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INVITED PAPER

Tests for multivariate normality—a critical review with emphasis on weighted L^2-statistics

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Abstract
This article gives a synopsis on new developments in affine invariant tests for multivariate normality in an i.i.d.-setting, with special emphasis on asymptotic properties of several classes of weighted L^2-statistics. Since weighted L^2-statistics typically have limit normal distributions under fixed alternatives to normality, they open ground for a neighborhood of model validation for normality. The paper also reviews several other invariant tests for this problem, notably the energy test, and it presents the results of a large-scale simulation study. All tests under study are implemented in the accompanying `R`-package `mnt`.

Keywords Test for multivariate normality · Weighted L^2-statistic · Affine invariance · Consistency

Mathematics Subject Classification 62H1510 · 62G20

1 Introduction

Testing for multivariate normality (for short MVN) is a topic of ongoing interest. A survey of dozens of MVN tests, including graphical procedures for assessing multivariate normality, is provided by Mecklin and Mundfrom (2004). The review of Henze (2002) concentrates on affine invariant and consistent procedures, and the book of Thode (2002) contains a chapter on testing for MVN.

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In a standard setting, let $X, X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) d-variate random (column) vectors, which are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The distribution of X will be denoted by $\mathbb{P}^X$. We write $\mathrm{N}_d(\mu, \Sigma)$ for the d-variate normal distribution with expectation μ and covariance matrix Σ, and we let

$$\mathcal{N}_d := \{\mathrm{N}_d(\mu, \Sigma) : \mu \in \mathbb{R}^d, \Sigma \text{ positive definite}\}$$

denote the class of all non-degenerate d-variate normal distributions. Testing for d-variate normality means testing the hypothesis

$$H_0 : \mathbb{P}^X \in \mathcal{N}_d,$$

against general alternatives, on the basis of $X_1, \ldots, X_n$. At the outset, it should be stressed that each model can merely hold approximately in practice. In particular, there can only be approximate normality, in whatever sense. Consequently, there is the following basic drawback inherent in any goodness-of-fit test, not only of H_0, but also of other families of distributions: If a level-α-test of H_0 does not lead to a rejection of H_0, the null hypothesis is by no means 'validated' or 'confirmed.' Presumably, there is merely not enough evidence to reject it! A further fundamental point is that there cannot be an optimal test of H_0, if one really wants to detect general alternatives. In this respect, Janssen (2000) shows that the global power function of *any* nonparametric test is flat on balls of alternatives, except for alternatives coming from a finite-dimensional subspace. Thus, loosely speaking, each test of H_0 has its own 'non-centrality.'

Regarding the task of reviewing MVN tests here in 2020, we cite Mecklin and Mundfrom (2004), who write 'the continuing proliferation of papers with new methods of assessing MVN makes it virtually impossible for any single survey article to cover all available tests.' And they continue: 'When compared to the amount of work that has been done in developing these tests, relatively little work has been done in evaluating the quality and power of the procedures.'

This review can also only be partial. We will take the above testing problem seriously and concentrate on genuine tests of H_0 that have been proposed since the review Henze (2002), and we will judge each of these according to the following points of view:

- affine invariance
- theoretical properties (limit distributions under H_0 and under fixed and contiguous alternatives to H_0, consistency)
- feasibility with respect to sample size and dimension.

Thus, e.g., we will not deal with tests for H_0 that allow for $n \le d$ (see Tan et al. 2005 or Yamada and Himeno 2019), since the condition $n \ge d + 1$ is necessary to decide whether the underlying covariance matrix is non-degenerate or not. Moreover, unlike the review of Mecklin and Mundfrom (2004), we will not discuss purely graphical procedures, as proposed in Holgersson (2006). We will also not embark upon a review of tests for normality in non-i.i.d.-settings, like testing for Gaussianity of the innovations in MGARCH processes (see, e.g., Lee and Ng 2011 or Lee et al. 2014), or situations with incomplete data (see, e.g., Yamada et al. 2015), since such a task would go beyond the scope of this review. We will also not review tests for Gaussianity in infinite-dimensional Hilbert spaces, see, e.g., Górecki et al. (2020) or Kellner and Celisse (2019).

Regarding affine invariance, notice that the class $\mathcal{N}_d$ is closed with respect to full rank affine transformations. Hence, any 'genuine' statistic $T_n = T_n(X_1, \ldots, X_n)$ (say) for testing H_0 should satisfy $T_n(AX_1 + b, \ldots, AX_n + b) = T_n(X_1, \ldots, X_n)$ for each regular $(d \times d)$-matrix A and each $b \in \mathbb{R}^d$. Otherwise, it would be possible to reject H_0 on given data and do not object against H_0 on the same data, after performing a rotation, which makes little, if any, sense. In the sequel, let

$$Y_{n,j} = S_n^{-1/2}(X_j - \overline{X}_n), \quad j = 1, \ldots, n, \tag{1.1}$$

denote the so-called *scaled residuals*. Here, $\overline{X}_n = n^{-1} \sum_{j=1}^n X_j$ is the sample mean, $S_n = n^{-1} \sum_{j=1}^n (X_j - \overline{X}_n)(X_j - \overline{X}_n)^\top$ stands for the sample covariance matrix of $X_1, \ldots, X_n$, and the superscript $\top$ denotes transposition of column vectors. The matrix $S_n^{-1/2}$ is the unique symmetric square root of S_n^{-1}. The latter matrix exists almost surely if $n \geq d + 1$ and $\mathbb{P}^X$ is absolutely continuous with respect to d-dimensional Lebesgue measure, see Eaton and Perlman (1973). These assumptions will be standing in what follows. We remark that S_n is sometimes defined with the factor $(n-1)^{-1}$ instead of n^{-1}, but this difference does not have implications for asymptotic considerations. A good account on finite-sample distribution theory of $Y_{n,1}, \ldots, Y_{n,n}$ under H_0 is provided by Takeuchi (2020).

Affine invariance is achieved if the test statistic T_n is a function of $Y_{n,i}^\top Y_{n,j}$, $i, j \in \{1, \ldots, n\}$, or if T_n is a function of (only) $Y_{n,1}, \ldots, Y_{n,n}$, and $T_n(OY_{n,1}, \ldots, OY_{n,n}) = T_n(Y_{n,1}, \ldots, Y_{n,n})$ for each orthogonal $(d \times d)$-matrix O. If a statistic T_n is affine invariant (henceforth *invariant* for the sake of brevity), the distribution of T_n under the null hypothesis H_0 does not depend on the parameters μ and Σ of the underlying normal distribution. Thus, regarding distribution theory under H_0, we can without loss of generality assume that $\mathbb{P}^X = \mathrm{N}_d(0, \mathrm{I}_d)$. Here, 0 is the origin in $\mathbb{R}^d$, and I_d is the unit matrix of order d. But invariance of a statistic T_n also entails that it is no restriction to assume $\mathbb{E}X = 0$ and $\mathbb{E}XX^\top = \mathrm{I}_d$ when studying the distribution of T_n under an alternative to H_0 that satisfies $\mathbb{E}\|X\|^2 < \infty$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

As for the second point, i.e., properties of a test of H_0 based on a statistic T_n that go beyond mere simulation results, there should be a sound rationale for the test, which means that there should be good knowledge of what is estimated by T_n if the underlying distribution is not normal. This rationale is intimately connected to the property of consistency. If T_n is some invariant statistic, it must be regarded—perhaps after some suitable normalization—as an estimator of some invariant functional $\mathcal{T}(P)$ of the unknown underlying distribution P, where $P = \mathbb{P}^X$. This means that $\mathcal{T}(P) = \mathcal{T}(\widetilde{P})$ if $\widetilde{P}$ is a full rank affine image of P, whence $\mathcal{T}(\cdot)$ is constant over the class $\mathcal{N}_d$. For such a functional, consistency of a test based on $\mathcal{T}$ against general alternatives can not be expected if $\mathcal{T}$ does not characterize the class $\mathcal{N}_d$, in the sense that there are $P_1 \in \mathcal{N}_d$ and $P_2 \notin \mathcal{N}_d$ such that $\mathcal{T}(P_1) = \mathcal{T}(P_2)$. Examples of non-characterizing functionals are time-honored measures of multivariate skewness and kurtosis, see Sect. 8. The most prominent of this group of tests is Mardia's invariant nonnegative skewness functional

$$\mathcal{T}(P) = \beta_d^{(1)}(P) = \mathbb{E}\Big[\Big((X_1 - \mu)^\top \Sigma^{-1}(X_2 - \mu)\Big)^3\Big]. \tag{1.2}$$

Here, X_1, X_2 are i.i.d. with distribution P, mean μ and non-singular covariance matrix Σ. The functional $\beta_d^{(1)}$ does not characterize the class $\mathcal{N}_d$ since it does not only vanish on $\mathcal{N}_d$, but in particular also for each non-normal elliptically symmetric distribution for which the expectation figuring in (1.2) exists. This fact has striking consequences for a standard test of H_0 that rejects H_0 for large values of the sample counterpart of $\beta_d^{(1)}$, see Sect. 8.

The paper is organized as follows: Sect. 2 gives a thorough account on general aspects of weighted L^2-statistics for testing H_0, and besides the class of BHEP tests, it reviews five recently proposed tests for multivariate normality that are based on either the characteristic function, the moment generating function, or a combination thereof. Section 3 reviews the Henze–Zirkler test with bandwidth depending on sample size and dimension, which is not a weighted L^2-statistic in the sense of Sect. 2. In Sect. 4, we summarize the most important features of the meanwhile well-established energy test of Székely and Rizzo (2005), and Section 5 deals with the test of Pudelko (2005). Section 6 reviews new theoretical results on a time-honored test of Cox and Small (1978), while Sect. 7 considers the test of Manzotti and Quiroz (2001), which is based on functions of spherical harmonics. In Sect. 8, we review tests based on skewness and kurtosis, and in Sect. 9, we try to give a brief account on further work on the subject. Section 10 presents the results of a large-scale simulation study that comprises each of the tests treated in Sects. 2–8. The final Sect. 11 draws some conclusions, and it gives an outlook for further research.

We conclude this section by pointing out some general notation. Throughout the paper, $\mathcal{B}^d$ stands for the σ-field of Borel sets in $\mathbb{R}^d$, $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ is the surface of the unit sphere in $\mathbb{R}^d$, and $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution. The symbol $\xrightarrow{\mathcal{D}}$ stands for convergence in distribution of random elements (variables, vectors and processes), and $\xrightarrow{\mathbb{P}}$, $\xrightarrow{\text{a.s.}}$ denote convergence in probability and almost sure convergence, respectively. Each limit refers to the setting $n \to \infty$. The symbol $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Throughout the paper, each unspecified integral will be over $\mathbb{R}^d$. The acronyms (E)MGF and (E)CF stand for the (empirical) moment generating function and the (empirical) characteristic function, respectively. Finally, we write $\mathbf{1}\{A\}$ for the indicator function of an event A.

2 Weighted L^2-statistics

In this chapter, we review the state of the art of weighted L^2-statistics for testing H_0. These statistics have a long history, and they are also in widespread use for goodness-of-fit problems with many other distributions, see, e.g., Baringhaus et al. (2017). A weighted L^2-statistic for testing H_0 takes the form

$$T_n = \int Z_n^2(t) w(t) \,\mathrm{d}t. \tag{2.1}$$

Here, $Z_n(t) = z_n(X_1, \ldots, X_n, t)$, z_n is a real-valued measurable function defined on the $(n+1)$-fold Cartesian product of $\mathbb{R}^d$, and $w : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative weight function satisfying

$$\int z_n^2(x_1, \ldots, x_n, t) w(t)\, \mathrm{d}t < \infty \quad \text{for each } (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n.$$

The function z_n can also be vector-valued; then $Z_n^2(t)$ in (2.1) is replaced with $\|Z_n(t)\|^2$. Typically, $Z_n(t)$ takes the form

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \ell\left(t^\top Y_{n,j}\right), \quad t \in \mathbb{R}^d, \tag{2.2}$$

where $\ell(\cdot)$ is some measurable function satisfying $\int \mathbb{E}\big[\ell^2(t^\top X)\big] w(t)\, \mathrm{d}t < \infty$, and $\mathbb{E}\big[\ell(t^\top X)\big] = 0$, $t \in \mathbb{R}^d$, if $X \overset{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$.

In view of (2.1), a natural setting to study asymptotic properties of T_n is the separable Hilbert space $\mathbb{H} := \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^d, w(t)\mathrm{d}t)$ of (equivalence classes) of measurabe functions on $\mathbb{R}^d$ that are square-integrable with respect to $w(t)\mathrm{d}t$. If $\|f\|_{\mathbb{H}} := \left(\int f^2(t) w(t)\, \mathrm{d}t\right)^{1/2}$ denotes the norm of $f \in \mathbb{H}$, then $T_n = \|Z_n\|_{\mathbb{H}}^2$. The general approach to derive the limit distribution of T_n under H_0 is to prove $Z_n \xrightarrow{\mathcal{D}} Z$ for some centered Gaussian random element of $\mathbb{H}$, whence $T_n \xrightarrow{\mathcal{D}} \|Z\|_{\mathbb{H}}^2$ by the continuous mapping theorem. To this end, it is indispensable to approximate Z_n figuring in (2.2) by a suitable random element $Z_{n,0}$ of $\mathbb{H}$ of the form

$$Z_{n,0}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \ell_0(t^\top X_j), \tag{2.3}$$

where $\mathbb{E}[\ell_0(t^\top X)] = 0, t \in \mathbb{R}^d$, $\int \mathbb{E}[\ell_0^2(t^\top X)] w(t)\mathrm{d}t < \infty$, and $\|Z_n - Z_{n,0}\|_{\mathbb{H}} \xrightarrow{\mathbb{P}} 0$. The central limit theorem in Hilbert spaces (see, e.g., Theorem 2.7 in Bosq 2000) then yields $Z_{n,0} \xrightarrow{\mathcal{D}} Z$ for some centered Gaussian element of $\mathbb{H}$ having covariance kernel

$$K(s,t) = \mathbb{E}\big[\ell_0(s, X)\ell_0(t, X)\big], \quad s, t \in \mathbb{R}^d.$$

The distribution of Z is uniquely determined by the kernel $K(\cdot,\cdot)$, and the distribution of $\|Z\|_{\mathbb{H}}^2$ is that of $\sum_{j=1}^{\infty} \lambda_j N_j^2$, where the N_j are i.i.d. standard normal random variables, and λ_j, $j = 1, 2, \ldots$, are the positive eigenvalues corresponding to eigenfunctions f_j of the (linear second-order homogeneous Fredholm) integral equation

$$\lambda f(s) = \int K(s,t) f(t) w(t)\, \mathrm{d}t, \quad s \in \mathbb{R}^d, \tag{2.4}$$

see, e.g., Kac and Siegert (1947). The problem of finding the eigenvalues and associated eigenfunctions of (2.4) is called the *kernel eigenproblem*. In this respect, hitherto none of the integral equations corresponding to the test presented in this section has been solved explicitly. Notice that knowledge of the largest eigenvalue λ_{max} (say) opens ground for the calculation of the approximate Bahadur slope and hence for statements on the Bahadur efficiency which, for asymptotically normal statistics, typically coincides with the Pitman efficiency, for details see Bahadur (1960) and Nikitin (1995).

To find a random element $Z_{n,0}$ of the form (2.3) that approximates Z_n, one has to evaluate the effect of replacing $Y_{n,j}$ in (2.2) with X_j. Putting $\Delta_{n,j} = Y_{n,j} - X_j$, $j = 1, \ldots, n$, the following result, taken from Dörr et al. (2020), is helpful.

Proposition 1 *Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors satisfying $\mathbb{E}\|X\|^4 < \infty$, $\mathbb{E}(X) = 0$ and $\mathbb{E}(XX^\top) = \mathrm{I}_d$. We then have*

$$\sum_{j=1}^{n} \|\Delta_{n,j}\|^2 = O_{\mathbb{P}}(1), \quad \frac{1}{n}\sum_{j=1}^{n} \|\Delta_{n,j}\|^2 \xrightarrow{a.s.} 0, \quad \max_{j=1,\ldots,n} \|\Delta_{n,j}\| = o_{\mathbb{P}}\left(n^{-1/4}\right).$$

Since $\ell(t^\top Y_{n,j}) = \ell(t^\top X_j + t^\top \Delta_{n,j})$, the function $\ell(\cdot)$ must be smooth enough to allow for a Taylor expansion. To tackle the linear part in this expansion, it is crucial to have some information on $\Delta_{n,j} = (S_n^{-1/2} - \mathrm{I}_d)X_j - S_n^{-1/2}\overline{X}_n$. Such information is provided by display (2.13) of Henze and Wagner (1997), according to which

$$\sqrt{n}(S_n^{-1/2} - \mathrm{I}_d) = -\frac{1}{2\sqrt{n}}\sum_{j=1}^{n}\left(X_j X_j^\top - \mathrm{I}_d\right) + O_{\mathbb{P}}\left(n^{-1/2}\right).$$

Since Proposition 1 holds under general assumptions, one may often obtain asymptotic normality of weighted L^2-statistics under fixed alternatives. To this end, notice that

$$\frac{T_n}{n} = \int \left(\frac{1}{n}\sum_{j=1}^{n} \ell(t^\top Y_{n,j})\right)^2 w(t)\,\mathrm{d}t.$$

Under suitable conditions, we will have $T_n/n \xrightarrow{\mathbb{P}} \Delta$, where $\Delta = \|z\|^2_{\mathbb{H}}$, and $z(t) = \mathbb{E}\big[\ell(t^\top X)\big]$, $z \in \mathbb{R}^d$. An immediate consequence of this stochastic convergence is the consistency of a test for H_0 based on T_n against each alternative distribution that satisfies $\Delta > 0$. But we have more! Writing $\langle u, v\rangle_{\mathbb{H}} = \int u(t)v(t)w(t)\,\mathrm{d}t$ for the inner product in $\mathbb{H}$, there is the decomposition

$$\begin{aligned}\sqrt{n}\left(\frac{T_n}{n} - \Delta\right) &= \sqrt{n}\left(\|Z_n\|^2_{\mathbb{H}} - \|z\|^2_{\mathbb{H}}\right)\\ &= \sqrt{n}\big\langle Z_n - z, Z_n + z\big\rangle_{\mathbb{H}}\\ &= \sqrt{n}\big\langle Z_n - z, 2z + Z_n - z\big\rangle_{\mathbb{H}}\\ &= 2\big\langle \sqrt{n}(Z_n - z), z\big\rangle_{\mathbb{H}} + \frac{1}{\sqrt{n}}\|\sqrt{n}(Z_n - z)\|^2_{\mathbb{H}}.\end{aligned}$$

These lines carve out the quintessence of asymptotic normality of weighted L^2-statistics under fixed alternatives. Namely, if one can show that the sequence $V_n := \sqrt{n}(Z_n - z)$ of random elements of $\mathbb{H}$ converges in distribution to some centered Gaussian random element V of $\mathbb{H}$, then, by the continuous mapping theorem and Slutski's lemma, we have

$$\sqrt{n}\left(\frac{T_n}{n} - \Delta\right) \stackrel{\mathcal{D}}{\longrightarrow} \mathrm{N}(0, \sigma^2), \tag{2.5}$$

where

$$\sigma^2 = 4 \iint K(s,t) z(s) z(t) w(s) w(t)\, \mathrm{d}s \mathrm{d}t,$$

and $K(\cdot,\cdot)$ is the covariance kernel of V, see Theorem 1 of Baringhaus et al. (2017). As a consequence, if $\widehat{\sigma}_n^2$ is a consistent estimator of σ^2 based on $X_1, \ldots, X_n$, then, for given $\alpha \in (0,1)$,

$$I_{n,1-\alpha} = \left[\frac{T_n}{n} - \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\frac{\widehat{\sigma}_n}{\sqrt{n}}, \frac{T_n}{n} + \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\frac{\widehat{\sigma}_n}{\sqrt{n}}\right] \tag{2.6}$$

is an asymptotic confidence interval for Δ of level $1 - \alpha$. Moreover, from (2.5) and Slutski's lemma, we have

$$\frac{\sqrt{n}}{\widehat{\sigma}_n}\left(\frac{T_n}{n} - \Delta\right) \stackrel{\mathcal{D}}{\longrightarrow} \mathrm{N}(0, 1), \tag{2.7}$$

which opens the ground for a validation of a certain neighborhood of H_0. Namely, suppose that we want to tolerate a given 'distance' Δ_0 to the class $\mathcal{N}_d$. We may then consider the 'inverse' testing problem

$$H_{\Delta_0} : \Delta(\mathbb{P}^X) \geq \Delta_0 \ \text{ against } \ K_{\Delta_0} : \Delta(\mathbb{P}^X) < \Delta_0.$$

Here, the dependence of Δ on the underlying distribution $\mathbb{P}^X$ has been made explicit.

From (2.7), the test which rejects H_{Δ_0} if

$$\frac{T_n}{n} \leq \Delta_0 - \frac{\widehat{\sigma}_n}{\sqrt{n}} \Phi^{-1}(1 - \alpha),$$

has asymptotic level α, and it is consistent against general alternatives, see Section 3.3 of Baringhaus et al. (2017). Notice that this test is in the spirit of bioequivalence testing (see, e.g., Czado et al. 2007; Dette and Munk 2003 or Wellek 2010), since it aims at validating a certain neighborhood of a hypothesized model.

We now review the time-honored class of BHEP tests and several recently suggested L^2-statistics for testing H_0. Each of these statistics has an upper rejection region, and it is invariant, because it is a function of $Y_{n,j}^\top Y_{n,k}$, where $j, k \in \{1, \ldots, n\}$.

2.1 The BHEP tests

Generalizing a test for univariate normality based on the ECF due to Epps and Pulley (1983), the first proposals for weighted L^2-statistics for testing H_0 are due to Baringhaus and Henze (1988) and Henze and Zirkler (1990), who considered the statistic

$$\mathrm{BHEP}_{n,\beta} = n \int \left|\Psi_n(t) - \Psi_0(t)\right|^2 w_\beta(t)\,\mathrm{d}t. \tag{2.8}$$

Here,

$$\Psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(\mathrm{i}t^\top Y_{n,j}), \qquad t \in \mathbb{R}^d, \tag{2.9}$$

denotes the ECF of $Y_{n,1}, \ldots, Y_{n,n}$, $\Psi_0(t) = \exp(-\|t\|^2/2)$ is the CF of the distribution $\mathrm{N}_d(0, \mathrm{I}_d)$, and the weight function w_β is given by

$$w_\beta(t) = \left(2\pi\beta^2\right)^{-d/2} \exp\left(-\frac{\|t\|^2}{2\beta^2}\right), \tag{2.10}$$

where $\beta > 0$ is a fixed constant. That $\mathrm{BHEP}_{n,\beta}$ is indeed of the type (2.1) will become clear from the representation (2.13).

Whereas Baringhaus and Henze (1988) studied the special case $\beta = 1$, the general case was treated by Henze and Zirkler (1990). An extremely appealing feature of the weight function w_β in (2.10) is that $\mathrm{BHEP}_{n,\beta}$ takes the feasible form

$$\begin{aligned}\mathrm{BHEP}_{n,\beta} &= \frac{1}{n}\sum_{j,k=1}^{n} \exp\left(-\frac{\beta^2\|Y_{n,j} - Y_{n,k}\|^2}{2}\right) \\ &\quad - \frac{2}{(1+\beta^2)^{d/2}}\sum_{j=1}^{n} \exp\left(-\frac{\beta^2\|Y_{n,j}\|^2}{2(1+\beta^2)}\right) + \frac{n}{(1+2\beta^2)^{d/2}}.\end{aligned} \tag{2.11}$$

The BHEP test is the most thoroughly studied class of tests for multivariate normality. Csörgő (1989) coined the acronym BHEP for this class of tests for H_0, after early developers of the idea, and he proved that $\liminf_{n\to\infty} n^{-1}\mathrm{BHEP}_{n,\beta} \ge C(\mathbb{P}^X, \beta) > 0$ almost surely for some constant $C(\mathbb{P}^X, \beta)$ if $\mathbb{P}^X$ does not belong to $\mathcal{N}_d$. As a consequence, a test for normality based on $\mathrm{BHEP}_{n,\beta}$ is consistent against any alternative.

If $\mathbb{E}\|X\|^2 < \infty$ and $\mathbb{E}X = 0$, $\mathbb{E}XX^\top = \mathrm{I}_d$ (the last two assumptions entail no loss of generality in view of invariance), then

$$\frac{1}{n}\mathrm{BHEP}_{n,\beta} \xrightarrow{\text{a.s.}} \Delta_\beta := \int \left|\Psi(t) - \Psi_0(t)\right|^2 w_\beta(t)\,\mathrm{d}t \tag{2.12}$$

(Baringhaus and Henze 1988), where $\Psi(t) = \mathbb{E}\exp(\mathrm{i}t^\top X)$, $t \in \mathbb{R}^d$, is the CF of X. Hence, $\Delta_\beta = \Delta_\beta(\mathbb{P}^X)$ is the functional associated with the BHEP test. Using a Hilbert space setting, Gürtler (2000) proved (2.5) for $T_n = \text{BHEP}_{n,\beta}$, where $\Delta = \Delta_\beta$ and $\sigma^2 = \sigma_\beta^2$ depend on β, under each alternative distribution satisfying $\mathbb{E}\|X\|^4 < \infty$. Moreover, Gürtler (2000) obtained a sequence $\widehat{\sigma}_{n,\beta}^2$ of consistent estimators of σ_β^2 and thus an asymptotic confidence interval of the type (2.6).

In view of the representation (2.11), Baringhaus and Henze (1988) and Henze and Zirkler (1990) obtained the limit null distribution of $\text{BHEP}_{n,\beta}$ as $n \to \infty$ by means of the theory of V-statistics with estimated parameters. Upon observing that

$$\text{BHEP}_{n,\beta} = \int Z_n^2(t)\, w_\beta(t)\, \mathrm{d}t, \tag{2.13}$$

where $Z_n(t) = n^{-1/2} \sum_{j=1}^n \big(\cos(t^\top Y_{n,j}) + \sin(t^\top Y_{n,j}) - \Psi_0(t)\big)$, Henze and Wagner (1997) considered $Z_n(\cdot)$ as a random element in a certain Fréchet space of random functions, and they showed that Z_n converges in distribution in that space to some centered Gaussian random element Z, see Theorem 2.1 of Henze and Wagner (1997). Moreover, $\text{BHEP}_{n,\beta} \xrightarrow{\mathcal{D}} \int Z^2(t)\, w_\beta(t)\, \mathrm{d}t$, and the test is able to detect a sequence of contiguous alternatives that approach H_d at the rate $n^{-1/2}$. Henze and Wagner (1997) also obtained the first three moments of the limit null distribution of $\text{BHEP}_{n,\beta}$. Finally, the class of BHEP tests is 'closed at the boundaries' $\beta \to 0$ and $\beta \to \infty$ since, elementwise on the underlying probability space, we have

$$\lim_{\beta \to 0} \frac{\text{BHEP}_{n,\beta}}{\beta^6} = \frac{n}{6} \cdot b_{n,d}^{(1)} + \frac{n}{4} \cdot \widetilde{b}_{n,d}^{(1)}, \tag{2.14}$$

where $b_{n,d}^{(1)}$ and $\widetilde{b}_{n,d}^{(1)}$ are given in (8.1) and (8.3), respectively, see Henze (1997b). Thus, as $\beta \to 0$, a scaled version of $\text{BHEP}_{n,\beta}$ is approximately a linear combination of two measures of multivariate skewness. The limit distribution of the right-hand side of (2.14) under general distributional assumptions on X has been studied by Henze (1997b). Last but not least, we have

$$\lim_{\beta \to \infty} \beta^d \left(\text{BHEP}_{n,\beta} - 1\right) = \frac{n}{2^{d/2}} - 2\sum_{j=1}^n \exp\left(-\frac{\|Y_{n,j}\|^2}{2}\right), \tag{2.15}$$

see Henze (1997b). Hence, as $\beta \to \infty$, rejection of H_0 for large values of $\text{BHEP}_{n,\beta}$ means rejection of H_0 for *small* values of $\sum_{j=1}^n \exp(-\|Y_{n,j}\|^2/2)$. The latter statistic, like Mardia's measure of multivariate kurtosis $b_{n,d}^{(2)}$ (see (8.1)), merely investigates an aspect of the 'radial part' of the underlying distribution.

Guided by theoretical and simulation based results in the univariate case, Tenreiro (2009) performed an extensive simulation study on the power of the BHEP test for dimensions $d \in \{2, 3, \ldots, 10, 12, 15\}$ and sample sizes $n \in \{20, 40, 60, 80, 100\}$. He concluded that the choice $\beta = 0.5$ gives 'the best results for long tailed or moderately skewed alternatives, but it also produces very poor results for short tailed alternatives.' If no relevant information about the tail of the alternatives is available, he strongly recommends the use of $\beta = \sqrt{2}/(1.376 + 0.075d)$ (in fact, his recommendation is in terms of $h = 1/(\beta\sqrt{2}))$, and there are similar recommendations for short tailed alternatives and long tailed or moderately skewed alternatives, respectively.

2.2 A weighted L^2-statistic via the moment generating function

Henze and Jiménez-Gamero (2019) generalized results of Henze and Koch (2020) to the multivariate case and considered a MGF analogue to the BHEP-test statistic. Letting

$$M_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp\left(t^\top Y_{n,j}\right), \quad t \in \mathbb{R}^d, \tag{2.16}$$

denote the EMGF of $Y_{n,1}, \ldots, Y_{n,n}$, and writing $M_0(t) = \exp(\|t\|^2/2)$, $t \in \mathbb{R}^d$, for the MGF of the standard normal distribution $\mathrm{N}_d(0, \mathrm{I}_d)$, the test statistic is

$$\mathrm{HJ}_{n,\gamma} = n \int (M_n(t) - M_0(t))^2 \, \widetilde{w}_\gamma(t)\,\mathrm{d}t, \tag{2.17}$$

where

$$\widetilde{w}_\gamma(t) = \exp\left(-\gamma \|t\|^2\right), \tag{2.18}$$

and $\gamma > 2$ is some fixed parameter. Notice that the condition $\gamma > 1$ is necessary for the integral in (2.17) to be finite, and the more stringent condition $\gamma > 2$ is needed for asymptotics under H_0.

The test statistic $\mathrm{HJ}_{n,\gamma}$ has a representation analogous to (2.11) (see display (1.4) of Henze and Jiménez-Gamero 2019). Elementwise on the underlying probability space, we have

$$\lim_{\gamma\to\infty} \gamma^{3+d/2} \frac{6\mathrm{HJ}_{n,\gamma}}{\pi^{d/2}} = \frac{n}{6}\cdot b_{n,d}^{(1)} + \frac{n}{4}\widetilde{b}_{n,d}^{(1)} \tag{2.19}$$

which, interestingly, is the same limit as in (2.14). By working in the Hilbert space $\mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^d, \widetilde{w}_\gamma(t)\mathrm{d}t)$ of (equivalence classes) of measurabe functions on $\mathbb{R}^d$ that are square-integrable with respect to $\widetilde{w}_\gamma(t)\mathrm{d}t$, Henze and Jiménez-Gamero (2019) derived the limit null distribution of $\mathrm{HJ}_{n,\gamma}$, which is that of $\mathrm{HJ}_{\infty,\gamma} := \int W^2(t)\widetilde{w}_\gamma(t)\,\mathrm{d}t$, where W is some centered Gaussian random element of that space. Henze and Jiménez-Gamero (2019) also obtained the expectation and the variance of $\mathrm{HJ}_{\infty,\gamma}$. Moreover, if X is a (standardized) alternative distribution with the property $M(t) := \mathbb{E}(\exp(t^\top X)) < \infty$, $t \in \mathbb{R}^d$, then

$$\liminf_{n\to\infty} \frac{\mathrm{HJ}_{n,\gamma}}{n} \geq \int (M(t) - M_0(t))^2 \, \widetilde{w}_\gamma(t)\,\mathrm{d}t \qquad \mathbb{P}\text{-almost surely.} \tag{2.20}$$

This inequality implies the consistency of the MVN test based on $\mathrm{HJ}_{n,\gamma}$ against those alternatives that have a finite MGF. Indeed, one may conjecture that this test is consistent against *any* alternative to H_0.

2.3 A test based on a characterization involving the MGF and the CF

Volkmer (2014) proved a characterization of the univariate centered normal distribution, which involves both the CF and the MGF. Henze et al. (2019) generalized this result as follows: If X is a centered d-variate non-degenerate random vector with MGF $M(t) = \mathbb{E}[\exp(t^\top X)] < \infty$, $t \in \mathbb{R}^d$, and $R(t) := \mathbb{E}[\cos(t^\top X)]$ denotes the real part of the CF of X, then

$$R(t)\, M(t) - 1 \; = \; 0 \quad \text{for each } t \in \mathbb{R}^d \tag{2.21}$$

holds true if and only if X follows some zero-mean normal distribution.

Since $Y_{n,1}, \ldots, Y_{n,n}$ provide an empirical standardization of $X_1, \ldots, X_n$, a natural test statistic based on (2.21) is

$$\mathrm{HJM}_{n,\gamma} \; := \; n \int \left(R_n(t)\, M_n(t) - 1\right)^2 \, \widetilde{w}_\gamma(t)\, \mathrm{d}t,$$

where

$$R_n(t) \; := \; \frac{1}{n} \sum_{j=1}^{n} \cos\left(t^\top Y_{n,j}\right), \quad t \in \mathbb{R}^d,$$

is the empirical cosine transform of the scaled residuals, and $M_n(t)$ and $\widetilde{w}_\gamma(t)$ are given in (2.16) and (2.18), respectively. There is a representation of $\mathrm{HJM}_{n,\gamma}$ similar to (2.11), but involving a fourfold sum (see display (3.7) of Henze et al. 2019). The main results about $\mathrm{HJM}_{n,\gamma}$ are as follows: Elementwise on the underlying probability space, we have

$$\lim_{\gamma \to \infty} \gamma^{3+d/2} \, \frac{8\mathrm{HJM}_{n,\gamma}}{\pi^{d/2}} = \frac{n}{6} \cdot b_{n,d}^{(1)} + \frac{n}{4} \cdot \widetilde{b}_{n,d}^{(1)}.$$

Interestingly, this is the same linear combination of two measures of skewness as in (2.14) and (2.19). If $\gamma > 1$, then the limit null distribution of $\mathrm{HJM}_{n,\gamma}$ is that of $\mathrm{HJM}_{\infty,\gamma} := \int W^2(t) \widetilde{w}_\gamma(t)\, \mathrm{d}t$, where W is a centered random element of the Hilbert space $\mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^d, \widetilde{w}(t)\mathrm{d}t)$ with a covariance kernel given in Theorem 5.1 of Henze et al. (2019). Moreover, that paper also states a formula for $\mathbb{E}[\mathrm{HJM}_{\infty,\gamma}]$ and, under the assumption $M(t) < \infty$, $t \in \mathbb{R}^d$, obtains the inequality

$$\liminf_{n \to \infty} \frac{\mathrm{HJM}_{n,\gamma}}{n} \; \geq \; \int \left(R(t) M(t) - 1\right)^2 \, w_\gamma(t)\, \mathrm{dt} \quad \mathbb{P}\text{-almost surely}, \tag{2.22}$$

which is analogous to (2.20), see Theorem 6.1 of Henze et al. (2019). We conjecture that also the MVN test based on $\mathrm{HJM}_{n,\gamma}$ is consistent against *any* non-normal alternative distribution.

2.4 A test based on a system of partial differential equations for the MGF

The novel idea of Henze and Visagie (2020) for constructing a test of H_0 is the following: Suppose that the MGF $M(t) = \mathbb{E}[\exp(t^\top X)]$ of a random vector X exists for each $t \in \mathbb{R}^d$ and satisfies the system of partial differential equations

$$\frac{\partial M(t)}{\partial t_j} = t_j M(t), \quad t = (t_1, \ldots, t_d)^\top \in \mathbb{R}^d, \quad j = 1, \ldots, d. \tag{2.23}$$

Since $M(0) = 1$, it is easily seen that the only solution to (2.23) is $M_0(t) = \exp(\|t\|^2/2)$, $t \in \mathbb{R}^d$, which is the MGF of $\mathrm{N}_d(0, \mathrm{I}_d)$.

If H_0 holds, the scaled residuals $Y_{n,1}, \ldots, Y_{n,n}$ should be approximately independent, with a distribution close to $\mathrm{N}_d(0, \mathrm{I}_d)$, at least for large n. Hence, a natural approach for testing H_0 is to consider the EMGF M_n of $Y_{n,1}, \ldots, Y_{n,n}$, defined in (2.16), and to employ the weighted L^2-statistic

$$\mathrm{HV}_{n,\gamma} := n \int \|\nabla M_n(t) - t M_n(t)\|^2 \, \widetilde{w}_\gamma(t) \, \mathrm{d}t,$$

where ∇f stands for the gradient of a function $f : \mathbb{R}^d \to \mathbb{R}$, and $\widetilde{w}_\gamma$ is given in (2.18). Putting $Y^+_{n,j,k} = Y_{n,j} + Y_{n,k}$, $\mathrm{HV}_{n,\gamma}$ takes the feasible form

$$\mathrm{HV}_{n,\gamma} = \frac{1}{n}\left(\frac{\pi}{\gamma}\right)^{d/2} \sum_{j,k=1}^{n} \exp\left(\frac{\|Y^+_{n,j,k}\|^2}{4\gamma}\right) \left(Y_{n,j}^\top Y_{n,k} - \frac{\|Y^+_{n,j,k}\|^2}{2\gamma} + \frac{d}{2\gamma} + \frac{\|Y^+_{n,j,k}\|^2}{4\gamma^2}\right).$$

To derive the limit null distribution of $\mathrm{HV}_{n,\gamma}$, put $W_n(t) := \sqrt{n}\,(\nabla M_n(t) - t M_n(t))$. Since $W_n(t)$ is $\mathbb{R}^d$-valued, Henze and Visagie (2020) consider the Hilbert space $\mathbb{H}$, which is the d-fold (orthogonal) direct sum $\mathbb{H} := \mathrm{L}^2 \oplus \cdots \oplus \mathrm{L}^2$, where $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^d, \widetilde{w}(t)\mathrm{d}t)$. If $\gamma > 2$, there is some centered Gaussian random element W of $\mathbb{H}$ with a covariance (matrix) kernel given in display (11) of Henze and Visagie (2020), so that $W_n \xrightarrow{\mathcal{D}} W$ as $n \to \infty$. By the continuous mapping theorem, we then have $\mathrm{HV}_{n,\gamma} \xrightarrow{\mathcal{D}} \mathrm{HV}_{\infty,\gamma} := \int \|W(t)\|^2 \, \widetilde{w}_\gamma(t) \, \mathrm{d}t$. Henze and Visagie (2020) also obtain a closed form expression for $\mathbb{E}[T_{\infty,\gamma}]$. Moreover, if the MGF $M(t)$ of X exists for each $t \in \mathbb{R}^d$ and X is standardized, we have

$$\liminf_{n\to\infty} \frac{\mathrm{HV}_{n,\gamma}}{n} \geq \int \|\nabla M'(t) - t M(t)\|^2 \, \widetilde{w}_\gamma(t) \, \mathrm{d}t \quad \mathbb{P}\text{-almost surely,}$$

which parallels (2.20) and (2.22).

We remark in passing that a differential equation involving the moment generating function has been employed by Meintanis and Hlávka (2010) in connection with testing for bivariate and multivariate *skew*-normality.

2.5 A test based on the harmonic oscillator in characteristic function spaces

Dörr et al. (2020) noticed that the CF $\Psi_0(t) = \exp(-\|t\|^2/2)$ of the distribution $\mathrm{N}(0, \mathrm{I}_d)$ is the unique solution of the partial differential equation

$$\Delta f(x) - (\|x\|^2 - d) f(x) = 0 \tag{2.24}$$

subject to $f(0) = 1$, where Δ is the Laplace operator, see Theorem 1 of Dörr et al. (2020). The operator $-\Delta + \|x\|^2 - d$ is called the *harmonic oscillator*, which is a special case of a Schrödinger operator. A suitable statistic for testing H_0 that reflects this characterization is

$$\begin{aligned}\mathrm{DEH}_{n,\gamma} &= n\int_{\mathbb{R}^d} |\Delta\Psi_n(t) - \Delta\Psi_0(t)|^2\, \widetilde{w}_\gamma(t)\mathrm{d}t\\ &= n\int \left|\frac{1}{n}\sum_{j=1}^n \|Y_{n,j}\|^2 \exp(\mathrm{i}t^\top Y_{n,j}) + (\|t\|^2 - d)\Psi_0(t)\right|^2 \widetilde{w}_\gamma(t)\,\mathrm{d}t,\end{aligned} \tag{2.25}$$

where $\widetilde{w}_\gamma$ is given in (2.18) and $\gamma > 0$. The test statistic has the feasible form

$$\begin{aligned}\mathrm{DEH}_{n,\gamma} = &\left(\frac{\pi}{\gamma}\right)^{\frac{d}{2}} \frac{1}{n}\sum_{j,k=1}^n \|Y_{n,j}\|^2\|Y_{n,k}\|^2 \exp\left(-\frac{1}{4\gamma}\|Y_{n,j} - Y_{n,k}\|^2\right)\\ &- \frac{2(2\pi)^{\frac{d}{2}}}{(2\gamma+1)^{2+\frac{d}{2}}}\sum_{j=1}^n \|Y_{n,j}\|^2\left(\|Y_{n,j}\|^2 + 2d\gamma(2\gamma+1)\right)\exp\left(-\frac{1}{2}\frac{\|Y_{n,j}\|^2}{2\gamma+1}\right)\\ &+ n\frac{\pi^{\frac{d}{2}}}{(\gamma+1)^{2+\frac{d}{2}}}\left(\gamma(\gamma+1)d^2 + \frac{d(d+2)}{4}\right).\end{aligned}$$

Like the class of BHEP tests, also the class of tests based on $\mathrm{DEH}_{n,\gamma}$ is closed at the boundaries $\gamma \to 0$ and $\gamma \to \infty$, since—elementwise on the underlying probability space—we have

$$\lim_{\gamma\to 0}\left(\frac{\gamma}{\pi}\right)^{d/2}\mathrm{DEH}_{n,\gamma} = b_{n,d}^{(2)}, \quad \lim_{\gamma\to\infty}\frac{2}{n\pi^{\frac{d}{2}}}\gamma^{\frac{d}{2}+1}\mathrm{DEH}_{n,\gamma} = \widetilde{b}_{n,d}^{(1)}.$$

Here, $b_{n,d}^{(2)}$ is multivariate kurtosis in the sense of Mardia (1970), defined in (8.1), and $\widetilde{b}_{n,d}^{(1)}$ is skewness in the sense of Móri et al. (1993), see (8.3). Dörr et al. (2020) proved a Hilbert space central limit theorem for the sequence of random elements

$$V_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Big(\|Y_{n,j}\|^2 \big\{ \cos(t^\top Y_{n,j}) + \sin(t^\top Y_{n,j}) \big\} - \mu(t) \Big), \quad t \in \mathbb{R}^d,$$

where $\mu(t) = \mathbb{E}[\|X\|^2(\cos(t^\top X) + \sin(t^\top X))]$ and X is a standardized random vector satisfying $\mathbb{E}\|X\|^4 < \infty$. Since $\mu(t) = -\Delta\Psi_0(t)$ if $X \stackrel{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$ and $\mathrm{DEH}_{n,\gamma} = \int V_n^2(t)\widetilde{w}_\gamma(t)\,\mathrm{d}t$ for that choice of $\mu(t)$, the authors obtained the limit distribution of $\mathrm{DEH}_{n,\gamma}$ under H_0 as well as under contiguous and fixed alternatives to H_0. Under H_0, we have $\mathrm{DEH}_{n,\gamma} \stackrel{\mathcal{D}}{\longrightarrow} \int V^2(t)\widetilde{w}_\gamma(t)\,\mathrm{d}t$, where V is the centered limit Gaussian random element of the sequence (V_n) (with $\mu(t) = -\Delta\Psi_0(t)$). Under contiguous alternatives that approach H_0 at the rate $n^{-1/2}$, the limit distribution of $\mathrm{DEH}_{n,\gamma}$ is that of $\int(V(t) + c(t))^2\widetilde{W}_\gamma(t)\,\mathrm{d}t$, where $c(\cdot)$ is a shift function (see Section 6 of Dörr et al. 2020). Under a fixed (and because of invariance without loss of generality standardized) alternative distribution satisfying $\mathbb{E}\|X\|^4 < \infty$, we have

$$\frac{\mathrm{DEH}_{n,\gamma}}{n} \to \mathrm{D}_\gamma := \int \big|\Delta\Psi(t) - \Delta\Psi_0(t)\big|^2 \widetilde{w}_\gamma(t)\,\mathrm{d}t \quad \mathbb{P}\text{-almost surely},$$

where Ψ is the CF of X. Moreover, the limit distribution of $\sqrt{n}(\mathrm{DEH}_{n,\gamma}/n - \mathrm{D}_\gamma)$ is a centered normal distribution with a variance that, under the stronger condition $\mathbb{E}\|X\|^6 < \infty$, can be consistently estimated from the data. Thus, by analogy with (2.6), an asymptotic confidence interval for D_γ is available. Notice that, when compared with (2.12), the almost sure limits above are 'Laplacian analogues' of (2.12).

2.6 A test based on a double estimation in a characterizing PDE

Dörr et al. (2020a) suggested to replace *both* of the functions f occurring in (2.24) by the ECF Ψ_n. Since, under H_0, $\Delta\Psi_n(t)$ and $(\|t\|^2 - d)\Psi_n(t)$ should be close to each other for large n, it is tempting to see what happens if, instead of $\mathrm{DEH}_{n,\gamma}$ defined in (2.25), we base a test of H_0 on the weighted L^2-statistic

$$\mathrm{DEH}^*_{n,\gamma} = n \int \Big|\Delta\Psi_n(t) - \Big(\|t\|^2 - d\Big)\Psi_n(t)\Big|^2 \widetilde{w}_\gamma(t)\,\mathrm{d}t$$

and reject H_0 for large values of $\mathrm{DEH}^*_{n,\gamma}$. Putting $D^2_{n,j,k} := \|Y_{n,j} - Y_{n,k}\|^2$, $E_{n,j,k} = \exp(-D^2_{n,j,k}/(4\gamma))$, $a_{d,\gamma} = 2\gamma d(2\gamma - 1)$, $b_{d,\gamma} = 16d^2\gamma^3(\gamma - 1) + 4d(d+2)\gamma^2$, $c_{d,\gamma} = (\pi/\gamma)^{d/2}$, and $e_{d,\gamma} = 8d\gamma^2 - 4(d+2)\gamma$, the statistic $\mathrm{DEH}^*_{n,\gamma}$ has the feasible representation

$$\mathrm{DEH}^*_{n,\gamma} = \frac{c_{d,\gamma}}{n} \sum_{j,k=1}^{n} \Bigg[\|Y_{n,j}\|^2\|Y_{n,k}\|^2 E_{n,j,k} - \frac{\|Y_{n,j}\|^2 + \|Y_{n,k}\|^2}{4\gamma^2}\big(D^2_{n,j,k} + a_{d,\gamma}\big)E_{n,j,k} + \frac{E_{n,j,k}}{16\gamma^4}\Big(b_{d,\gamma} + (D^2_{n,j,k})^2 + e_{d,\gamma}D^2_{n,j,k}\Big)\Bigg].$$

Also the class of tests based on $\mathrm{DEH}^*_{n,\gamma}$ is 'closed at the boundaries $\gamma \to 0$ and $\gamma \to \infty$' since, elementwise on the underlying probability space, we have

$$\lim_{\gamma\to 0}\left[\left(\frac{\gamma}{\pi}\right)^{d/2}\mathrm{DEH}^*_{n,\gamma} - \frac{d(d+2)}{4\gamma^2}\right] = b^{(2)}_{n,d} - d^2, \quad \lim_{\gamma\to\infty}\frac{2\gamma^{d/2+1}}{n\pi^{d/2}}\mathrm{DEH}^*_{n,\gamma} = \widetilde{b}^{(1)}_{n,d}, \tag{2.26}$$

where $b^{(2)}_{n,d}$ and $\widetilde{b}^{(1)}_{n,d}$ are given in (8.1) and (8.3), respectively. Under H_0, we have $\mathrm{DEH}^*_{n,\gamma} \xrightarrow{\mathcal{D}} \mathrm{DEH}^*_{\infty,\gamma} := \int \mathcal{S}^2(t)\widetilde{w}_\gamma(t)\,\mathrm{d}t$, where $\mathcal{S}$ is some centered Gaussian random element of $\mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^d, \widetilde{w}_\gamma(t)\mathrm{d}t)$. Dörr et al. (2020a) also obtain a closed-form expression for $\mathbb{E}[\mathrm{DEH}^*_{\infty,\gamma}]$.

If X has a standardized alternative distribution satisfying $\mathbb{E}\|X\|^4 < \infty$, we have

$$\frac{\mathrm{DEH}^*_{n,\gamma}}{n} \xrightarrow{\text{a.s.}} D^*_\gamma := \int |-\Delta\Psi^+(t) + (\|t\|^2 - d)\Psi^+(t)|^2\widetilde{w}_\gamma(t)\,\mathrm{d}t,$$

where $\Psi^+(t) = \mathbb{E}[\cos(t^\top X)] + \mathbb{E}[\sin(t^\top X)]$. Hence, D^*_γ is the measure of distance from H_0 associated with $\mathrm{DEH}^*_{n\gamma}$. Interestingly, under the stronger condition $\mathbb{E}\|X\|^6 < \infty$, we have

$$\lim_{\gamma\to\infty}\frac{2\gamma^{d/2+1}}{\pi^{d/2}}D^*_\gamma = \left\|\mathbb{E}\left(\|X\|^2X\right)\right\|^2.$$

Since the right hand side is population skewness in the sense of Móri et al. (1993) (see Sect. 8), this result complements the second limit in (2.26). Dörr et al. (2020a) also show that, under a fixed alternative distribution satisfying $\mathbb{E}\|X\|^4 < \infty$, $\sqrt{n}\big(\mathrm{DEH}^*_{n,\gamma}/n - D^*_\gamma\big)$) has a centered limit normal distribution with a variance that can be consistently estimated from $X_1, \ldots, X_n$.

3 The Henze–Zirkler test

Henze and Zirkler (1990) observed that the BHEP-statistic defined in (2.8) may be written in the form

$$\mathrm{BHEP}_{n,\beta} = (2\pi)^{d/2}\beta^{-d}\int_{\mathbb{R}^d}\left(g_{n,\beta}(x) - \frac{1}{(2\pi\tau^2)^{d/2}}\exp\left(-\frac{\|x\|^2}{2\tau^2}\right)\right)^2\mathrm{d}x,$$

where $\tau^2 = (2\beta^2+1)/(2\beta^2)$, and

$$g_{n,\beta}(x) = \frac{1}{nh^d}\sum_{j=1}^n\frac{1}{(2\pi)^{d/2}}\exp\left(-\frac{\|x - Y_{n,j}\|^2}{2h^2}\right),$$

where $h^2 = 1/(2\beta^2)$. The function $g_{n,\beta}$ is a nonparametric kernel density estimator with Gaussian kernel w_1 (recall w_β from (2.10)) and bandwidth h, applied to $Y_{n,1}, \ldots, Y_{n,n}$. The choice $h = h_n = (4/(2d+1)n)^{1/(d+4)}$, taken from Silverman (1986), p. 87, yields $\beta = \beta_n$, where

$$\beta_n = 2^{-1/2}((2d+1)n/4)^{1/(d+4)}. \tag{3.1}$$

The Henze–Zirkler test statistic is given by $\mathrm{HZ}_n = \mathrm{BHEP}_{n,\beta_n}$. Notice that the optimal bandwidth that minimizes the asymptotic MISE of the kernel density estimator, when both the kernel and the underlying density are the standard d-variate normal density, is *not* h_n given above, but $\widetilde{h}_n = (4/(d+2)n)^{1/(d+4)}$.

Apparently unaware of the work of Henze and Zirkler (1990), Bowman and Foster (1993) proposed a test statistic BF_n that turned out to satisfy $\mathrm{BF}_n = \beta_n^d (2\pi)^{d/2} \mathrm{BHEP}_{n,\beta_n}$ (see Section 7 of Henze 2002). Thus, BF_n is equivalent to a BHEP-statistic with a smoothing parameter that depends on n. Gürtler (2000) proved that

$$\frac{nh^d 2^d \pi^{d/2} \mathrm{BF}_n - 1}{2^{1/2-d/4} h^{d/2}} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \tag{3.2}$$

as $n \to \infty$ under H_0. Under a fixed standardized alternative distribution with density f, Gürtler (2000) showed that

$$\frac{\sqrt{n}}{2} \left(\mathrm{BF}_n - \frac{1}{nh_n^d 2^d \pi^{d/2}} - C(f, h_n) \right) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma^2(f)) \tag{3.3}$$

for constants $\sigma^2(f)$ and $C(f, h_n)$, where $\lim_{n\to\infty} C(f, h_n) = \int (f(x) - w_1(x))^2 \mathrm{d}x$. In view of $nh_n^d \to \infty$, (3.3) entails $\mathrm{BF}_n \xrightarrow{\mathbb{P}} \int (f(x) - w_1(x))^2 \,\mathrm{d}x$ under f. Hence, the test of H_0 based on BF_n (or HZ_n) is consistent against general alternatives. However, since (3.2) remains true under contiguous alternatives that approach H_0 at the rate $n^{-1/2}$, the Henze–Zirkler (Bowman–Foster) test is not able to detect such alternatives, see also Tenreiro (2007) for more general results on Bickel–Rosenblatt-type statistics.

4 The energy test

For nearly 20 years now, the energy test has emerged as a strong genuine test for multivariate normality. It is based on the notion of *energy distance* between multivariate distributions. The naming *energy* stems from a close analogy with Newton's gravitational potential energy, see, e.g., Székely and Rizzo (2013). Besides goodness-of-fit testing, the concept of energy distance has found applications in many other fields, such as testing for equality of distributions, nonparametric extensions of analysis of variance, clustering, or testing for independence via distance covariance and distance correlation, see, e.g., Székely and Rizzo (2016).

If X and Y are independent random vectors with distributions $\mathbb{P}^X$ and $\mathbb{P}^Y$, and X' and Y' denote independent copies of X and Y, respectively, then the squared energy distance between $\mathbb{P}^X$ and $\mathbb{P}^Y$ is defined as

$$D^2(\mathbb{P}^X, \mathbb{P}^Y) := 2\mathbb{E}\|X - Y\| - \mathbb{E}\|X - X'\| - \mathbb{E}\|Y - Y\|,$$

provided these expectations exist (which is tacitly assumed). The energy distance $D(\mathbb{P}^X, \mathbb{P}^Y)$ satisfies all axioms of a metric. A proof of the fundamental inequality $D(\mathbb{P}^X, \mathbb{P}^Y) \geq 0$, with equality if and only if $\mathbb{P}^X = \mathbb{P}^Y$, follows from Zinger et al. (1992) or Mattner (1997), see also Székely and Rizzo (2005) for a different proof related to a result of Morgenstern (2001).

The energy test statistic for testing H_0 is

$$\mathcal{E}_n := n\left(\frac{2}{n}\sum_{j=1}^{n}\mathbb{E}\|\widetilde{Y}_{n,j} - N_1\| - \mathbb{E}\|N_1 - N_2\| - \frac{1}{n^2}\sum_{j,k=1}^{n}\|\widetilde{Y}_{n,j} - \widetilde{Y}_{n,k}\|\right).$$

Here, $\widetilde{Y}_{n,j} = \sqrt{n/(n-1)}Y_{n,j}$ with $Y_{n,j}$ given in (1.1) and N_1 and N_2 are independent random vectors with the normal distribution $\mathrm{N}_d(0, \mathrm{I}_d)$, which are independent of $X_1, \ldots, X_n$. The first expectation is with respect to N_1. Notice that $\mathbb{E}\|N_1 - N_2\| = 2\Gamma((d+1)/2)/\Gamma(d/2)$, where $\Gamma(\cdot)$ is the gamma function. Since, for $a \in \mathbb{R}^d$, the distribution of $\|a - N_1\|^2$ does only depend on $\|a\|^2$, the statistic $\mathcal{E}_n$ is seen to be invariant. The energy test for multivariate normality rejects H_0 for large values of $\mathcal{E}_n$. It is consistent against each fixed non-normal alternative, see Székely and Rizzo (2005), and it is fully implemented in the *energy* package for R, see Rizzo and Székely (2014). To the authors' knowledge, there are hitherto no results on the behavior of $\mathcal{E}_n$ with respect to contiguous alternatives to H_0. Since the intrinsic (quadratic) measure of distance between an alternative distribution $\mathbb{P}^X$ (which, because of invariance, may be taken as having zero mean and unit covariance matrix) and the standard d-variate normal distribution $\mathrm{N}_d(0, \mathrm{I}_d)$ is given by $\Delta_E(\mathbb{P}^X) := D^2(\mathbb{P}^X, \mathrm{N}_d(0, \mathrm{I}_d))$, say, it would be interesting to see whether $\sqrt{n}(\mathcal{E}_n - \Delta_E(\mathbb{P}^X))$ has a non-degenerate normal limit as $n \to \infty$, with a variance that can consistently be estimated from the data $X_1, \ldots, X_n$. Such a result would pave the way for an asymptotic confidence interval for $\Delta_E(\mathbb{P}^X)$.

5 The test of Pudelko

For a fixed $r > 0$, Pudelko (2005) suggested to reject H_0 for large values of the weighted supremum distance

$$\mathrm{PU}_{n,r} = \sqrt{n}\sup_{0<\|t\|\leq r}\frac{|\Psi_n(t) - \Psi_0(t)|}{\|t\|},$$

where $\Psi_n(t)$ is given in (2.9), and $\Psi_0(t) = \exp(-\|t\|/2)$. The test statistic $\mathrm{PU}_{n,r}$ is invariant, since it is a function of the scaled residuals $Y_{n,1}, \ldots, Y_{n,n}$ and rotation invariant. This statistic is similar in spirit as the statistic studied by Csörgő (1986), which is $\sup_{\|t\|\leq r}\left||\Psi_n(t)|^2 - \Psi_0^2(t)\right|$. Under H_0, $\mathrm{PU}_{n,r}$ converges in distribution to $\sup_{0<\|t\|\leq r}|\mathcal{P}(t)|/\|t\|$, where $\mathcal{P}(\cdot)$ is a centered Gaussian random element of the

Banach space $C(B_r)$ of complex-valued continuous functions, defined on $B_r := \{x \in \mathbb{R}^d : \|x\| \le r\}$, equipped with the supremum norm $\|f\|_{C(B_r)} := \sup_{x\in B_r} |f(x)|$. Pudelko (2005) also showed that the test is able to detect contiguous alternatives that approach H_0 at the rate $n^{-1/2}$. The consistency of the test based on $\mathrm{PU}_{n,r}$ follows easily from Csörgő (1989). A drawback of this test is its lack of feasibility, since one has to calculate the supremum of a function inside a d-dimensional sphere.

6 The test of Cox and Small

According to Cox and Small (1978), a main objective of tests of H_0 is 'to see whether an estimated covariance matrix provides an adequate summary of the interrelationships among a set of variables,' and that departure from multivariate normality 'is often the occurrence of appreciable nonlinearity of dependence.' To obtain an affine invariant test that assesses the degree of nonlinearity, they propose to find that pair of linear combinations of the original variables, such that one has maximum curvature in its regression on the other. The population functional which underlies the test of Cox and Small is $T_{CS}(\mathbb{P}^X) = \max_{b\in\mathcal{S}^{d-1}} \eta^2(b)$, where

$$\eta^2(b) = \frac{\left\|\mathbb{E}\left(X(b^\top X)^2\right)\right\|^2 - \left(\mathbb{E}\left(b^\top X\right)^3\right)^2}{\mathbb{E}\left(b^\top X\right)^4 - 1 - \left(\mathbb{E}\left(b^\top X\right)^3\right)^2},$$

see Cox and Small (1978), p. 268. The test statistic is $T_{n,CS} = \max_{b\in\mathcal{S}^{d-1}} \eta_n^2(b)$, where

$$\eta_n^2(b) = \frac{\left\|n^{-1}\sum_{j=1}^n Y_{n,j}(b^\top Y_{n,j})^2\right\|^2 - \left(n^{-1}\sum_{j=1}^n (b^\top Y_{n,j})^3\right)^2}{n^{-1}\sum_{j=1}^n (b^\top Y_{n,j})^4 - 1 - \left(n^{-1}\sum_{j=1}^n (b^\top Y_{n,j})^3\right)^2}$$

is the empirical counterpart of $\eta^2(b)$. Rejection of H_0 will be for large values of $T_{n,CS}$. The statistic $T_{n,CS}$ is affine invariant, since it is both a function of $Y_{n,1},\ldots,Y_{n,n}$ and rotation invariant. Notice that the functional T_{CS} vanishes on the set $\mathcal{N}_d$, but $T_{CS}(\mathbb{P}^X) = 0$ does not necessarily imply that $\mathbb{P}^X \in \mathcal{N}_d$. Some missing distributional properties of the statistic $T_{n,CS}$ were provided by Ebner (2012). If $\mathbb{P}^X$ is elliptically symmetric and satisfies $\mathbb{E}\|X\|^6 < \infty$, then

$$nT_{n,CS} \xrightarrow{\mathcal{D}} \frac{d(d+2)}{3m_4 - d(d+2)} \max_{b\in\mathcal{S}^{d-1}} W(b)^\top B W(b),$$

where $m_4 = E\|X\|^4$, B is the $(d+1)\times(d+1)$-matrix $\mathrm{diag}(1,\ldots,1,-1)$, and $W(\cdot)$ is a centered $(d+1)$-variate Gaussian process in $C(\mathcal{S}^{d-1}, \mathbb{R}^{d+1})$, the space of continuous functions from $\mathcal{S}^{d-1}$ to $\mathbb{R}^{d+1}$ (see Theorem 2.4 of Ebner 2012, where the covariance matrix kernel of W is given explicitly). As a consequence, the test of Cox and Small is not able to detect such elliptical alternatives to normality. Next, writing $\mu(b) = \mathbb{E}((b^\top X)^2(X, (b^\top X))^\top)$, we have

$$T_{n,CS} \xrightarrow{\mathbb{P}} \max_{b \in \mathcal{S}^{d-1}} \frac{\mu(b)^\top B \mu(b)}{\mathbb{E}(b^\top X)^4 - 1 - (\mathbb{E}(b^\top X)^3)^2}$$

if $\mathbb{E}\|X\|^6 < \infty$. Thus, the test based on $T_{n,CS}$ is consistent against each alternative distribution for which the above stochastic limit $\delta(\mathbb{P}^X)$ (say) is positive. Ebner (2012) also provides the limit distribution of $T_{n,CS}$ under contiguous alternatives to H_0, but it is still an open problem whether $\sqrt{n}(T_{n,CS} - \delta(\mathbb{P}^X))$ has a non-degenerate limit distribution as $n \to \infty$. From a practical point of view, the test of Cox and Small has the drawback that finding the maximum of $\eta_n^2(b)$ over $b \in \mathcal{S}^{d-1}$ is a computationally extensive task.

7 The test of Manzotti and Quiroz

Manzotti and Quiroz (2001) propose to test H_0 by means of averages over the standardized sample of multivariate spherical harmonics, radial functions and their products. For $k \in \mathbb{N}$ let $f_1, \ldots, f_k : \mathbb{R}^d \to \mathbb{R}$, such that $\mathbb{E} f_j^2(X) < \infty$ if $X \stackrel{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$, $j = 1, \ldots, k$. Let $V = (v_{ij})$ be the $(k \times k)$-matrix with entries

$$v_{ij} = \mathbb{E}[f_i(X) f_j(X)] - \mathbb{E} f_i(X)\, \mathbb{E} f_j(X), \quad X \stackrel{\mathcal{D}}{=} \mathrm{N}_d(0, I_d),$$

where V is assumed to be invertible. For $\mathbf{f} = (f_1, \ldots, f_k)^\top$, let

$$\nu_n(f_j) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \left\{ f_j(Y_{n,\ell}) - \mathbb{E} f_j(X) \right\} \quad \text{and} \quad \nu_n(\mathbf{f}) = (\nu_n(f_1), \ldots, \nu_n(f_k))^\top.$$

The general type of test statistic of Manzotti and Quiroz (2001) is the quadratic form

$$T_{n,MQ}(\mathbf{f}) = \nu_n(\mathbf{f})^\top V^{-1} \nu_n(\mathbf{f}).$$

To be more specific, let $\mathcal{H}_j$, $j \geq 0$, be the set of spherical harmonics of degree j in the orthonormal basis of spherical harmonics in d dimensions with respect to the uniform measure on $\mathcal{S}^{d-1}$, and put $\mathcal{G}_j = \bigcup_{i=0}^{j} \mathcal{H}_i$. The number of linear independent spherical harmonics of degree j in dimension d is $\binom{d+j-1}{j} - \binom{d+j-3}{j-2}$. A suitable orthonormal basis can be found using Theorem 5.25 in Axler et al. (2001) or Manzotti and Quiroz (2001), see also Groemer (1996) or Müller (1998) for details on spherical harmonics. Manzotti and Quiroz (2001) suggest two different choices for $\mathbf{f}$. Putting $r_j(x) = \|x\|^j$, $x \in \mathbb{R}^d$, and $u(x) = x/\|x\|$, $x \neq 0$, the first statistic $T_{n,MQ}(\mathbf{f}_1)$ uses f_j of the form $g \circ u$ for $g \in \mathcal{G}_4 \setminus \mathcal{H}_0$, giving a total of $k = \binom{d+3}{4} - \binom{d+2}{3} - 1$ functions. Due to orthonormality we have $V = \mathrm{I}_k$, and since no radial functions are considered, $T_{n,MQ}(\mathbf{f}_1)$ only tests for aspects of spherical symmetry. The second statistic $T_{n,MQ}(\mathbf{f}_2)$ uses the functions r_1 and $r_3(g \circ u)$, where $g \in \mathcal{G}_2$, which comprise a totality of $k = \binom{d+1}{2} + d + 1$ functions.

Both statistics are affine invariant, and Manzotti and Quiroz (2001) derive their limit null distributions, which are sums of weighted independent χ_1^2 random variables. Although the authors do not deal with the question of consistency of their tests, it is easily seen that, under an alternative distribution $\mathbb{P}^X$ (which, in view of invariance, is assumed to be standardized), and suitable conditions on $f_1, \ldots, f_k$, we have

$$\frac{1}{n} T_{n,MQ} \xrightarrow{\mathbb{P}} \delta(f)^\top V^{-1} \delta(f)$$

as $n \to \infty$, where $\delta(f) = (\mathbb{E} f_1(X) - \mathbb{E}_0 f_1, \ldots, \mathbb{E} f_k(X) - \mathbb{E}_0 f_k)^\top$, and $\mathbb{E}_0 f_j$ is the expectation $\mathbb{E} f_j(N)$, where $N \stackrel{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$. Since there are non-normal distributions for which the above (nonnegative) stochastic limit vanishes, the tests of Manzotti and Quiroz (2001) are not consistent against general alternatives. To the best of our knowledge, there are no further asymptotic properties of $T_{n,MQ}$ under alternatives to H_0.

8 Tests based on skewness and kurtosis

A still very popular group of tests for H_0 employ measures of multivariate skewness and kurtosis. The popularity of these tests stems from the widespread belief that, in case of rejection of H_0, there is some evidence regarding the kind of departure from normality of the underlying distribution.

The state of the art regarding this group of tests has been reviewed in Henze (2002), but for the sake of completeness, we revisit the most important facts. The classical invariant measures of multivariate sample skewness and kurtosis due to Mardia (1970) are defined by

$$b_{n,d}^{(1)} = \frac{1}{n^2} \sum_{j,k=1}^{n} \left(Y_{n,j}^\top Y_{n,k} \right)^3, \qquad b_{n,d}^{(2)} = \frac{1}{n} \sum_{j=1}^{n} \|Y_{n,j}\|^4, \tag{8.1}$$

respectively. The functional (population counterpart) corresponding to $b_{n,d}^{(1)}$ is $\beta_d^{(1)} = \beta_d^{(1)}(\mathbb{P}^X) = \mathbb{E}(X_1^\top X_2)^3$, where X is standardized, X_1, X_2 are i.i.d. copies of X, and $\mathbb{E}\|X\|^6 < \infty$. The functional accompanying kurtosis is $\beta_d^{(2)} = \beta_d^{(2)}(\mathbb{P}^X) = \mathbb{E}\|X\|^4$, where, like above, $\mathbb{E}(X) = 0$ and $\mathbb{E}(XX^\top) = \mathrm{I}_d$. When used as statistics to test H_0, $b_{n,d}^{(1)}$ has an upper rejection region, whereas the test based on $b_{n,d}^{(2)}$ is two-sided. If the distribution of X is elliptically symmetric, we have

$$n b_{n,d}^{(1)} \xrightarrow{\mathcal{D}} \alpha_1 \chi_d^2 + \alpha_2 \chi_{d(d-1)(d+4)}^2, \tag{8.2}$$

where

$$\alpha_1 = \frac{3}{d}\left[\frac{\mathbb{E}\|X\|^6}{d+2} - 2\mathbb{E}\|X\|^4 + d(d+2)\right], \qquad \alpha_2 = \frac{6\mathbb{E}\|X\|^6}{d(d+2)(d+4)},$$

where χ^2_d, $\chi^2_{d(d-1)(d+4)}$ are independent χ^2-variables with d and $d(d-1)(d+4)$ degrees of freedom, respectively, see Baringhaus and Henze (1992), and Klar (2002). Notice that $\alpha_1 = \alpha_2 = 6$ under H_0, whence $nb^{(1)}_{n,d} \xrightarrow{\mathcal{D}} 6\chi^2_{d(d+1)(d+2)/6}$ under normality, see Mardia (1970). From (8.2), it follows that the test of H_0 based on $b^{(1)}_{n,d}$ is not consistent against spherically symmetric alternatives satisfying $\mathbb{E}\|X\|^6 < \infty$. If $\beta^{(1)}_d > 0$, then $\sqrt{n}(b^{(1)}_{n,d} - \beta^{(1)}_d)$ has a centered non-degenerate limit normal distribution as $n \to \infty$, see Theorem 3.2 of Baringhaus and Henze (1992). The skewness functional $\beta^{(1)}_d(\cdot)$ does not characterize the class $\mathcal{N}_d$ of normal distributions since, although $\beta^{(1)}_d(\cdot)$ vanishes on $\mathcal{N}_d$, there are (notably elliptically *symmetric*) non-normal distributions that share this property. Since the critical value of $b^{(1)}_{n,d}$ as a test statistic for assessing multivariate normality is computed under the very assumption of *normality*, the inclination to impute supposedly diagnostic properties to $b^{(1)}_{n,d}$ in case of rejection of H_0 in the sense that 'there is evidence that the underlying distribution is skewed' is not justified, at least not in terms of statistical significance. In fact, the limit distribution of $nb^{(1)}_{n,d}$ under certain classes of elliptically symmetric distributions is stochastically much larger than the limit null distribution of $nb^{(1)}_{n,d}$ (see Baringhaus and Henze 1992), and so rejection of H_0 based on $b^{(1)}_{n,d}$ may be due to an underlying long-tailed elliptically *symmetric* distribution.

Regarding kurtosis, we have $\sqrt{n}(b^{(2)}_{n,d} - \beta^{(2)}_d) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \sigma^2)$ as $n \to \infty$, where σ^2 depends on mixed moments of X up to order 8, see Henze (1994a). Under H_0, we have $\beta^{(2)}_d = d(d+2)$ and $\sigma^2 = 8d(d+2)$, and the limit distribution was already obtained by Mardia (1970), see also Klar (2002) for the case that $\mathbb{P}^X$ is elliptically symmetric. It follows that, under the condition $\mathbb{E}\|X\|^8 < \infty$, Mardia's kurtosis test for normality is consistent if and only if $\beta^{(2)}_d \neq d(d+2)$. The critical remarks made above on alleged diagnostic capabilities of tests for H_0 based on measures of skewness apply mutatis mutandis to a test for normality based on $b^{(2)}_{n,d}$ or any other measure of multivariate kurtosis.

Among the many measures of multivariate skewness, we highlight skewness in the sense of Móri et al. (1993), because it emerges in connection with several weighted L^2-statistics for testing H_0. This measure is defined by

$$\widetilde{b}^{(1)}_{n,d} := \frac{1}{n^2}\sum_{j,k=1}^{n} \|Y_{n,j}\|^2 \|Y_{n,k}\|^2 Y_{n,j}^\top Y_{n,k}. \tag{8.3}$$

The corresponding functional (population counterpart) is $\widetilde{\beta}^{(1)}_d = \left\|\mathbb{E}(\|X\|^2 X)\right\|^2$, where X is assumed to be standardized and $\mathbb{E}\|X\|^6 < \infty$. Limit distributions for

$\widetilde{b}^{(1)}_{n,d}$ have been obtained by Henze (1997a) both for the case that $\mathbb{P}^X$ is elliptically symmetric (which implies $\widetilde{\beta}^{(1)}_d = 0$) and the case that $\widetilde{\beta}^{(1)}_d > 0$, see also Klar (2002).

A further measure of multivariate skewness that has been reviewed in Henze (2002) is skewness in the sense of Malkovich and Afifi (1973), which is defined as

$$b^{(1)}_{n,d,M} = \max_{u \in \mathcal{S}^{d-1}} \frac{\left\{n^{-1}\sum_{j=1}^n (u^\top X_j - u^\top \overline{X}_n)^3\right\}^2}{(u^\top S_n u)^3}.$$

General limit distribution theory for $b^{(1)}_{n,d,M}$ is given in Baringhaus and Henze (1991).

As for further measures of multivariate kurtosis, we mention the measure

$$\widetilde{b}^{(2)}_{n,d} = \frac{1}{n^2}\sum_{j,k=1}^n \left(Y_{n,j}^\top Y_{n,k}\right)^4,$$

introduced by Koziol (1989). The corresponding functional is $\widetilde{\beta}^{(2)}_d = \mathbb{E}(X_1^\top X_2)^4$, where X_1, X_2 are i.i.d. copies of the standardized vector X, and $\mathbb{E}\|X\|^8 < \infty$. General asymptotic distribution theory for $\widetilde{b}^{(2)}_{n,d}$ is provided by Henze (1994b) and Klar (2002). Henze (2002) also reviewed kurtosis in the sense of Malkovich and Afifi (1973), which is defined as

$$b^{(2)}_{n,d,M} = \max_{u \in \mathcal{S}^{d-1}} \frac{n^{-1}\sum_{j=1}^n (u^\top X_j - u^\top \overline{X}_n)^4}{(u^\top S_n u)^2}.$$

Limit distribution theory for $b^{(2)}_{n,d,M}$ has been obtained by Baringhaus and Henze (1991) and Naito (1998).

Since the review Henze (2002), there have been the following suggestions to test H_0 by means of measures of multivariate skewness and kurtosis (which, however, do not lead to consistent tests and share the drawback stated at the beginning of this section): Kankainen et al. (2007) consider invariant tests of multivariate normality that are based on the Mahalanobis distance between two multivariate location vector estimates (as a measure of skewness) and on the (matrix) distance between two scatter matrix estimates (as a measure of kurtosis). Special choices of these estimates yield generalizations of Mardia's skewness and kurtosis. The authors obtain asymptotic distribution theory of their test statistics both under normality and certain contiguous alternatives to H_0, and they compare the limiting Pitman efficiencies to those of Mardia's tests based on $b^{(1)}_{n,d}$ and $b^{(2)}_{n,d}$.

Doornik and Hansen (2008) propose a non-invariant test based on skewness and kurtosis.

Enomoto et al. (2020) consider a transformation of Mardia's kurtosis statistic, with the aim of improving the finite-sample approximation with respect to a normal limit distribution.

9 Miscellaneous results

Arcones (2007) proposed two invariant test statistics that are based on the following characterizations, see, e.g., Cramér (1936). Let $m \geq 2$ be a fixed integer, and let $X_1, \ldots, X_m$ be i.i.d. d-dimensional vectors satisfying $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1 X_1^\top) = \mathrm{I}_d$. Then, $m^{-1/2} \sum_{j=1}^m X_j \overset{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$ if and only if $X_1 \overset{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$. Furthermore, $m^{-1/2} \sum_{j=1}^m X_j \overset{\mathcal{D}}{=} X_1$ if and only if $X_1 \overset{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$. A statistic that corresponds to the first characterization is $\widehat{D}_{n,m} = \int \left|\widehat{\Psi}_{n,m}(t) - \Psi_0(t)\right|^2 w_\beta(t)\,\mathrm{d}t$, where $\widehat{\Psi}_{n,m}(t) = n!^{-1}(n-m)! \sum_{\neq} \exp\left(\mathrm{i}t^\top m^{-1/2} \sum_{p=1}^m Y_{n,j_p}\right)$, and $\Sigma_{\neq}$ means summation over all $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $j_p \neq j_q$ if $p \neq q$. Notice that this approach is a generalization of the BHEP-statistic given in (2.8). The statistic which is tailored to the second characterization is $\widehat{E}_{n,m} = \int \left|\Psi_{n,m}(1) - \Psi_{n,1}(t)\right|^2 w_\beta(t)\,\mathrm{d}t$. Both statistics have representations in form of multiple sums. By using the theory of U-statistics with estimated parameters, Arcones (2007) derives almost sure limits of $\widehat{D}_{n,m}$ and $\widehat{E}_{n,m}$ as well as the limit distributions of $n\widehat{D}_{n,m}$ and $n\widehat{E}_{n,m}$ under H_0. Some very limited simulations, performed for $n \leq 15$ and $d = 2$, indicate that the power of these tests is comparable to that of the BHEP test. However, the computational burden involved increases rapidly with m. Bakshaev and Rudzkis (2017) propose a test for multivariate normality that is based on the supremum of $\left|\Psi_n(t) - \Psi_0(t)\right|^2$ figuring in (2.8), where the supremum is taken over a fixed d-dimensional cube without zero origin.

Without providing any distribution theory, Hwu et al. (2002) suggest an invariant two-stage test procedure for testing H_0. This procedure combines a modified correlation coefficient related to a Q–Q plot of the ordered values of $\|Y_{n,j}\|^2$, $j = 1, \ldots, n$, against ordered quantiles of the χ^2_d-distribution, and a test based on Mardia's nonnegative invariant measure of skewness $b^{(1)}_{n,d}$ given in (8.1).

Liang et al. (2004) deal with Q–Q plots based on functions of $(j(j+1))^{-1/2}(X_1 + \ldots + X_j - jX_{j+1})$, $j = 1, \ldots, n-1$, and hence recommend procedures that are not even invariant with respect to permutations of $X_1, \ldots, X_n$. The latter objection also holds for the procedures suggested by Fang et al. (1998) and Liang and Bentler (1999).

Tan et al. (2005) extend the projection procedure of Liang et al. (2000) to test for multivariate normality with incomplete longitudinal data with small sample size, including cases when the sample size n is smaller than d.

Hanusz and Tarasińska (2008) correct an inaccuracy of the (non-invariant) test of Srivastava and Hui (1987), and Maruyama (2007) derives approximations of expectations and variances related to that test under alternative distributions.

Without providing any theoretical results, Hanusz and Tarasińska (2012) aim at transforming two graphical methods for assessing H_0 into formal statistical tests. A variant of this approach was considered by Madukaife and Okafor (2018).

Cardoso de Oliveira and Ferreira (2010) suggest to perform a chi-quare test based on $\|Y_{n,1}\|^2, \ldots, \|Y_{n,n}\|^2$ (see also Moore and Stubblebine 1981), and Batsidis et al. (2013) extend this approach to include more general power divergence type of test statistics. Madukaife and Okafor (2019) consider ℓ_1- and ℓ_2-type measures of deviation between $\|Y_{n,j}\|^2$ and corresponding approximate expected order statistics of a χ^2_d-

distribution (for tests based on $\|Y_{n,1}\|^2, \ldots, \|Y_{n,n}\|^2$, see also Section 5.2 of Henze 2002). Voinov et al. (2016) compare several test statistics that, for fixed $r \geq 2$, are quadratic forms in the vector $(V_{n,1}, \ldots, V_{n,r})^\top$. Here, $V_{n,j} = (N_{n,j} - n/r)/(\sqrt{n/r})$, $N_{n,j} = \sum_{k=1}^n \mathbf{1}\{c_{j-1} < \|Y_{n,k}\|^2 \leq c_j\}$, and $0 < c_1 < \ldots < c_{r-1} < c_r = \infty$, where c_j is the (j/r)-quantile of the χ^2_d-distribution, $j = 1, \ldots, r-1$.

Jönsson (2011) investigates the finite-sample performance of the Jarque–Bera test for H_0 in order to improve the size of the test. Koizumi et al. (2014) improve upon multivariate Jarque–Bera-type tests by means of transformations. Simulations show that such transformations essentially improves test accuracy when d is close to n. Kim (2016) generalizes the univariate Jarque–Bera test and its modifications to the multivariate versions using an orthogonalization of data and compares it with competitors in a simulation study.

Kim and Park (2018) propose a non-invariant test based on univariate Anderson–Darling-type statistics that are averaged out over the d coordinates. Villasenor Alva and González Estrada (2009) suggest a non-invariant test that is based on the average of Shapiro–Wilk statistics, applied to each of the components of $Y_{n,1}, \ldots, Y_{n,n}$.

By using an idea of Fromont and Laurent (2006), Tenreiro (2011) proposes an invariant consistent multiple test procedure that combines Mardia's measures of skewness and kurtosis and two members of the family of BHEP tests. The combined procedure rejects H_0 if one of the statistics is larger than its $(1 - u_{n,\alpha})$-quantile under H_0, where $u_{n,\alpha}$ is calibrated so that the combined test has a desired level of significance α. In the same spirit, Tenreiro (2017) combines two BHEP tests and the 'extreme' BHEP tests, the statistics of which are given by the right hand sides of (2.14) and (2.15).

Majerski and Szkutnik (2010) consider the problem of testing H_0 against some alternatives that are invariant with respect to a subgroup of the full group of affine transformations and obtain approximations to the most powerful invariant tests. Special emphasis is given to exponential and uniform alternatives in the case $d = 2$, whereas the case $d \geq 3$ is only sketched.

In the spirit of projection pursuit tests (see Section 8.1 of Henze 2002), which are based on Roy's union-intersection principle (Roy 1953), Zhou and Shao (2014) propose a non-invariant test that combines the Shapiro–Wilk test and Mardia's kurtosis test. In the same spirit, Wang and Hwang (2011) suggest a statistic that considers solely the Shapiro–Wilk statistic.

Wang (2014) provides a MATLAB package for testing H_0, which is implemented as an interactive and graphical tool. The package comprises 12 different tests, among which are the energy test, the Henze–Zirkler test, and the tests based on Mardia's skewness and kurtosis.

Thulin (2014) proposes six invariant tests for H_0, the common basis of which are characterizations of independence of sample moments of the multivariate normal distribution.

10 Comparative simulation studies

10.1 Available simulation studies

Mecklin and Mundfrom (2005) perform an extensive simulation study with 13 tests for multivariate normality. From this study, they conclude that 'if one is going to rely on one and only one procedure, the Henze–Zirkler test is recommended. This recommendation is based on the relative ease of use (the test statistic has an approximately lognormal asymptotic distribution), good Monte Carlo simulation results, and mathematically proven consistency against all alternatives.'

Farrell et al. (2007) compare four tests of multivariate normality and conclude: 'The results of our simulation suggest that, relative to the other two tests considered, the Henze and Zirkler test generally possesses good power across the alternative distributions investigated, in particular for $n \geq 75$.'

Hanusz et al. (2018) compare four test of H_0 that are based on a combination of measures of multivariate skewness and kurtosis, and the Henze–Zirkler test. They concluded that 'the Henze–Zirkler test best preserves the nominal significance level' and that 'for the number of traits and sample sizes considered, it is not possible to indicate the most powerful test for all kinds of alternative distributions considered in the paper.'

Joenssen and Vogel (2014) investigate 15 tests of H_0, all of which freely available as R-functions. They find that some tests are unreliable and should either be corrected or removed, or their deficits should be commented upon in the documentation by the package maintainer. Moreover, they summarize: 'On the question of whether or not multivariate tests offer an advantage over simply testing each marginal distribution with a univariate test, the answer is a resounding yes. Not only are some multivariate tests able to detect deviations from normality that are not reflected in the marginals of the distribution, but these tests are also, in part, more powerful for distributions that do display the deviations in the marginals.'

10.2 New simulation study

This subsection compares the finite-sample power performance of the tests presented in this survey by means of a Monte Carlo simulation study. All simulations are performed using the statistical computing environment R, see R Core Team (2018). The tests were implemented in the accompanying `R` package `mnt`, see Butsch and Ebner (2020).

We consider the sample sizes $n = 20$, $n = 50$ and $n = 100$, the dimensions $d = 2$, $d = 3$ and $d = 5$, and the nominal level of significance is set to 0.05. Throughout, critical values for the tests have been simulated with 100 000 replications under H_0, see Table 1. Note that, in order to ease the comparison with the original articles, we state the empirical quantiles of $\left(16\gamma^{2+d/2}/\pi^{d/2}\right)\mathrm{HV}_{n,\gamma}$, $\pi^{-d/2}\mathrm{HJ}_{n,\gamma}$, $(\gamma/\pi)^{d/2}\mathrm{HJM}_{n,\gamma}$, $(\gamma/\pi)^{d/2}d^{-2}\mathrm{DEH}_{n,\gamma}$, and $(\gamma/\pi)^{d/2}d^{-2}\mathrm{DEH}^*_{n,\gamma}$ and chose whenever available the tuning parameter γ according to the suggestions of the authors, respectively. For the sake of readability, we suppress the index n for all tests in the tables. Note that $\overline{\mathrm{BHEP}}$ denotes the BHEP test with tuning parameter $\beta = \sqrt{2}/(1.376+0.075d)$, as suggested in Tenreiro (2009). The values given in Table 1 are also reported in package `mnt` in the data frame `Quantile095` for easy access. Each entry in a table that refers to empirical rejection rates as estimates of the power of the test is based on 10 000 replications, with the exception of the HJM test, where 1 000 replications have been considered, due to the heavy computation time of the procedure.

We consider a total of 32 alternatives as well as a representative of the multivariate normal distribution. By $\mathrm{NMix}(p, \mu, \Sigma)$ we denote the normal mixture distribution generated by

$$(1-p)\,\mathrm{N}_d(0, \mathrm{I}_d) + p\,\mathrm{N}_d(\mu, \Sigma), \quad p \in (0,1),\ \mu \in \mathbb{R}^d,\ \Sigma > 0,$$

where $\Sigma > 0$ stands for a positive definite matrix. In the notation of above, $\mu = 3$ stands for a d-variate vector of 3's and $\Sigma = \mathrm{B}_d$ for a $(d \times d)$-matrix containing 1's on the main diagonal and 0.9's for each off-diagonal entry. We write $t_\nu(0, \mathrm{I}_d)$ for the multivariate t-distribution with ν degrees of freedom, see Genz and Bretz (2009). By $\mathrm{DIST}^d(\vartheta)$ we denote the d-variate random vector generated by independently simulated components of the distribution DIST with parameter vector ϑ, where DIST is taken to be the uniform distribution U, the lognormal distribution LN, the beta distribution B, as well as the Pearson type II P_{II} and Pearson type VII distribution P_{VII}. For the latter distribution, we used the `R` package `PearsonDS`, see Becker and Klößner (2017). The spherical symmetric distributions were simulated using the `R` package `distrEllipse`, see Ruckdeschel et al. (2006), and they are denoted by $\mathcal{S}^d(\mathrm{DIST})$, where DIST stands for the distribution of the radii, which was chosen to be the exponential, the beta, the χ^2-distribution and the lognormal distribution. With $\mathrm{MAR}_d(\mathrm{DIST})$ we denote $\mathrm{N}_d(0, \mathrm{I}_d)$-distributed random vectors, where the dth component is independently replaced by a random variable following the distribution DIST. Here, we chose the exponential, the χ^2, student's t and the gamma distribution. With $\mathrm{NM}_d(\vartheta)$ we denote the normal mixture distributions generated by

$$0.5\,\mathrm{N}_d(0, \Sigma_\vartheta) + 0.5\,\mathrm{N}_d(0, \Sigma_{-\vartheta}),$$

where Σ_ϑ is a positive definite $(d \times d)$-matrix with 1's on the diagonal and the constant ϑ for each off diagonal entry. In this family of non-normal distributions each component follows a normal law. The symbol $\mathrm{S}|\mathrm{N}_d|$ stands for the distribution of $\pm|X|$, where $X \overset{\mathcal{D}}{=} \mathrm{N}_d(0, \mathrm{I}_d)$, the absolute value $|\cdot|$ is applied componentwise, and $\pm$ assigns, independently of each other and with equal probability 0.5, a random sign to each component of $|X|$. Finally, we consider the distribution $\mathrm{N}_d(\mu_d, \Sigma_{0.5})$, with $\mu_d = (1, 2, \ldots, d)^\top$ and the same covariance structure as reported for the NM-alternatives, in order to show that all tests under consideration are invariant and indeed have a type I error equal to the significance level of 5%.

The results of the weighted L^2-type tests in Tables 2, 3 and 4 are presented for the same tuning parameters as in Table 1, and in order to keep the tables concise the values are omitted.

First, we evaluate the results for $d = 2$. A close look at Table 2 reveals that, for the family of normal mixture distributions, the HZ test and the PU test perform best when the shifted standard normal distributions are mixed, whereas for different

Table 1 Empirical 95% quantiles of the test statistics under H_0 (100 000 replications)

d	n	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$	BHEP_1	HZ	HV_5	$\overline{\text{BHEP}}$
2	20	2.38	9.44	1.82	40.90	1.77	5.47	0.54	0.73	250	0.47
	50	1.09	9.44	0.84	37.28	0.87	4.94	0.55	0.88	358	0.48
	100	0.56	9.17	0.43	33.40	0.46	4.42	0.56	0.97	397	0.48
3	20	4.63	16.37	2.81	75.36	2.68	6.68	0.67	0.82	545	0.54
	50	2.11	16.73	1.25	67.38	1.39	5.81	0.68	0.92	823	0.56
	100	1.09	16.49	0.63	60.05	0.74	5.02	0.68	0.98	936	0.56
5	20	12.57	35.35	4.38	191	4.55	8.37	0.84	0.91	1750	0.65
	50	5.77	37.01	1.96	163	2.61	7.16	0.85	0.96	2993	0.66
	100	2.96	36.94	0.94	140	1.44	5.92	0.85	0.99	3530	0.66

d	n	$\text{HJ}_{1.5}$	$\text{HJM}_{1.5}$	$\text{DEH}_{0.25}$	$\text{DEH}^*_{0.5}$	$\mathcal{E}$	$T_{MQ}(\mathbf{f}_1)$	$T_{MQ}(\mathbf{f}_2)$	T_{CS}	PU_2
2	20	12.31	2.89	1.92	3.42	0.93	11.26	3.71	0.38	1.02
	50	42.83	3.40	1.98	3.50	0.96	11.17	4.39	0.16	1.02
	100	80.57	3.62	1.96	3.53	0.97	11.27	4.65	0.08	1.03
3	20	32.74	6.67	1.60	2.44	1.04	29.13	3.91	0.59	1.18
	50	148	9.32	1.66	2.52	1.07	29.09	4.67	0.26	1.19
	100	335	9.71	1.65	2.53	1.07	29.05	4.99	0.13	1.20
5	20	127	25.64	1.36	1.79	1.23	115	4.52	0.82	1.33
	50	1049	55.62	1.42	1.85	1.26	113	5.23	0.42	1.35
	100	3117	72.65	1.42	1.86	1.28	113	5.61	0.22	1.36

Table 2 Empirical rejection rates of the considered tests ($d = 2, \alpha = 0.05$)

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
NMix(0.5, 3, I_2)	20	18	14	24	2	3	7	3	16	20	11	19	5	13	2	1	4	2	3	2
	50	64	57	82	2	2	10	6	67	71	34	51	5	77	2	0	2	0	3	2
	100	99	98	100	2	2	70	38	99	99	76	88	5	96	2	0	3	0	3	1
NMix(0.79, 3, I_2)	20	42	41	42	14	10	24	17	34	39	15	13	21	46	18	11	20	11	23	11
	50	94	93	93	21	6	26	51	89	93	43	23	44	96	52	8	56	8	54	7
	100	100	100	100	48	5	33	96	100	100	82	50	75	99	91	7	94	6	87	6
NMix(0.9, 3, I_2)	20	38	40	34	32	26	44	34	37	37	13	22	27	44	34	23	38	24	35	24
	50	83	85	74	70	38	81	80	83	82	31	62	62	87	87	49	89	51	82	50
	100	99	99	96	97	45	97	99	99	99	59	95	92	99	100	69	100	72	99	73
NMix(0.5, 0, B_2)	20	15	15	14	17	16	34	19	18	16	10	13	12	16	17	18	17	20	16	19
	50	31	31	31	25	23	60	40	43	34	24	31	15	38	20	32	20	35	18	35
	100	59	58	61	36	31	84	70	75	63	45	55	16	72	23	52	23	59	20	60
NMix(0.9, 0, B_2)	20	20	21	18	27	28	40	28	24	21	10	22	19	22	26	29	26	30	24	30
	50	37	39	29	54	53	66	55	47	38	15	50	31	43	45	54	45	55	42	57
	100	59	62	46	78	77	89	78	70	60	24	75	39	66	56	80	56	83	53	85
$t_1(0, I_2)$	20	96	97	96	94	94	98	97	97	97	77	97	85	96	92	97	91	97	90	96
	50	100	100	100	100	100	100	100	100	100	97	100	95	100	99	100	99	100	99	100
	100	100	100	100	100	100	100	100	100	100	100	100	98	100	100	100	100	100	100	100
$t_3(0, I_2)$	20	48	49	45	54	52	67	56	53	49	16	53	36	46	52	59	50	58	48	54
	50	82	84	78	85	83	95	88	87	83	24	91	55	80	77	92	75	90	72	86
	100	98	98	97	97	96	100	99	99	98	31	100	96	78	92	99	91	100	87	98

Table 2 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$t_5(0, I_2)$	20	25	25	22	32	32	48	34	29	26	9	29	20	24	30	36	31	35	29	31
	50	49	49	42	58	58	78	63	58	51	11	66	33	46	53	71	52	68	48	62
	100	75	76	66	81	78	95	83	80	75	12	90	41	68	66	92	64	91	59	86
$t_{10}(0, I_2)$	20	10	12	9	16	16	25	16	13	11	5	13	10	11	15	17	15	16	15	14
	50	17	18	13	29	29	48	29	23	18	6	29	15	17	25	32	23	32	22	30
	100	26	28	20	44	42	69	42	34	27	6	49	18	24	33	57	32	57	30	50
$U^2(0, 1)$	20	12	10	18	0	0	1	1	10	11	8	33	3	5	0	0	0	0	1	0
	50	59	54	68	0	0	7	6	57	52	15	82	2	46	0	0	0	0	0	0
	100	98	97	98	0	0	92	79	98	96	27	100	2	96	0	0	0	0	0	0
$LN^2(0, 0.5)$	20	60	61	55	50	43	55	50	56	57	18	37	52	55	58	40	55	42	58	41
	50	97	97	93	92	75	88	93	96	96	45	83	94	95	97	76	95	77	96	73
	100	100	100	100	100	93	99	100	100	100	81	99	100	100	100	95	100	96	100	93
$B^2(1, 2)$	20	27	26	28	5	4	8	6	20	25	10	13	13	18	7	2	7	3	10	3
	50	81	80	78	5	1	6	30	73	77	23	41	34	68	21	0	16	1	30	1
	100	100	100	99	19	0	17	95	100	99	46	91	77	98	73	0	56	0	74	0
$B^2(2, 2)$	20	5	4	6	1	1	2	1	3	4	5	12	3	3	1	1	1	1	1	1
	50	13	11	17	0	0	0	1	10	10	7	34	2	9	0	0	0	0	0	0
	100	39	37	41	0	0	15	7	34	31	9	71	2	27	0	0	0	0	0	0
$P_{II}^2(0.5, 0, 1)$	20	46	37	59	0	0	1	1	45	48	20	67	5	24	0	0	1	0	1	0
	50	99	99	100	0	0	71	69	100	99	52	100	3	98	0	0	0	0	0	0
	100	100	100	100	0	0	100	100	100	100	85	100	3	100	0	0	0	0	0	0

Table 2 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$P^2_{VII}(5, 0, 1)$	20	20	22	18	28	27	42	28	25	21	8	23	18	21	28	31	27	30	26	28
	50	38	40	32	51	49	70	55	48	40	10	55	30	39	43	59	42	59	39	55
	100	63	63	53	72	71	91	78	72	64	13	82	37	57	56	85	54	85	50	78
$P^2_{VII}(10, 0, 1)$	20	9	10	8	14	13	25	13	11	10	6	11	9	10	15	17	16	17	14	15
	50	13	14	11	24	23	41	23	18	14	7	22	13	14	21	28	19	26	20	23
	100	19	20	14	35	34	57	35	27	20	6	38	16	17	24	44	24	44	22	37
$\mathcal{S}^2(\text{Exp}(1))$	20	77	76	78	68	64	86	75	82	83	32	81	38	67	68	81	64	77	63	70
	50	99	99	100	93	89	99	98	100	100	46	100	49	97	80	99	76	98	72	94
	100	100	100	100	99	98	100	100	100	100	56	100	55	100	91	100	89	100	80	100
$\mathcal{S}^2(\text{B}(1, 1))$	20	5	5	7	1	1	7	4	7	9	7	5	2	5	2	4	2	3	2	2
	50	5	5	15	0	0	2	3	11	18	8	4	1	5	1	0	0	0	0	0
	100	8	6	44	0	0	0	6	26	39	8	5	1	4	0	0	0	0	0	0
$\mathcal{S}^2(\text{B}(1, 2))$	20	25	24	27	14	13	39	22	30	34	11	28	9	18	14	30	12	24	14	17
	50	52	47	66	10	7	60	32	60	70	15	65	6	34	14	44	11	35	12	18
	100	84	81	95	7	2	82	52	90	95	17	93	4	62	9	64	9	57	8	25
$\mathcal{S}^2(\text{B}(2, 2))$	20	3	3	5	0	0	2	1	2	3	4	9	2	3	0	0	1	0	1	0
	50	8	7	10	0	0	0	0	5	6	5	23	1	6	0	0	0	0	0	0
	100	21	20	22	0	0	8	1	14	15	4	54	1	11	0	0	0	0	0	0
$\mathcal{S}^2(\chi^2_5)$	20	16	17	14	21	21	36	22	19	16	6	18	12	15	19	25	19	24	20	22
	50	29	31	25	38	36	63	42	37	31	7	46	17	26	32	50	29	47	29	40
	100	50	51	43	55	52	85	61	57	52	7	73	20	40	40	76	38	74	35	63

Table 2 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\mathcal{S}^2(\text{LN}(0, 0.5))$	20	12	11	11	16	16	21	15	14	13	6	13	12	10	16	14	16	15	16	15
	50	15	16	14	30	30	44	29	25	20	7	23	17	15	25	29	25	31	24	30
	100	19	20	22	47	47	64	45	38	31	6	34	22	18	38	50	36	50	34	47
$\text{MAR}_2(\text{Exp}(1))$	20	52	54	49	41	34	44	39	48	52	16	28	44	57	44	29	48	30	47	29
	50	95	96	92	81	60	81	86	94	95	40	70	83	97	94	60	96	62	93	62
	100	100	100	100	99	82	97	100	100	100	75	97	99	100	100	84	100	85	100	86
$\text{MAR}_2(\chi_3^2)$	20	39	39	35	31	26	36	30	35	37	11	20	32	41	34	23	36	23	36	24
	50	84	85	77	67	48	67	71	81	83	26	54	70	88	82	48	85	49	80	49
	100	99	100	98	95	70	89	98	99	100	52	90	95	100	100	68	100	70	98	72
$\text{MAR}_2(\chi_5^2)$	20	25	26	22	22	20	28	21	22	23	9	15	22	26	25	16	25	16	24	16
	50	61	63	51	50	35	54	50	57	60	16	38	51	66	64	33	66	35	64	36
	100	93	93	83	82	52	76	86	91	92	30	71	83	95	93	50	95	51	92	51
$\text{MAR}_2(t_3)$	20	23	24	21	29	28	42	29	27	24	12	24	21	25	27	30	27	31	26	30
	50	47	48	41	55	53	73	57	54	48	24	55	33	51	47	58	47	60	44	59
	100	73	75	66	78	76	91	81	79	74	42	81	45	76	59	84	58	85	55	85
$\text{MAR}_2(t_5)$	20	12	13	11	16	16	28	17	15	13	7	13	12	13	17	18	17	18	15	19
	50	21	21	17	31	30	49	31	27	22	10	29	18	23	28	36	27	36	25	36
	100	34	35	27	47	47	70	48	42	35	13	48	24	35	33	52	33	54	31	56
$\text{MAR}_2(\Gamma(5, 1))$	20	25	26	22	22	19	30	21	23	24	9	15	22	22	26	18	25	19	28	18
	50	64	65	53	53	36	55	54	60	62	16	39	60	57	68	36	64	37	68	36
	100	93	94	83	87	53	79	89	92	92	31	74	93	89	97	52	95	52	96	49

Table 2 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$NM_2(0.05)$	20	5	5	5	5	5	12	5	5	5	5	5	5	5	5	5	6	5	5	5
	50	5	5	5	5	5	15	5	5	5	5	5	5	5	5	6	4	6	5	6
	100	5	5	5	5	5	14	5	5	5	5	5	5	5	5	5	4	4	3	4
$NM_2(0.1)$	20	5	5	5	5	5	13	5	5	5	5	5	5	5	6	6	5	6	5	6
	50	5	5	5	5	5	13	5	5	5	5	5	5	5	6	5	5	5	6	5
	100	5	5	5	6	5	14	5	5	5	5	5	5	5	5	5	5	4	5	4
$NM_2(0.2)$	20	5	5	5	6	6	14	6	6	5	5	6	5	5	7	7	7	7	6	7
	50	5	5	5	6	6	16	6	6	6	5	6	6	6	7	7	7	7	8	6
	100	5	6	5	7	7	17	6	6	6	5	7	6	5	7	7	7	6	6	6
$S\|N_2\|$	20	14	13	15	11	11	25	15	21	17	33	8	12	13	12	12	12	13	11	12
	50	31	28	45	16	16	46	50	66	46	87	14	14	33	14	18	14	23	13	24
	100	74	68	95	23	19	72	96	99	93	100	20	16	66	14	27	15	37	13	38
$N_2(\mu_2, \Sigma_{0.5})$	20	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	50	5	5	5	5	5	5	5	6	5	5	5	5	5	5	5	5	5	5	5
	100	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 3 Empirical rejection rates of the considered tests ($d = 3, \alpha = 0.05$)

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
NMix(0.5, 3, I_3)	20	16	13	21	3	3	18	3	11	14	13	15	7	34	1	1	3	1	3	2
	50	61	47	81	3	3	17	5	51	57	39	36	6	77	2	1	2	2	2	3
	100	99	97	100	3	2	52	30	99	99	87	70	5	86	2	0	2	1	2	2
NMix(0.79, 3, I_3)	20	40	39	39	13	10	36	16	29	37	16	11	27	75	18	11	18	11	22	10
	50	96	95	95	14	6	32	46	87	94	49	20	59	96	41	7	47	7	47	6
	100	100	100	100	25	5	42	95	100	100	91	43	89	98	85	10	93	10	83	6
NMix(0.9, 3, I_3)	20	38	42	33	33	28	63	35	38	40	14	21	34	72	35	26	40	28	37	28
	50	89	91	81	66	35	89	84	89	91	40	63	82	96	91	50	94	53	86	52
	100	99	100	98	96	37	99	100	100	100	79	95	99	99	100	67	100	72	100	73
NMix(0.5, 0, B_3)	20	28	28	26	28	25	67	36	38	32	17	22	17	54	28	34	25	32	24	24
	50	67	66	68	45	37	91	77	82	71	45	64	24	83	40	66	35	66	32	50
	100	97	95	97	63	47	99	98	99	97	82	93	26	98	42	90	39	93	31	79
NMix(0.9, 0, B_3)	20	28	31	24	43	41	66	42	38	32	11	31	26	55	43	43	43	46	40	44
	50	59	62	49	80	78	91	80	75	66	20	77	54	80	72	80	73	82	66	81
	100	85	87	73	96	95	99	96	94	89	34	96	67	94	85	98	84	98	78	97
$t_1(0, I_3)$	20	99	99	98	98	97	100	99	99	99	87	99	93	99	98	99	96	99	96	97
	50	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
$t_3(0, I_3)$	20	56	58	52	65	62	88	70	67	62	19	61	43	74	64	71	60	69	56	62
	50	93	94	91	94	91	99	97	96	95	31	97	74	96	91	98	88	98	84	92
	100	100	100	100	100	99	100	100	100	100	41	100	87	100	98	100	96	100	94	99

Table 3 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$t_5(0, I_3)$	20	29	32	26	42	38	70	44	40	34	10	33	24	52	42	47	37	45	36	38
	50	61	64	54	74	69	92	80	75	67	13	80	44	77	68	82	64	80	59	72
	100	89	90	83	92	89	99	96	94	91	14	98	59	92	83	98	79	97	74	92
$t_{10}(0, I_3)$	20	12	13	10	20	19	46	21	17	14	6	13	11	35	18	21	16	19	17	16
	50	22	24	17	39	35	68	40	34	27	7	38	19	46	33	46	30	44	30	36
	100	37	40	28	57	52	86	60	52	43	7	66	25	55	44	73	40	71	35	54
$U^3(0, 1)$	20	11	8	15	0	0	4	0	4	6	8	37	3	18	0	0	1	0	0	1
	50	58	49	65	0	0	1	1	32	38	19	88	2	66	0	0	0	0	0	0
	100	98	97	98	0	0	84	34	95	94	46	100	2	99	0	0	0	0	0	0
$LN^3(0, 0.5)$	20	61	64	56	55	46	73	54	59	63	20	38	57	78	61	46	53	47	61	43
	50	98	99	96	93	78	96	96	98	99	54	88	97	99	99	86	96	86	97	79
	100	100	100	100	100	94	100	100	100	100	90	100	100	100	100	98	100	98	100	96
$B^3(1, 2)$	20	25	23	26	4	4	15	5	13	20	10	15	13	41	4	2	4	2	8	2
	50	81	81	78	3	1	11	18	58	75	28	46	33	88	16	0	9	0	30	1
	100	100	100	99	6	0	20	83	99	100	65	93	75	100	66	0	34	0	72	0
$B^3(2, 2)$	20	5	4	6	1	1	6	1	2	3	5	17	3	16	1	0	1	1	1	1
	50	14	11	17	0	0	0	0	4	7	7	41	2	26	0	0	0	0	0	0
	100	40	36	40	0	0	10	2	20	26	10	79	2	51	0	0	0	0	0	0
$P_{II}^3(0.5, 0, 1)$	20	38	27	48	0	0	3	1	20	26	22	69	6	39	0	0	0	0	0	0
	50	99	98	100	0	0	26	24	97	97	73	100	3	99	0	0	0	0	0	0
	100	100	100	100	0	0	100	100	100	100	99	100	2	100	0	0	0	0	0	0

Table 3 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\mathrm{P}^3_{VII}(5, 0, 1)$	20	19	21	17	30	28	58	32	28	23	8	22	18	45	29	32	28	32	27	28
	50	41	44	34	60	55	84	64	58	47	12	61	35	64	52	70	47	68	46	59
	100	68	71	58	81	76	97	87	83	73	18	88	48	81	70	92	66	92	59	85
$\mathrm{P}^3_{VII}(10, 0, 1)$	20	9	9	8	14	14	39	14	12	10	6	9	9	31	13	14	13	15	12	13
	50	13	15	10	26	24	56	27	22	16	6	23	14	36	26	32	22	31	21	26
	100	20	21	15	40	36	72	41	32	24	7	41	18	41	32	49	30	49	26	40
$\mathcal{S}^3(\mathrm{Exp}(1))$	20	95	94	95	88	84	99	95	97	97	54	95	63	95	89	97	82	94	81	85
	50	100	100	100	100	98	100	100	100	100	72	100	80	100	98	100	96	100	92	99
	100	100	100	100	100	100	100	100	100	100	82	100	86	100	100	100	99	100	95	100
$\mathcal{S}^3(\mathrm{B}(1, 1))$	20	27	24	31	11	9	48	24	35	37	13	25	7	40	15	27	9	18	11	10
	50	57	46	76	2	1	48	29	66	73	16	59	4	52	7	30	4	17	4	3
	100	88	79	99	0	0	65	45	94	97	19	89	2	67	5	36	2	22	1	2
$\mathcal{S}^3(\mathrm{B}(1, 2))$	20	63	62	65	45	37	87	63	72	73	24	64	22	71	50	72	36	60	41	37
	50	97	96	99	57	40	98	91	98	99	32	99	20	93	55	96	40	91	36	56
	100	100	100	100	70	35	100	99	100	100	38	100	16	100	58	100	43	100	36	80
$\mathcal{S}^3(\mathrm{B}(2, 2))$	20	5	5	5	3	3	20	4	5	5	5	4	4	21	4	6	3	4	3	4
	50	5	4	8	0	0	8	3	6	7	5	3	2	21	1	2	1	1	1	0
	100	6	5	15	0	0	3	3	9	9	5	3	1	19	1	1	0	0	0	0

Table 3 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\mathcal{S}^3(\chi_5^2)$	20	37	39	35	44	39	77	50	49	44	10	42	22	59	45	56	37	50	37	39
	50	80	80	78	73	67	97	86	87	84	11	91	37	83	68	91	58	87	54	71
	100	98	98	98	91	84	100	98	99	99	11	100	44	97	79	100	70	99	64	93
$\mathcal{S}^3(\text{LN}(0, 0.5))$	20	20	18	15	29	28	56	30	26	22	8	20	17	43	27	30	26	30	26	27
	50	36	38	28	58	55	85	62	54	43	8	56	32	59	54	69	50	67	47	57
	100	59	63	45	80	75	96	84	76	66	9	84	41	74	68	91	62	90	57	81
$\text{MAR}_3(\text{Exp}(1))$	20	34	37	31	30	27	53	28	32	36	12	19	36	69	34	23	37	24	36	26
	50	83	85	76	66	50	81	70	79	86	30	57	83	97	83	51	87	54	81	55
	100	100	100	98	94	70	96	97	99	100	64	91	99	100	100	73	100	76	99	79
$\text{MAR}_3(\chi_3^2)$	20	24	26	21	23	20	47	21	23	26	9	13	25	55	26	19	27	20	28	21
	50	66	68	56	51	39	68	53	60	69	20	43	67	91	65	35	72	37	64	39
	100	96	97	90	83	56	89	89	94	97	43	78	95	99	98	60	99	63	96	66
$\text{MAR}_3(\chi_5^2)$	20	16	17	14	16	15	38	15	15	16	7	10	16	42	18	15	19	15	20	16
	50	43	45	34	36	27	55	34	38	46	13	28	46	78	48	25	52	26	48	27
	100	79	83	65	65	40	73	66	73	83	24	56	80	96	82	38	88	40	81	41
$\text{MAR}_3(t_3)$	20	17	18	15	24	24	49	25	23	20	10	17	17	41	22	23	23	25	21	25
	50	35	37	30	48	47	74	50	45	39	19	45	31	62	39	47	39	49	35	50
	100	61	63	52	72	69	93	75	71	65	35	72	42	82	58	73	58	77	53	78

Table 3 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\text{MAR}_3(t_5)$	20	9	10	8	14	13	36	13	12	10	6	9	9	30	11	12	10	12	10	11
	50	15	17	13	27	25	52	26	23	18	8	22	15	39	23	26	22	26	20	25
	100	24	26	18	41	40	73	40	34	28	12	38	20	51	32	45	31	48	27	49
$\text{MAR}_3(\Gamma(5, 1))$	20	25	26	22	23	20	48	22	23	26	9	14	23	48	27	20	24	21	27	20
	50	66	69	54	53	37	68	55	60	68	18	42	65	82	70	36	58	38	68	32
	100	96	97	86	86	54	88	91	94	97	38	80	96	98	98	59	93	59	97	51
$\text{NM}_3(0.05)$	20	5	5	5	5	5	23	5	5	5	5	5	5	21	6	5	4	5	5	5
	50	5	5	5	5	5	26	5	5	5	5	5	5	22	5	6	6	7	5	7
	100	5	5	5	6	5	22	5	5	5	5	5	5	21	4	5	4	4	5	4
$\text{NM}_3(0.1)$	20	5	6	5	5	5	25	6	5	5	5	5	5	22	5	6	6	6	6	6
	50	5	5	5	6	6	22	6	6	5	5	5	5	22	6	6	5	6	5	6
	100	5	5	5	6	6	26	6	6	5	5	6	5	21	5	5	4	5	5	5
$\text{NM}_3(0.2)$	20	5	6	5	7	6	24	6	6	5	5	5	6	23	6	6	5	6	5	6
	50	6	6	6	8	7	27	8	7	6	5	6	7	24	6	6	6	6	5	6
	100	7	6	6	9	9	36	10	8	7	6	8	7	24	9	12	7	12	8	10
$\text{S}\lvert\text{N}_3\rvert$	20	16	17	17	14	13	43	20	26	19	36	8	12	39	13	13	11	14	11	12
	50	44	37	57	21	18	64	63	79	53	93	18	17	68	19	23	18	29	16	23
	100	90	81	99	30	24	90	99	100	96	100	33	19	84	22	37	21	53	18	40
$\text{N}_3(\mu_3, \Sigma_{0.5})$	20	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	50	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	100	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 4 Empirical rejection rates of the considered tests ($d = 5, \alpha = 0.05$)

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
NMix$(0.5, 3, \mathrm{I}_5)$	20	12	9	13	4	4	81	3	5	8	10	11	6	40	4	3	3	3	4	3
	50	41	24	52	3	4	43	3	12	21	27	19	6	45	2	1	2	2	3	3
	100	90	69	98	3	3	38	9	56	66	70	36	5	46	2	0	2	1	4	2
NMix$(0.79, 3, \mathrm{I}_5)$	20	25	25	24	12	10	87	12	15	24	13	7	26	46	12	9	14	9	16	9
	50	85	83	83	9	5	53	22	43	78	36	12	71	56	26	8	32	7	31	5
	100	100	100	100	11	5	47	66	96	100	82	25	60	95	63	7	86	6	67	3
NMix$(0.9, 3, \mathrm{I}_5)$	20	25	30	22	32	29	94	27	28	33	13	9	34	48	30	24	38	28	31	28
	50	85	92	75	50	30	93	65	75	94	41	53	95	59	82	46	95	48	81	49
	100	100	100	98	77	25	98	98	100	100	86	89	65	89	100	60	100	66	99	68
NMix$(0.5, 0, \mathrm{B}_5)$	20	56	58	54	53	45	100	72	74	66	30	37	26	68	61	71	36	66	46	39
	50	98	98	99	78	61	100	99	100	99	79	96	43	87	78	97	55	95	57	64
	100	100	100	100	93	72	100	100	100	100	99	100	94	30	83	100	66	100	61	90
NMix$(0.9, 0, \mathrm{B}_5)$	20	32	41	28	62	61	97	56	54	48	15	33	35	60	58	58	57	61	53	58
	50	75	83	66	96	95	99	95	94	89	26	93	82	84	95	96	92	96	90	95
	100	96	98	92	100	100	100	100	100	99	45	100	96	95	100	100	99	100	97	100
$\mathrm{t}_1(0, \mathrm{I}_5)$	20	99	100	99	100	99	100	100	100	100	94	99	96	93	100	100	98	100	99	99
	50	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
$\mathrm{t}_3(0, \mathrm{I}_5)$	20	62	69	59	79	75	100	85	83	76	23	64	50	70	82	86	72	83	73	73
	50	98	99	97	99	98	100	100	100	99	40	100	89	93	98	100	96	99	95	98
	100	100	100	100	100	100	100	100	100	100	53	100	98	99	100	100	99	100	99	100

Table 4 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$t_5(0, I_5)$	20	32	39	28	54	49	99	59	57	47	12	29	27	58	55	60	49	57	49	50
	50	78	82	72	89	83	100	95	94	87	15	93	63	80	89	96	81	95	78	85
	100	98	99	96	99	97	100	100	100	99	19	100	81	92	98	100	94	100	91	98
$t_{10}(0, I_5)$	20	13	15	12	26	22	95	28	26	20	7	8	12	50	28	31	23	29	23	24
	50	28	34	23	54	48	93	65	60	44	7	54	28	61	55	67	46	63	44	48
	100	54	61	43	79	68	98	89	84	69	7	87	39	71	73	92	62	90	55	73
$U^5(0, 1)$	20	9	5	11	0	1	58	0	1	2	7	37	3	33	0	0	1	0	1	1
	50	49	34	52	0	0	2	0	1	13	20	91	2	44	0	0	0	0	0	0
	100	96	93	95	0	0	0	0	25	75	57	100	1	69	0	0	0	0	0	0
$LN^5(0, 0.5)$	20	54	61	49	55	48	98	54	56	65	20	26	51	67	62	50	50	50	59	46
	50	98	99	96	94	81	100	97	98	100	63	91	99	93	100	91	96	91	99	83
	100	100	100	100	100	96	100	100	100	100	97	100	100	99	100	100	100	100	100	98
$B^5(1, 2)$	20	18	16	19	3	4	78	2	4	12	9	14	11	45	3	2	3	2	6	3
	50	73	74	68	2	1	26	3	13	62	27	43	24	70	8	0	4	1	23	2
	100	100	100	98	1	0	12	22	74	99	71	91	65	91	44	0	13	0	65	0
$B^5(2, 2)$	20	5	3	6	1	2	66	1	1	2	5	19	4	35	1	0	1	0	1	1
	50	14	9	15	0	0	6	0	0	3	8	50	2	37	0	0	0	0	0	0
	100	37	30	34	0	0	1	0	1	13	13	85	2	46	0	0	0	0	0	0
$P^5_{II}(0.5, 0, 1)$	20	25	14	29	0	0	55	0	1	6	17	64	4	35	0	0	0	0	0	0
	50	97	90	98	0	0	1	0	21	63	71	100	2	63	0	0	0	0	0	0
	100	100	100	100	0	0	16	24	100	100	100	100	1	86	0	0	0	0	0	0

Table 4 continued

Distribution	n	$\frac{\text{BH}}{\text{EP}}$	$\frac{\overline{\text{BH}}}{\overline{\text{EP}}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\mathrm{P}^5_{VII}(5, 0, 1)$	20	16	20	14	32	29	96	34	32	25	9	12	16	51	32	36	29	35	26	29
	50	39	46	32	67	61	97	77	73	57	14	66	41	66	64	78	56	76	54	63
	100	71	77	60	88	82	100	95	94	83	24	94	58	79	84	96	75	95	70	87
$\mathrm{P}^5_{VII}(10, 0, 1)$	20	8	9	7	14	13	92	15	14	12	6	4	8	44	15	16	12	16	12	13
	50	11	14	9	28	25	80	32	29	19	6	21	15	51	27	37	22	34	20	26
	100	18	22	14	45	38	87	54	46	28	8	45	19	54	35	56	31	55	28	40
$\mathcal{S}^5(\mathrm{Exp}(1))$	20	99	99	99	99	97	100	100	100	100	80	99	83	91	99	100	95	100	95	94
	50	100	100	100	100	100	100	100	100	100	93	100	98	99	100	100	100	100	99	100
	100	100	100	100	100	100	100	100	100	100	97	100	100	100	100	100	100	100	100	100
$\mathcal{S}^5(\mathrm{B}(1, 1))$	20	74	71	76	52	38	99	80	85	81	33	63	22	70	63	81	29	69	43	31
	50	99	98	100	45	20	100	97	100	99	37	99	19	82	66	98	27	90	36	30
	100	100	100	100	40	5	100	100	100	100	42	100	14	91	61	100	24	99	23	29
$\mathcal{S}^5(\mathrm{B}(1, 2))$	20	94	93	94	86	75	100	97	98	97	52	90	47	82	93	98	70	95	77	72
	50	100	100	100	98	90	100	100	100	100	62	100	63	96	98	100	83	100	81	92
	100	100	100	100	100	96	100	100	100	100	69	100	64	99	99	100	92	100	85	100
$\mathcal{S}^5(\mathrm{B}(2, 2))$	20	26	25	27	22	16	96	42	43	34	9	16	9	53	29	43	11	33	18	14
	50	65	57	70	14	6	90	62	73	67	8	68	9	61	26	66	9	40	13	9
	100	94	89	97	6	1	95	82	95	93	7	95	8	68	22	88	8	69	11	5
$\mathcal{S}^5(\chi^2_5)$	20	72	73	70	77	67	100	89	88	82	22	69	39	72	78	88	61	82	64	62
	50	100	100	100	98	93	100	100	100	100	23	100	72	91	98	100	89	100	87	94
	100	100	100	100	100	99	100	100	100	100	24	100	84	98	100	100	98	100	93	100

Table 4 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$\mathcal{S}^5(\text{LN}(0, 0.5))$	20	33	40	30	55	49	99	62	60	49	11	31	26	62	57	66	45	61	47	48
	50	82	86	77	91	83	100	97	96	90	14	95	61	81	88	97	80	94	76	83
	100	99	99	98	99	97	100	100	100	100	15	100	78	93	98	100	93	100	88	98
$\text{MAR}_5(\text{Exp}(1))$	20	16	18	14	19	19	90	16	17	20	9	6	22	50	19	16	22	17	22	18
	50	47	58	39	45	37	83	42	44	62	17	36	72	67	59	35	67	39	59	39
	100	88	94	76	76	55	93	76	80	96	40	72	98	74	94	56	97	64	91	66
$\text{MAR}_5(\chi_3^2)$	20	11	13	11	15	14	88	12	13	14	7	5	15	48	14	11	17	13	15	14
	50	32	41	26	34	28	80	30	31	45	12	25	52	64	45	28	56	30	45	34
	100	71	79	56	59	41	88	58	61	84	25	54	91	71	84	40	94	47	80	53
$\text{MAR}_5(\chi_5^2)$	20	9	9	8	11	11	88	9	9	10	6	5	10	44	11	10	13	11	12	11
	50	21	25	16	24	19	68	19	19	28	9	16	31	58	28	15	35	17	30	21
	100	46	55	33	41	29	75	37	38	61	16	35	71	68	60	27	78	33	61	37
$\text{MAR}_5(t_3)$	20	9	11	9	18	17	90	15	15	14	8	7	12	45	17	15	21	18	16	21
	50	20	25	16	39	38	81	38	37	30	13	31	25	54	31	34	36	38	29	40
	100	38	43	30	62	60	91	63	59	51	22	58	38	61	50	62	56	67	47	70
$\text{MAR}_5(t_5)$	20	6	7	6	11	10	86	9	8	7	5	5	7	42	9	8	10	9	8	9
	50	9	11	8	20	19	69	19	17	13	6	13	12	46	18	20	18	22	16	22
	100	13	15	10	32	30	73	31	27	20	8	24	17	51	24	32	28	35	21	37
$\text{MAR}_5(\Gamma(5, 1))$	20	19	22	17	21	18	93	18	19	23	9	7	18	52	20	17	18	18	22	17
	50	59	69	47	50	36	87	50	53	73	19	39	62	79	73	42	54	42	71	37
	100	95	98	84	83	54	95	88	91	99	46	80	97	91	98	67	90	68	97	53

Table 4 continued

Distribution	n	BH EP	$\overline{\text{BH}}$ $\overline{\text{EP}}$	HZ	HV	HJ	HJM	DEH	DEH*	$\mathcal{E}$	$T(\mathbf{f}_1)$	$T(\mathbf{f}_2)$	T_{CS}	PU	$b^{(1)}$	$b^{(2)}$	$b_M^{(1)}$	$\widetilde{b}^{(2)}$	$\widetilde{b}^{(1)}$	$b_M^{(2)}$
$NM_5(0.05)$	20	5	5	5	6	6	83	6	5	5	5	5	5	40	5	5	5	5	4	6
	50	5	5	5	5	6	49	6	5	5	5	5	5	41	6	5	6	5	6	6
	100	4	5	5	5	6	37	5	5	5	5	5	5	41	5	4	5	5	5	5
$NM_5(0.1)$	20	5	5	5	6	6	85	6	6	5	5	5	5	39	4	5	6	6	5	5
	50	6	5	5	6	6	49	6	6	6	5	5	6	41	6	6	6	6	7	6
	100	5	6	5	7	6	50	7	7	6	5	5	6	42	8	9	8	8	8	8
$NM_5(0.2)$	20	6	7	6	9	9	87	10	9	8	6	4	7	42	8	9	8	9	7	7
	50	10	10	9	12	11	68	16	16	12	10	8	9	45	12	15	12	14	11	13
	100	13	14	12	17	14	75	26	25	16	16	15	10	47	14	22	14	26	11	22
$S\lvert N_5\rvert$	20	16	15	16	15	13	95	20	23	18	26	4	10	48	16	17	12	16	12	11
	50	41	35	47	24	19	88	54	67	42	86	19	17	57	26	33	17	38	18	20
	100	87	75	95	36	26	95	97	99	82	100	40	21	65	28	53	22	68	19	35
$N_5(\mu_5, \Sigma_{0.5})$	20	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	50	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	100	5	5	5	5	4	5	6	5	5	5	5	5	5	5	5	5	5	5	5

covariance matrices, the strongest procedure is HJM. The HJM test performs also best throughout the multivariate t-distributions. For the independently simulated components, $T_{MQ}(\mathbf{f}_2)$ is strong, especially for marginal distributions with bounded support. Interestingly, each of the tests that are based on measures of skewness and kurtosis, as well as the HV and HJ tests, completely fails to detect these alternatives. For the Pearson type VII alternatives, HJM again has the strongest power, while $\overline{\text{BHEP}}$ shows the strongest performance for $\text{LN}^2(0, 0.5)$ and BHEP for $\text{B}^2(1, 2)$. The spherically symmetric alternatives with bounded support of the radial distributions are well detected by the HZ and the $\mathcal{E}$ tests. For the case of unbounded support of the radial distribution, the strongest test is again HJM. This test is also strongest for the marginally perturbed alternatives $\text{MAR}_2(\text{DIST})$, where it is just outperformed by the PU test for the perturbation by Exp(1)- and χ^2-random variables. The $\text{NM}_d(\vartheta)$-distributions are uniformly best detected by HJM, although the power is not very strong, whereas all other tests almost completely fail to detect these alternatives. Notably, the $\text{S}|\text{N}_2|$ alternatives are best detected by $T_{MQ}(\mathbf{f}_1)$. Overall, for the chosen alternatives HJM performs best, but it also lacks power especially when the support of the distribution is bounded. From a robust point of view, the weighted L^2 procedures, like DEH*, the HZ test as well as the energy test $\mathcal{E}$ perform very well, especially if the focus is on consistency.

In dimensions $d = 3$ and $d = 5$, one can paint the same picture for the allocation of the best procedures to the alternatives. Interestingly, the power of the procedures increases compared to the lower-dimensional setting, which appears to be counterintuitive in view of the curse of dimensionality. Some noticeable phenomena arise: For the $\mathcal{S}^d(\text{B}(2, 2))$ distribution, some of the tests, like HV, HJ and T_{CS}, $b_M^{(1)}$, $b_M^{(2)}$ seem to loose power when the sample size is increased. An explanation for this behavior for the latter tests might be that these procedures use an approximation of the maximum on the unit sphere, which might be harder to approximate for larger samples. In the case $d = 3$, we also observe this behavior for the HJM test. Interestingly, the HJM test and the PU test increase the power against $\text{NM}_d(\vartheta)$-alternatives in comparison with the case $d = 2$, whereas the other procedures nearly uniformly fail to distinguish them from the null hypothesis in each dimension considered.

11 Conclusions and outlook

From a practical point of view, we recommend to use the computationally efficient weighted L^2-type procedures, like BHEP (or versions of it like HZ) and DEH*, or the energy test $\mathcal{E}$, since they show a good balance between fast computation time and robust power against many alternatives, and they do not exhibit any particular weakness. If computation time is not an issue, we suggest to employ the HJM test, as it outperforms most of the other procedures. Note that by choosing other tuning parameters, the weighted L^2-procedures are expected to benefit in terms of power against specific alternatives, especially if one is able to choose the tuning parameter in a data dependent way. For a first step in this direction for univariate goodness-of-fit tests, see Tenreiro (2019). In general, it would be nice to have explicit solutions of the Fredholm integral equation (2.4). For some recent cases in which such integral equations have witnessed explicit solutions in the context of goodness-of-fit testing, see, e.g., Theorem 3.2 of Baringhaus and Taherizadeh (2010) or Theorems 3 and 5 of Hadjicosta and Richards (2019). High-dimensional L^2-statistics for testing normality have not been considered so far in the literature. The efficient implementation of the tests in the package `mnt` admits first simulations, which indicate that new interesting phenomena arise.

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