left(1 - e^{-x}\right)\right) e^{-x}\,\mathrm{d}x\\ &= \frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n (Y_{(j)} - 1)\exp(-Y_{(j)} \vee Y_{(k)}) - \frac{1}{n}\sum_{j=1}^n (Y_{(j)} - 1)\left(1 - \frac{1}{2}e^{-Y_{(j)}}\right) e^{-Y_{(j)}}.\end{aligned}$$

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