Geometriae Dedicata **55:** 319–340, 1995.

On the Mean Number of Normals Through a Point in the Interior of a Convex Body

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(Received: 30 December 1993; revised version: 4 May 1994)

Abstract. Recently, Kathy Hann established bounds on the average number of normals through a point in a convex body K, in the cases where K is either a polytope or sufficiently smooth. In addition, an Euler-type theorem was obtained for these particular classes of convex bodies. In the present work we show that all these statements are true for an arbitrary convex body K. For this purpose measure geometric tools and a general approximation technique will be essential.

Mathematics Subject Classifications (1991): Primary 52A40, 52A38; Secondary 53C65, 52A22.

1. Introduction and Statement of Results

The general setting throughout this paper will be the d-dimensional Euclidean space $\mathbb{R}^d$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We assume $d \geq 2$. The class of all compact, convex sets with nonempty interiors will be denoted by $\mathcal{K}_0^d$, its members will be referred to as convex bodies. We define the support function $h(K, \cdot)$ of $K \in \mathcal{K}_0^d$ by $h(K, u) := \max\{\langle u, y \rangle \mid y \in K\}$ for all $u \in \mathbb{R}^d$. Let $K \in \mathcal{K}_0^d$ be a convex body and let $x \in \operatorname{bd} K$ be a point in the boundary of K. Then the set of all vectors $u \in \mathbb{R}^d$ such that $\langle u, x \rangle = h(K, u)$ makes up the cone $\mathrm{N}(K, x)$ of exterior normal vectors of K at x. Alternatively we could write

$$\mathrm{N}(K, x) := \{u \in \mathbb{R}^d \setminus \{o\} \mid x \in H(K, u)\} \cup \{o\},$$

where $H(K, u) := \{y \in \mathbb{R}^d \mid \langle u, y \rangle = h(K, u)\}$ is the support plane of K with exterior normal vector $u \in \mathbb{R}^d \setminus \{o\}$.

Let us fix $p \in \mathbb{R}^d$ and $K \in \mathcal{K}_0^d$ for the moment. As usual we set $S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. Now we can introduce the number $n(K, p) \in [0, \infty]$ of normals $x + \mathbb{R}u$ that contain the point p, where $x \in \operatorname{bd} K$ and $u \in \mathrm{N}(K, x) \cap S^{d-1}$. More precisely, if we define the generalized (unit) normal bundle of K by

$$\mathcal{N}(K) := \{(x, u) \in \operatorname{bd} K \times S^{d-1} \mid u \in \mathrm{N}(K, x)\},$$

then

$$n(K, p) := \operatorname{card}\{(x, u) \in \mathcal{N}(K) \mid p \in x + \mathbb{R}u\}.$$

To begin with, we report on some local properties of the function $n(K, \cdot)$.

An obvious estimate is $2 \leq n(K, p) \leq \infty$ for arbitrary $K \in \mathcal{K}_0^d$ and $p \in \mathbb{R}^d$. No better inequalities are available in general. This is already shown by the example of a ball. Heil [14, Th. 1] proved that there is at least one point $p \in \operatorname{int} K$ (the interior of K) with $n(K, p) \geq 6$, if $d \geq 3$. For centrally symmetric $K \in \mathcal{K}_0^d$ he sketches a proof that $n(K, p) \geq 2d$ for a suitably chosen point $p \in \operatorname{int} K$. In contrast to the situation for the ball, an amazing result by Zamfirescu [24, Th.] implies that for most convex bodies $K \in \mathcal{K}_0^d$ and for most points $p \in \mathbb{R}^d$ we even have $n(K, p) = \infty$. In this context 'most' has to be understood in the sense of Baire category. Recall that $\mathcal{K}_0^d$ is a Baire space in the topology which is induced by the Hausdorff metric. See, e.g., [12] for the details. This is, however, not a contradiction to our Corollary 1.2, saying that for an arbitrary convex body $K \in \mathcal{K}_0^d$ we have $n(K, p) < \infty$ for almost all $p \in \mathbb{R}^d$ with respect to d-dimensional Lebesgue measure λ^d.

Another way to study normals of a convex body, which is of a global nature, is motivated by some work of Santaló [18, pp. 533–534]. There it is suggested to find the limits between which the integral

$$I(K) := \int_K n(K, y)\, \mathrm{d}\lambda^d(y)$$

varies. The problem of determining the behaviour of the motion invariant functional $I\colon \mathcal{K}_0^d \to \bar{\mathbb{R}}^{\geq 0} := [0, \infty]$ may also be reinterpreted as estimating the mean value (expectation, first moment) of the random variable $n(K, \cdot)$. For this to make sense we have to consider the completion of the measure space $(K, \mathfrak{B}(K), V(K)^{-1}\lambda^d \llcorner K)$ as underlying probability space. In this situation $\mathfrak{B}(K)$ denotes the σ-algebra of Borel sets in K and $V(K) := \lambda^d(K) > 0$.

Only recently, Kathy Hann [13] was able to establish estimates for the average $n(K) := I(K)/V(K)$, when K is either a polytope or bd K is a C^2-submanifold of $\mathbb{R}^d$ with everywhere positive Gauss-Kronecker curvature (in the latter case, K is said to be of class C^2_+). Unfortunately it is not possible to extend her results to the case of a general convex body via standard continuity arguments, since the functional $n : \mathcal{K}_0^d \to \bar{\mathbb{R}}^{\geq 0}$ actually is discontinuous. In fact, for regular $2m$-gons $P_{2m} \in \mathcal{K}_0^2$ in the plane we obtain $n(P_{2m}) = 8$ for all $m \in \mathbb{N}$ [13, p. 38], whereas obviously $n(B^2(o, 1)) = 2$. We use $B^d(x, r)$ to denote the closed Euclidean ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. Nevertheless, in the following we shall generalize the main estimates from [13] and at the same time unify the earlier approach.

THEOREM 1.1. *Let $K \in \mathcal{K}_0^d$ be an arbitrary convex body. Then the similarity invariant functional $n : \mathcal{K}_0^d \to \bar{\mathbb{R}}^{\geq 0}$ satisfies the inequalities*

$$2 \leq n(K) \leq \frac{V(K + DK)}{V(K)} - 1, \tag{1}$$

where $DK := K - K$ denotes the difference body of K.

The left inequality in (1) is sharp. This is shown by the example of a ball $K = B^d(o, 1)$. If, in addition, K is centrally symmetric, the inequality can be expressed as

$$2 \leq n(K) \leq 3^d - 1. \tag{2}$$

The example of the d-cube C_d shows that the right estimate is sharp in this restricted situation.

COROLLARY 1.2. *For an arbitrary convex body $K \in \mathcal{K}_0^d$ the number $n(K, p)$ is finite for λ^d-almost every point $p \in \mathbb{R}^d$.*

The other main result in [13] refers to an Euler-type identity which turned out to be useful in deriving estimates for the quantity $n(K)$. Again the statement in [13, (4.1.5) and (4.2.1)] was subject to severe regularity assumptions for the boundary of the convex body K. In order to be able to state a general version of this result, we first have to introduce some notations. For a convex body $K \in \mathcal{K}_0^d$ and $(x, u) \in \mathcal{N}(K)$ let

$$\sigma(K, x, u) := \mathcal{H}^1(K \cap (x + \mathbb{R}u))$$

denote the length of the secant in K, generated by the line $x + \mathbb{R}u$. Here and in the following we denote by $\mathcal{H}^k$, $k \geq 0$, the k-dimensional Hausdorff measure in $\mathbb{R}^d$. This measure is independent of the dimension of the surrounding space. In particular we have $\lambda^d = \mathcal{H}^d$ in $\mathbb{R}^d$. For a definition and details see [9] or [17]. Next, for $K \in \mathcal{K}_0^d$ we introduce the nearest-point map (metric projection) $p(K, \cdot) : \mathbb{R}^d \to K$, which assigns to an arbitrary $x \in \mathbb{R}^d$ the uniquely determined point $p(K, x) \in K$ satisfying

$$\|x - p(K, x)\| = \min\{\|x - y\| \mid y \in K\}$$

(see [20, 1.2 and Lemma 1.8.9]). A boundary point $x \in \operatorname{bd} K$ is called regular, if $\dim \mathrm{N}(K, x) = 1$, and the set of regular boundary points of K is denoted by $\operatorname{reg} K$. In [20, 2.2, p. 78] the maping $\sigma_K : \operatorname{reg} K \to S^{d-1}$ is defined by letting $\sigma_K(x)$, $x \in \operatorname{reg} K$, be the unique unit outer normal vector of K at x. We extend this definition of σ_K by putting

$$\sigma_K(x) := \frac{x - p(K, x)}{\|x - p(K, x)\|}$$

for $x \in \mathbb{R}^d \backslash K$. Thus σ_K is defined on $\operatorname{reg} K \cup (\mathbb{R}^d \backslash K)$. If we define the parallel body K^ϵ, $\epsilon > 0$, as the Minkowski sum $K + B^d(o, \epsilon)$, we get $\sigma_{K^\epsilon}(x) = \sigma_K(x)$ for $x \in \operatorname{bd} K^\epsilon = \operatorname{reg} K^\epsilon$. We also need the composed mapping

$$F_K : \mathbb{R}^d \backslash K \to \mathbb{R}^d \times S^{d-1}, \; F_K := (p(K, \cdot), \sigma_K).$$

For a Borel subset $\eta \subset \mathbb{R}^d \times S^{d-1}$ and $\rho > 0$ we can define the local parallel set $M_\rho(K, \eta)$ by

$$M_\rho(K, \eta) := \{x \in \mathbb{R}^d \,|\, 0 < \|x - p(K, x)\| \leq \rho,\ F_K(x) \in \eta\}.$$

For these sets a generalized Steiner formula holds, i.e. a polynomial expansion of $\lambda^d(M_\rho(K, \eta))$ in powers of $\rho > 0$, the coefficients of which are positive measures $\Theta_r(K, \eta)$ on the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ for $r \in \{0, \dots, d-1\}$:

$$\lambda^d(M_\rho(K, \eta)) = \frac{1}{d}\sum_{r=0}^{d-1} \rho^{d-r} \binom{d}{r} \Theta_r(K, \eta). \tag{3}$$

This is shown in [20, Th. 4.2.1, (4.2.4)], where further properties of these measures can be found.

We are now prepared to state the announced general Euler-type identity.

THEOREM 1.3. *For an arbitrary convex body $K \in \mathcal{K}_0^d$*

$$(1 - (-1)^d)V(K)$$
$$= \sum_{r=0}^{d-1}(-1)^r \frac{1}{d} \binom{d}{r+1} \int_{\mathcal{N}(K)} \sigma(K, x, u)^{r+1}\, \mathrm{d}\Theta_{d-1-r}(K, x, u). \tag{4}$$

In [13] this was proved in the cases where K is either a polytope or of differentiability class C_+^2. Relation (4) is a generalization of an identity, which was treated by Blaschke [4], Dinghas [7], Santaló [19], Debrunner [6], and Chakerian and Groemer [5], in the special case of a convex body of constant width.

Although our results can be formulated entirely within convexity theory, we will have to apply tools from geometric measure theory. Part of the motivation for the present approach stems from work of Zähle [22] and Kohlmann [15], [16]. Their work may be understood as an effort to transfer concepts and theorems which originally pertained to convex differential geometry to the more general setting of convexity theory. The measure geometric methods of Federer [8], [9], as well as the fundamental investigations of Aleksandrov [1], [2] and Fenchel and Jessen [10] have been essential for this process.

2. Preliminaries

In the present section we gather some material which appears at various places in the literature [21], [22], [15], [11], [16], but mainly in a more general guise than is needed for our purpose. We therefore streamline the current presentation to the Euclidean, convex situation, on which we are focusing in this paper. As regards proofs – the restricted situation would allow for considerable simplifications – we partially refer to the literature cited above for the details. For general terminology with respect to geometric measure theory [9] is our reference, for convexity theory we follow [20].

In Section 1 we already defined mappings $p(K, \cdot)$, σ_K, and F_K which are connected with an arbitrary convex body $K \in \mathcal{K}_0^d$. Let $\mathcal{D}_K \subset \mathbb{R}^d \backslash K$ be the common set of differentiability points for these three mappings, and let the coordinate projection $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be given by $\pi_1(x, y) := x$. We then have for all $\epsilon > 0$

$$\mathcal{N}(K) = F_K(\mathbb{R}^d \backslash K) = F_K(\text{bd } K^\epsilon).$$

Thus $\mathcal{N}(K)$ is a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$, and $F_K \mid \text{bd } K^\epsilon : \text{bd } K^\epsilon \to \mathcal{N}(K)$ is a bi-Lipschitz homeomorphism. This follows from the Lipschitz property of $p(K, \cdot)$ as well as from the representation of its inverse $t_\epsilon := (F_K \mid \text{bd } K^\epsilon)^{-1}$:

$$t_\epsilon : \mathcal{N}(K) \to \text{bd } K^\epsilon, \ (x, u) \mapsto x + \epsilon u.$$

A more general investigation may be found in [21, §2 and Th. 4.3]. For each $\epsilon > 0$ one can show that $\mathcal{H}^{d-1}(\text{bd } K^\epsilon \backslash \mathcal{D}_K) = 0$ [21, Th. 3.3]. An essential step in the proof of this fact is the observation that $p(K, \cdot)$ is differentiable in $x \in \mathbb{R}^d \backslash K$ if and only if $p(K, \cdot)$ is differentiable in $p(K, x) + \lambda \sigma_K(x)$ for an arbitrary $\lambda > 0$. In addition, a symmetric, positive semidefinite bilinear form is defined for $y \in \mathcal{D}_K \cap \text{bd } K^\epsilon$ on the $(d-1)$-dimensional linear space $T_y \text{ bd } K^\epsilon = \text{Tan}(\text{bd } K^\epsilon, y) = \sigma_K(y)^\perp$ [9, §3.1.21] by

$$\text{II}_y(u, v) = \langle D\sigma_K(y)(u), v \rangle, \quad u, v \in T_y \text{ bd } K^\epsilon.$$

For a proof the relation

$$\nabla d_K(y) = \frac{y - p(K, y)}{d_K(y)} = \sigma_K(y), \quad y \in \mathbb{R}^d \backslash K,$$

may be used, where $d_K(y) := \|y - p(K, y)\|$. The distance function d_K is continuously differentiable on $\mathbb{R}^d \backslash K$ [8, Th. 4.8, (3), (5)] and its second differential $D^2 d_K(y)$ exists exactly for $y \in \mathcal{D}_K$. The asserted symmetry then follows from

$$\text{II}_y(u, v) = D^2 d_K(y)(u, v)$$

together with [9, 3.1.11]. Since d_K is convex for $K \in \mathcal{K}_0^d$, it follows that II_y is positive semidefinite. Alternatively this can be viewed as a special case of [3, (4.6), (4.7), (4.9)]. See also [11, Prop. 3.5] or [16, §1, Lemmas 1.8 and 1.9].

In the following survey we denote by $u_1, \ldots, u_{d-1}$ an orthonormal basis of eigenvectors with corresponding eigenvalues $k_1(y), \ldots, k_{d-1}(y) \in [0, \infty)$ for

II_y, $y \in \mathcal{D}_K$. (Here we deviate from the notation used in [15], [16].) Then the limit

$$k_i(K, F_K(y)) := k_i(F_K(y)) := \lim_{t \downarrow 0} \frac{k_i(y)}{1 + (t - \epsilon)k_i(y)}$$
$$= \begin{cases} \infty, & \text{if } k_i(y) = \epsilon^{-1} \\ \dfrac{k_i(y)}{1 - \epsilon k_i(y)}, & \text{if } k_i(y) < \epsilon^{-1} \end{cases},$$

$i \in \{1, \dots, d-1\}$, exists and depends only on $F_K(y)$ and not on $\epsilon := d_K(y)$. This was obtained in [22, §1] and [16, 2.3 Def. and Th.]. From this definition one can derive that for $\mathcal{H}^{d-1}$-almost all $(x, u) = v \in \mathcal{N}(K)$ and for $y := t_\epsilon(v)$ the relation

$$k_i(y) = k_i(K^\epsilon, x + \epsilon u, u) = \frac{k_i(K, v)}{1 + \epsilon k_i(K, v)} \le \frac{1}{\epsilon} < \infty \tag{5}$$

holds for $i \in \{1, \dots, d-1\}$. Moreover, for such $y \in \mathcal{D}_K \cap \operatorname{bd} K^\epsilon$, $\epsilon > 0$, $v = F_K(y)$, an orthonormal basis of the $(d-1)$-dimensional linear subspace $\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), v) \subset \mathbb{R}^d \times \mathbb{R}^d$ of $(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), d-1)$ approximate tangent vectors at v is given by

$$\left(\left(\frac{1 - \epsilon k_i(y)}{\sqrt{(1 - \epsilon k_i(y))^2 + k_i(y)^2}} \cdot u_i, \right.\right.$$
$$\left.\left. \frac{k_i(y)}{\sqrt{(1 - \epsilon k_i(y))^2 + k_i(y)^2}} \cdot u_i \right) \,\middle|\, i = 1, \dots, d-1 \right).$$

This is shown by an application of the tangential map

$$DF_K(y) \,|\, \mathrm{Tan}(\operatorname{bd} K^\epsilon, y) : \mathrm{Tan}(\operatorname{bd} K^\epsilon, y) \to \mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), v),$$

which proves that

$$\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), v) = \mathrm{Tan}(\mathcal{N}(K), v)$$
$$= \mathrm{im}(DF_K(y) \,|\, \mathrm{Tan}(\operatorname{bd} K^\epsilon, y))$$

according to [9, Lemma 3.2.17]. The right equality is also implied by [21, Prop. 4.2]. It will be convenient to use

$$\frac{1 - \epsilon k_i(y)}{\sqrt{(1 - \epsilon k_i(y))^2 + k_i(y)^2}} = \frac{1}{\sqrt{1 + k_i(v)^2}},$$

and similarly

$$\frac{k_i(y)}{\sqrt{(1-\epsilon k_i(y))^2 + k_i(y)^2}} = \frac{k_i(v)}{\sqrt{1+k_i(v)^2}}.$$

In order to verify these relations one has to distinguish the two cases $k_i(y) < \epsilon^{-1}$ and $k_i(y) = \epsilon^{-1}$. See also [22, §3, Proof of Prop.]. It is now easy to compute the following $(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K),\, d-1)$ approximate Jacobians for $\mathcal{H}^{d-1}$-almost all $(x,\, u) \in \mathcal{N}(K)$:

$$\text{ap}\, J_{d-1}(\pi_1 \,|\, \mathcal{N}(K))(x,\, u) = \frac{1}{\prod_{i=1}^{d-1} \sqrt{1+k_i(K,\, x,\, u)^2}}, \tag{6}$$

$$\text{ap}\, J_{d-1} t_\epsilon(x,\, u) = \prod_{i=1}^{d-1} \frac{1+\epsilon k_i(K,\, x,\, u)}{\sqrt{1+k_i(K,\, x,\, u)^2}}. \tag{7}$$

Here we should recall [9, §3.2.16]. In the same way we derive for the mapping

$$f : \mathcal{N}(K) \times [0,\, \text{diam}\, K] \to \mathbb{R}^d,\ (x,\, u,\, \lambda) \mapsto x - \lambda u,$$

for $\mathcal{H}^d$-almost every $(x,\, u,\, \lambda) \in \mathcal{N}(K) \times [0,\, \text{diam}\, K]$, the equation

$$\text{ap}\, J_d f(x,\, u,\, \lambda) = \prod_{i=1}^{d-1} \frac{|1-\lambda k_i(K,\, x,\, u)|}{\sqrt{1+k_i(K,\, x,\, u)^2}}. \tag{8}$$

We shall need these approximate Jacobians in the following in order to be able to apply Federer's general area/coarea formula [9, §3.2.22].

A comparison of the coefficients in Steiner formulae for local parallel sets, given in [20, Th. 4.2.1, (4.2.4)] respectively [22, Th. 2, (10)], and essentially also in [16, Th. 2.7], after an obvious extension of the reasoning there, leads finally to

$$\binom{d-1}{r} \Theta_{d-1-r}(K,\, \eta)$$
$$= \int_{\mathcal{N}(K)\cap\eta} \frac{\sum_{1\le i_1<\dots<i_r\le d-1} k_{i_1}(K,\, v)\dots k_{i_r}(K,\, v)}{\prod_{i=1}^{d-1}\sqrt{1+k_i(K,\, v)^2}}\, d\mathcal{H}^{d-1}(v) \tag{9}$$

for $r \in \{0, \dots,\, d-1\}$ and Borel sets $\eta \subset \mathbb{R}^d \times S^{d-1}$. If $r = 0$, the empty sum in the numerator has to be interpreted as 1. The connection between these generalized curvature measures $\Theta_r(K,\, \cdot)$ and Federer's [8] curvature measures

$C_r(K, \cdot)$, respectively the surface area measures $S_r(K, \cdot)$ of Aleksandrov [1] and Fenchel and Jessen [10], is described by the relations

$$\Theta_r(K, \beta \times S^{d-1}) = C_r(K, \beta), \quad \beta \in \mathfrak{B}(\mathbb{R}^d), \tag{10}$$

and

$$\Theta_r(K, \mathbb{R}^d \times \omega) = S_r(K, \omega), \quad \omega \in \mathfrak{B}(S^{d-1}), \tag{11}$$

$r \in \{0, \dots, d-1\}$. From (5), (6), (9), (10), and also from [9, Th. 3.2.22], one derives for $\beta \in \mathfrak{B}(\mathbb{R}^d)$ and $\epsilon > 0$

$$\binom{d-1}{r} C_{d-1-r}(K^\epsilon, \beta) = \int_{\mathrm{bd}\, K^\epsilon \cap \beta} \sum_{1 \le i_1 < \dots < i_r \le d-1}$$

$$\times k_{i_1}(K^\epsilon, x, \sigma_{K^\epsilon}(x)) \dots k_{i_r}(K^\epsilon, x, \sigma_{K^\epsilon}(x))\, \mathrm{d}\mathcal{H}^{d-1}(x), \tag{12}$$

where $\sigma_{K^\epsilon}(x) = \sigma_K(x) \in \mathrm{N}(K^\epsilon, x) \cap S^{d-1}$ is uniquely determined for all $x \in \mathrm{bd}\, K^\epsilon$.

3. Proof of the Main Results

Let $K \in \mathcal{K}_0^d$ be a convex body and $\mathcal{N}(K)$ its corresponding generalized normal bundle. Instead of the cylindrical set M, considered by Hann [13] in the smooth case, we define $M(K) \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ by

$$M(K) := \{(x, u, \lambda) \mid (x, u) \in \mathcal{N}(K), \lambda \in [0, \sigma(K, x, u)]\}.$$

Therefore the first problem we encounter is to show the measurability of this newly defined set $M(K)$. A verification is provided in the course of the next two lemmas. The first of these will also be relevant to our proof of Theorem 1.3.

LEMMA 3.1. *Let $K \in \mathcal{K}_0^d$ be a convex body, $(x, u) \in \mathcal{N}(K)$, and $\epsilon > 0$. Then we have $(x + \epsilon u, u) \in \mathcal{N}(K^\epsilon)$ and*

$$\lim_{\epsilon \downarrow 0} \sigma(K^\epsilon, x + \epsilon u, u) = \sigma(K, x, u). \tag{13}$$

In addition, the mapping

$$\sigma(K, \cdot) : \mathcal{N}(K) \to \mathbb{R}^{\ge 0}, \ (x, u) \mapsto \sigma(K, x, u),$$

is Borel measurable.

Proof. For each $\epsilon > 0$ the equality $K^\epsilon \cap (x + \epsilon u + \mathbb{R}u) = K^\epsilon \cap (x + \mathbb{R}u)$ holds, and we have according to [20, Lemma 1.8.1]

$$\lim_{\epsilon \downarrow 0}[K^\epsilon \cap (x + \mathbb{R}u)] = \bigcap_{\epsilon > 0}[K^\epsilon \cap (x + \mathbb{R}u)] = K \cap (x + \mathbb{R}u).$$

Viewing $K^\epsilon \cap (x + \mathbb{R}u)$, $\epsilon > 0$, as convex bodies in $x + \mathbb{R}u$, the first statement is implied by [20, Th. 1.8.16]. To show the measurability statement, we observe that t_ϵ and

$$\text{bd } K^\epsilon \to \mathbb{R}^{\geq 0},\ y \mapsto \sigma(K^\epsilon,\ y,\ \sigma_{K^\epsilon}(y)),$$

are continuous maps. See [20, Th. 1.8.8] for the second map. Hence, the composition of these two maps,

$$\mathcal{N}(K) \to \mathbb{R}^{\geq 0},\ (x,\ u) \mapsto \sigma(K^\epsilon,\ x + \epsilon u,\ u),$$

is continuous as well. According to the first part of the present proof, the mapping

$$\sigma(K,\ \cdot) : \mathcal{N}(K) \to \mathbb{R}^{\geq 0},\ (x,\ u) \mapsto \sigma(K,\ x,\ u),$$

is obtained as limit of a sequence of continuous functions. Hence this limit is Borel measurable. □

LEMMA 3.2. *The set $M(K) \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ is closed and thus Borel measurable.*

Proof. For $\epsilon > 0$ define the approximating sets

$$M^\epsilon(K) := \{(x,\ u,\ \lambda) \mid (x,\ u) \in \mathcal{N}(K),\ \lambda \in [0,\ \sigma(K^\epsilon,\ x + \epsilon u,\ u)]\}.$$

From Lemma 3.1 we can conclude that

$$\bigcap_{n \in \mathbb{N}} M^{1/n}(K) = M(K).$$

Therefore it is sufficient to prove that $M^{1/n}(K)$ is a closed set for each $n \in \mathbb{N}$. The mapping

$$H^\epsilon : \begin{cases} \mathcal{N}(K) \times [0,\ 1] \to \mathcal{N}(K) \times \mathbb{R} \\ \qquad (x,\ u,\ t) \mapsto (x,\ u,\ t\sigma(K^\epsilon,\ x + \epsilon u,\ u)) \end{cases}$$

is continuous. Since $\mathcal{N}(K) \times [0,\ 1]$ is compact, the image set

$$H^\epsilon(\mathcal{N}(K) \times [0,\ 1]) = M^\epsilon(K)$$

is a compact subset of the topological Hausdorff space $\mathcal{N}(K) \times \mathbb{R}$ and hence a closed set. □

The number $n(K, p)$, $p \in K$, has already been defined. It is not self-evident that the map $p \mapsto n(K, p)$ is $\mathcal{H}^d$-measurable on K. However, this statement is implicitly contained in Proposition 3.3.

PROPOSITION 3.3. *Let $K \in \mathcal{K}_0^d$. Then we have*

$$I(K) = \int_K n(K, y) \, d\mathcal{H}^d(y)$$

$$= \int_{\mathcal{N}(K)} \int_0^{\sigma(K, x, u)} \prod_{i=1}^{d-1} \frac{|1 - \lambda k_i(K, x, u)|}{\sqrt{1 + k_i(K, x, u)^2}} \, d\mathcal{H}^1(\lambda) \, d\mathcal{H}^{d-1}(x, u). \quad (14)$$

Proof. Let us define $W := \mathcal{N}(K) \times [0, \text{diam } K] \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. From [9, Th. 3.2.23] we obtain that W is $(\mathcal{H}^d, d)$-rectifiable and trivially W is Borel measurable. We choose $R > 0$ sufficiently large, so that $x - \lambda u \in B^d(o, R)$ for $(x, u) \in \mathcal{N}(K)$ and $\lambda \in [0, \text{diam } K]$. The set $Z := B^d(o, R) \subset \mathbb{R}^d$ is $(\mathcal{H}^d, d)$-rectifiable and $\mathcal{H}^d$-measurable. The situation described in [9, Th. 3.2.22] concerns a transforming map f, which is in our case given by

$$f : \begin{cases} \mathcal{N}(K) \times [0, \text{diam } K] \to Z \\ (x, u, \lambda) \mapsto x - \lambda u \end{cases},$$

and a $\mathcal{H}^d \llcorner W$-integrable function g chosen as

$$g : W \to \bar{\mathbb{R}}, \; g := \mathbf{1}_{M(K)}.$$

The characteristic function $\mathbf{1}_{M(K)}(x)$ is equal to 1 for $x \in M(K)$ and zero otherwise. It is also sufficient to assume that g is $\mathcal{H}^d \llcorner W$-measurable and non-negative. An application of Federer's general area/coarea formula [9, Th. 3.2.22] yields

$$\int_W \mathbf{1}_{M(K)} \text{ ap } J_d f \, d\mathcal{H}^d = \int_Z \int_{f^{-1}(\{z\})} \mathbf{1}_{M(K)} \, d\mathcal{H}^0 \, d\mathcal{H}^d(z).$$

Since we explained in Section 2 that for $\mathcal{H}^d$-almost all $(x, u, \lambda) \in W$ the relation

$$\text{ap } J_d f(x, u, \lambda) = \prod_{i=1}^{d-1} \frac{|1 - \lambda k_i(K, x, u)|}{\sqrt{1 + k_i(K, x, u)^2}}$$

holds and because of

$$\int_{f^{-1}(\{z\})} \mathbf{1}_{M(K)}(x, u, \lambda)\, d\mathcal{H}^0(x, u, \lambda) = \begin{cases} n(K, z) & \text{if } z \in K \\ 0 & \text{otherwise} \end{cases}$$

(this confirms our measurability statement), we arrive at equation (14) after just another Fubini-like application of Federer's coarea theorem, i.e. [9, Th. 3.2.23]. □

For a convex body $L \in \mathcal{K}_0^d$ with $o \in \operatorname{int} L$, the radial function $\rho(L, \cdot) : \mathbb{R}^d \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho(L, x) := \max\{\lambda \geq 0 \mid \lambda x \in L\}, \quad x \in \mathbb{R}^d \backslash \{o\}.$$

Our next lemma extends and simplifies an argument in [13, (3.1.7) and pp. 45–46].

LEMMA 3.4. *Let K, $L \in \mathcal{K}_0^d$ be convex bodies and $o \in \operatorname{int} L$. Then*

$$\sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{S^{d-1}} \rho(L, u)^{r+1}\, dS_{d-1-r}(K, u)$$
$$\leq V(K + L) - V(K). \tag{15}$$

Equality holds, e.g., if $K = C_d$ and $L = DC_d$.

Proof. First, let K be a polytope. We use $\mathcal{F}_r(K)$ to denote the system of r-dimensional faces of K, $r \in \{0, \ldots, d-1\}$. The relative interior of $F \in \mathcal{F}_r(K)$ is denoted by relint F. Then the following disjoint partitioning of $(K + L) \backslash \operatorname{int} K$ holds

$$(K + L) \backslash \operatorname{int} K = \bigcup_{r=0}^{d-1} \bigcup_{F \in \mathcal{F}_r(K)} [(K + L) \cap p(K, \cdot)^{-1}(\operatorname{relint} F)].$$

Further on, for $r \in \{0, \ldots, d-1\}$ and $F \in \mathcal{F}_r(K)$, we have the inclusion

$$\operatorname{relint} F + \{tu \mid t \in [0, \rho(L, u)],\ u \in \mathrm{N}(K, F) \cap S^{d-1}\}$$
$$\subset (K + L) \cap p(K, \cdot)^{-1}(\operatorname{relint} F). \tag{16}$$

Notice that $\mathrm{N}(K, F) := \mathrm{N}(K, x)$, $x \in \operatorname{relint} F$, is independent of the particular choice of x, see [20, p. 72]. Therefore we obtain from Fubini's theorem

$$\mathcal{H}^r(F) \frac{1}{d-r} \int_{\mathrm{N}(K,F) \cap S^{d-1}} \rho(L, u)^{d-r}\, d\mathcal{H}^{d-1-r}(u)$$

$$\leq \lambda^d((K+L)\cap p(K,\,\cdot)^{-1}(\text{relint } F)). \tag{17}$$

Equality holds in (16) respectively (17) for $K = C_d$ and $L = DC_d$. If we use

$$S_r(K,\,\cdot) = \frac{d}{\binom{d}{r}(d-r)} \sum_{F\in\mathcal{F}_r(K)} \mathcal{H}^r(F)\mathcal{H}^{d-1-r}(\mathrm{N}(K,\,F)\cap\cdot),$$

see [20, (4.2.11) and (4.2.18)], we get for $r \in \{0,\ldots,\,d-1\}$

$$\frac{1}{d}\binom{d}{r}\int_{S^{d-1}} \rho(L,\,u)^{d-r}\,\mathrm{d}S_r(K,\,u)$$

$$\leq \sum_{F\in\mathcal{F}_r(K)} \lambda^d((K+L)\cap p(K,\,\cdot)^{-1}(\text{relint } F)).$$

This yields the lemma for a polytope K. The general case now is an immediate consequence of the weak continuity of the surface area measures. □

Proof of Theorem 1.1. The convexity of K implies for $\mathcal{H}^{d-1}$-almost all $(x,\,u) \in \mathcal{N}(K)$ that $k_i(K,\,x,\,u) \in [0,\,\infty]$, $i \in \{1,\,\ldots,\,d-1\}$. In the sequel we shall use the abbreviation

$$H_r(k_1,\ldots,\,k_{d-1}) := \sum_{1\leq i_1<\cdots<i_r\leq d-1} k_{i_1}\ldots k_{i_r},$$

where $k_1,\ldots,\,k_{d-1} \in [0,\,\infty]$ and $r \in \{1,\ldots,\,d-1\}$. In the special case $r=0$ we define

$$H_0(k_1,\,\ldots,\,k_{d-1}) := 1.$$

We even write $H_r(k_i)$, if it is clear from the context that i runs through the index set $i = 1,\ldots,\,d-1$. Similar to [13] we introduce a rough estimate in (14):

$$I(K) \leq \int_{\mathcal{N}(K)}\int_0^{\sigma(K,x,u)} \prod_{i=1}^{d-1} \frac{1+\lambda k_i(K,\,x,\,u)}{\sqrt{1+k_i(K,\,x,\,u)^2}}\,\mathrm{d}\mathcal{H}^1(\lambda)\,\mathrm{d}\mathcal{H}^{d-1}(x,\,u)$$

$$= \sum_{r=0}^{d-1}\int_{\mathcal{N}(K)}\int_0^{\sigma(K,x,u)} \frac{H_r(k_i(K,\,x,\,u))}{\prod_{i=1}^{d-1}\sqrt{1+k_i(K,\,x,\,u)^2}}$$

$$\times\lambda^r\,\mathrm{d}\mathcal{H}^1(\lambda)\,\mathrm{d}\mathcal{H}^{d-1}(x,\,u)$$

$$= \sum_{r=0}^{d-1} \frac{1}{r+1} \int_{\mathcal{N}(K)} \frac{H_r(k_i(K, x, u))}{\prod_{i=1}^{d-1} \sqrt{1 + k_i(K, x, u)^2}} \times \sigma(K, x, u)^{r+1} \, \mathrm{d}\mathcal{H}^{d-1}(x, u)$$

$$= \sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{\mathcal{N}(K)} \sigma(K, x, u)^{r+1} \, \mathrm{d}\Theta_{d-1-r}(K, x, u)$$

$$\leq \sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{\mathcal{N}(K)} \rho(DK, u)^{r+1} \, \mathrm{d}\Theta_{d-1-r}(K, x, u).$$

Again, analogous to [13, (3.1.4)], we have used the elementary estimate $\sigma(K, x, u) \leq \rho(DK, u)$ for all $(x, u) \in \mathcal{N}(K)$. This latter estimate enables us to get rid of the dependence on $x \in \operatorname{bd} K$, so that according to (11) we may transform the integrals over the generalized normal bundle into integrals over the unit sphere with respect to Aleksandrov's surface area measures:

$$I(K) \leq \sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{S^{d-1}} \rho(DK, u)^{r+1} \, \mathrm{d}S_{d-1-r}(K, u).$$

From our unified approach we get with the help of Lemma 3.4, i.e.

$$\sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{S^{d-1}} \rho(DK, u)^{r+1} \, \mathrm{d}S_{d-1-r}(K, u) \leq V(K + DK) - V(K), \tag{18}$$

the desired implication

$$n(K) = \frac{I(K)}{V(K)} \leq \frac{V(K + DK) - V(K)}{V(K)} \tag{19}$$

for an arbitrary convex body $K \in \mathcal{K}_0^d$. For centrally symmetric K we even obtain the sharp estimate (equality holds for a d-cube)

$$n(K) \leq 3^d - 1. \qquad \square \tag{20}$$

Whereas Lemma 3.1 merely provides pointwise convergence, Lemma 3.5 is formulated in terms of uniform convergence. In the proof of Theorem 1.3 this will be essential for the transition from convex bodies of class C_+^∞ to convex bodies all of whose boundary points are ϵ-smooth. For the notion of an ϵ-smooth boundary point, see [20, Notes for §2.5, 7].

LEMMA 3.5. *Let* $K \in \mathcal{K}_0^d$ *be a convex body,* $\epsilon > 0$, *and let* $K_m \in \mathcal{K}_0^d$, $m \in \mathbb{N}$, *be a sequence of convex bodies of class* C_+^∞ *which satisfy* $K^\epsilon \subset K_m \subset K^\epsilon + B^d(o, \epsilon/m)$ *for each* $m \in \mathbb{N}$. *Then there is a sequence* $\beta(m)$, $m \in \mathbb{N}$, *with* $\lim_{m\to\infty}\beta(m) = 0$ *and, for all* $x \in \text{bd } K_m$,

$$|\sigma(K_m, x, \sigma_{K_m}(x)) - \sigma(K^\epsilon, p(K^\epsilon, x), \sigma_{K^\epsilon}(p(K^\epsilon, x)))| \leq \beta(m).$$

Proof. For $x \in \mathbb{R}^d \backslash \text{int } K^\epsilon$ we define $s(x) \in \text{bd } K^\epsilon$ by the relation

$$\{s(x)\} = \text{bd } K^\epsilon \cap (p(K^\epsilon, x) + \mathbb{R}^{>0}(-\sigma_{K^\epsilon}(p(K^\epsilon, x)))).$$

The mapping $s \mid \text{bd } K^\epsilon : \text{bd } K^\epsilon \to \text{bd } K^\epsilon$, $x \mapsto s(x)$, is continuous, since for all $x \in \text{bd } K^\epsilon$

$$[x + \mathbb{R}\sigma_{K^\epsilon}(x)] \cap \text{int } K^\epsilon \neq \emptyset.$$

Moreover, for each $x \in \text{bd } K^\epsilon$ we have

$$\measuredangle(\sigma_{K^\epsilon}(s(x)), -\sigma_{K^\epsilon}(x)) < \frac{\pi}{2}.$$

As the spherical Gauss map σ_{K^ϵ} is continuous on bd K^ϵ as well, there is a constant β such that, for all $x \in \text{bd } K^\epsilon$,

$$\measuredangle(\sigma_{K^\epsilon}(s(x)), -\sigma_{K^\epsilon}(x)) \leq \beta < \frac{\pi}{2}.$$

This immediately implies for all $x \in \mathbb{R}^d \backslash \text{int } K^\epsilon$ that

$$\measuredangle(\sigma_{K^\epsilon}(s(x)), -\sigma_{K^\epsilon}(p(K^\epsilon, x))) \leq \beta < \frac{\pi}{2}. \tag{21}$$

Let $x \in \text{bd } K_m$. According to our assumption on K_m, i.e. $K^\epsilon \subset K_m \subset K^{\epsilon(1+(1/m))}$, we have

$$\|x - p(K, x)\| \leq \epsilon\left(1 + \frac{1}{m}\right) \quad \text{and} \quad B^d(p(K, x), \epsilon) \subset K_m.$$

Let $C(x, B^d(p(K, x), \epsilon))$ be the minimal closed cone with apex x that contains the ball $B^d(p(K, x), \epsilon)$, and let $\alpha(x, m)$ be the angle between the axis of this cone and any of its extremal rays. Then

$$\sin \alpha(x, m) = \frac{\epsilon}{\|x - p(K, x)\|} \geq \frac{\epsilon}{\epsilon\left(1 + \frac{1}{m}\right)} = \frac{m}{m+1},$$

so that

$$\measuredangle(\sigma_{K_m}(x), \sigma_{K^\epsilon}(p(K^\epsilon, x))) \leq \frac{\pi}{2} - \arcsin\frac{m}{m+1} =: \alpha(m).$$

This last argument is taken from [15]. Similar to $s(x)$ we define the boundary point $t_m(x) \in \text{bd } K_m$, for $x \in \text{bd } K_m$, by the relation

$$\{t_m(x)\} = \text{bd } K_m \cap [x + \mathbb{R}^{>0}(-\sigma_{K_m}(x))].$$

As a simple consequence of the triangle inequality we see for $x \in \text{bd } K_m$:

$$\|x - t_m(x)\| \leq \|x - p(K^\epsilon, x)\| + \|p(K^\epsilon, x) - s(x)\| + \|s(x) - t_m(x)\|,$$

that is,

$$\|x - t_m(x)\| - \|p(K^\epsilon, x) - s(x)\| \leq \frac{\epsilon}{m} + \|s(x) - t_m(x)\|,$$

as well as

$$\|p(K^\epsilon, x) - s(x)\| \leq \|x - s(x)\| \leq \|x - t_m(x)\| + \|t_m(x) - s(x)\|.$$

Together this yields

$$\big|\|p(K^\epsilon, x) - s(x)\| - \|x - t_m(x)\|\big| \leq \frac{\epsilon}{m} + \|s(x) - t_m(x)\|.$$

To estimate the difference $\|s(x) - t_m(x)\|$, $x \in \text{bd } K_m$, let us denote by $C(x, m)$ the cone

$$C(x, m) := x + \{\lambda u \mid \langle u, -\sigma_{K^\epsilon}(p(K^\epsilon, x))\rangle \geq \cos \alpha(m), \lambda \geq 0, u \in S^{d-1}\},$$

where $x \in \text{bd } K_m$ and $m \in \mathbb{N}$. If we choose $m \geq m_0$, m_0 sufficiently large, and $x \in \text{bd } K_m$, then all extremal rays of the cone $C(x, m)$ hit the ball $B^d(s(x) - \epsilon\sigma_{K^\epsilon}(s(x)), \epsilon)$, according to (21) and $\lim_{m\to\infty} \alpha(m) = 0$. The set $C(x, m)\backslash B^d(s(x) - \epsilon\sigma_{K^\epsilon}(s(x)), \epsilon)$, $m \geq m_0$, is the union of two connected components. We denote the unbounded component by $C^+(x, m)$. Obviously we have

$$s(x), t_m(x) \in C^+(x, m) \cap \left(K^{\epsilon(1+(1/m))}\backslash K^\epsilon\right).$$

If we define

$$\gamma(m) := \sup_{x \in \text{bd } K_m} \text{diam}\left[C^+(x, m) \cap \left(K^{\epsilon(1+(1/m))}\backslash K^\epsilon\right)\right],$$

we see that $\lim_{m\to\infty} \gamma(m) = 0$. Therefore

$$\big|\|p(K^\epsilon, x) - s(x)\| - \|x - t_m(x)\|\big| \leq \frac{\epsilon}{m} + \gamma(m) =: \beta(m)$$

is the desired estimate. □

Proof of Theorem 1.3. First, we choose a sequence of convex bodies K_m, $m \in \mathbb{N}$, as described in Lemma 3.5. For these convex bodies a mapping-degree argument yields:

$$\begin{aligned}
&(1-(-1)^d)V(K_m)\\
&=\sum_{r=0}^{d-1}\frac{(-1)^r}{r+1}\int_{\operatorname{bd}K_m}\sigma(K_m,\,x,\,\sigma_{K_m}(x))^{r+1}H_r(k_i(K_m,\,x))\,\mathrm{d}\mathcal{H}^{d-1}(x)\\
&=\sum_{r=0}^{d-1}\frac{(-1)^r}{r+1}\binom{d-1}{r}\int_{\operatorname{bd}K_m}\\
&\times\sigma(K_m,\,x,\,\sigma_{K_m}(x))^{r+1}\,\mathrm{d}C_{d-1-r}(K_m,\,x). \qquad (22)
\end{aligned}$$

This was essentially stated in [13, (4.1.5)]. To get (22), we used the representation of the curvature measures in the C^2_+ case [20, (4.2.19)] as integrals of the elementary symmetric functions of the principal curvatures $k_i(K_m,\,x)$ of bd K_m. We set

$$I_r(m):=\int_{\operatorname{bd}K_m}\sigma(K_m,\,x,\,\sigma_{K_m}(x))^{r+1}\,\mathrm{d}C_{d-1-r}(K_m,\,x).$$

The mapping

$$x\mapsto\sigma(K^\epsilon,\,p(K^\epsilon,\,x),\,\sigma_{K^\epsilon}(p(K^\epsilon,\,x)))$$

is defined on $\mathbb{R}^d\backslash\operatorname{int}K^\epsilon$ and continuous, since

$$(p(K^\epsilon,\,x)+\mathbb{R}\sigma_{K^\epsilon}(p(K^\epsilon,\,x)))\cap\operatorname{int}K^\epsilon\neq\emptyset.$$

With the help of Lemma 3.5, the following estimate is justified

$$\begin{aligned}
&\left|I_r(m)-\int_{\operatorname{bd}K_m}\sigma(K^\epsilon,\,p(K^\epsilon,x),\,\sigma_{K^\epsilon}(p(K^\epsilon,\,x)))^{r+1}\,\mathrm{d}C_{d-1-r}(K_m,\,x)\right|\\
&\quad\le\int_{\operatorname{bd}K_m}\mathrm{const}\cdot\beta(m)\,\mathrm{d}C_{d-1-r}(K_m,\,x)\\
&\quad=\mathrm{const}\cdot\beta(m)C_{d-1-r}(K_m,\,\mathbb{R}^d).
\end{aligned}$$

The last expression tends to zero for $m \to \infty$, as $K_m \to K^\epsilon$ implies $C_{d-1-r}(K_m, \mathbb{R}^d) \to C_{d-1-r}(K^\epsilon, \mathbb{R}^d)$. Because of the weak continuity of Federer's curvature measures, we observe that

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\mathrm{bd}\, K_m} \sigma(K^\epsilon, p(K^\epsilon, x), \sigma_{K^\epsilon}(p(K^\epsilon, x)))^{r+1}\, \mathrm{d}C_{d-1-r}(K_m, x) \\
&= \int_{\mathrm{bd}\, K^\epsilon} \sigma(K^\epsilon, p(K^\epsilon, x), \sigma_{K^\epsilon}(p(K^\epsilon, x)))^{r+1}\, \mathrm{d}C_{d-1-r}(K^\epsilon, x) \\
&= \int_{\mathrm{bd}\, K^\epsilon} \sigma(K^\epsilon, x, \sigma_{K^\epsilon}(x))^{r+1}\, \mathrm{d}C_{d-1-r}(K^\epsilon, x).
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&(1-(-1)^d)V(K^\epsilon) \\
&= \sum_{r=0}^{d-1} \frac{(-1)^r}{r+1} \binom{d-1}{r} \\
&\times \int_{\mathrm{bd}\, K^\epsilon} \sigma(K^\epsilon, x, \sigma_{K^\epsilon}(x))^{r+1}\, \mathrm{d}C_{d-1-r}(K^\epsilon, x). \qquad (23)
\end{aligned}$$

Obviously $\lim_{\epsilon\downarrow 0} V(K^\epsilon) = V(K)$. Therefore we have to show that the right side of (23) converges to the right side of (4) for $\epsilon \downarrow 0$. To this end the integrals on the right side of (23) will first be transformed into integrals over $\mathcal{N}(K)$ by repeatedly using Federer's coarea formula:

$$\begin{aligned}
&\binom{d-1}{r} \int_{\mathrm{bd}\, K^\epsilon} \sigma(K^\epsilon, x, \sigma_{K^\epsilon}(x))^{r+1}\, \mathrm{d}C_{d-1-r}(K^\epsilon, x) \\
&\overset{(12)}{=} \int_{\mathrm{bd}\, K^\epsilon} \sigma(K^\epsilon, x, \sigma_{K^\epsilon}(x))^{r+1} H_r(k_i(K^\epsilon, x, \sigma_{K^\epsilon}(x)))\, \mathrm{d}\mathcal{H}^{d-1}(x) \\
&\overset{(7)}{=} \int_{\mathcal{N}(K)} \mathrm{ap}\, J_{d-1} t_\epsilon(x, u) \sigma(K^\epsilon, x+\epsilon u, u)^{r+1} \\
&\times H_r(k_i(K^\epsilon, x+\epsilon u, u))\, \mathrm{d}\mathcal{H}^{d-1}(x, u).
\end{aligned}$$

Finally, an application of Lebesgue's bounded convergence theorem [9, §2.4.9] shows that it is sufficient to prove that

$$\mathrm{ap}\, J_{d-1} t_\epsilon(x, u) \sigma(K^\epsilon, x+\epsilon u, u)^{r+1} H_r(k_i(K^\epsilon, x+\epsilon u, u)) \qquad (24)$$

is bounded almost everywhere on $\mathcal{N}(K)$ by a constant independent of ϵ, and further on that this expression converges almost everywhere on $\mathcal{N}(K)$ to

$$\sigma(K, x, u)^{r+1} \frac{H_r(k_1(K, x, u), \ldots, k_{d-1}(K, x, u))}{\prod_{i=1}^{d-1} \sqrt{1+k_i(K, x, u)^2}} \qquad (25)$$

for $\epsilon \downarrow 0$. Boundedness may be seen from $\sigma(K^\epsilon, x+\epsilon u, u) \leq \operatorname{diam} K^\epsilon$ and from the subsequent estimates, where we assume without loss of generality that $0<\epsilon<1$ and $k_1(K, x, u) \geq \cdots \geq k_{d-1}(K, x, u) \geq 0$:

$$\begin{aligned}
&\operatorname{ap} J_{d-1} t_\epsilon(x, u) H_r(k_i(K^\epsilon, x+\epsilon u, u)) \\
&\overset{(7)}{\leq} \prod_{i=1}^{d-1} \frac{1+\epsilon k_i(K, x, u)}{\sqrt{1+k_i(K, x, u)^2}} \binom{d-1}{r} \prod_{j=1}^{r} \frac{k_j(K, x, u)}{1+\epsilon k_j(K, x, u)} \\
&= \binom{d-1}{r} \prod_{i=1}^{r} \frac{k_i(K, x, u)}{\sqrt{1+k_i(K, x, u)^2}} \prod_{i=r+1}^{d-1} \frac{1+\epsilon k_i(K, x, u)}{\sqrt{1+k_i(K, x, u)^2}} \\
&\leq \binom{d-1}{r} \sqrt{2}^{d-1-r}.
\end{aligned}$$

Besides we notice, if we write $k_i := k_i(K, x, u)$ and $k_i(K^\epsilon) := k_i(K^\epsilon, x+\epsilon u, u)$ for the moment and $\{j_1, \ldots, j_{d-1-r}\}$, $1 \leq j_1 < \cdots < j_{d-1-r} \leq d-1$, for the index set complementary to $\{i_1, \ldots, i_r\}$, $1 \leq i_1 < \cdots < i_r \leq d-1$, with respect to $\{1, \ldots, d-1\}$:

$$\begin{aligned}
&\prod_{i=1}^{d-1} \frac{1+\epsilon k_i}{\sqrt{1+k_i^2}} \sum_{1 \leq i_1 < \cdots < i_r \leq d-1} k_{i_1}(K^\epsilon) \ldots k_{i_r}(K^\epsilon) \\
&= \sum_{1 \leq i_1 < \cdots < i_r \leq d-1} \left\{ \left(\prod_{i=1}^{d-1} \frac{1+\epsilon k_i}{\sqrt{1+k_i^2}} \right) k_{i_1}(K^\epsilon) \ldots k_{i_r}(K^\epsilon) \right\} \\
&\overset{(5)}{=} \sum_{1 \leq i_1 < \cdots < i_r \leq d-1} \left\{ \left(\prod_{i=1}^{d-1} \frac{1+\epsilon k_i}{\sqrt{1+k_i^2}} \right) \frac{k_{i_1}}{1+\epsilon k_{i_1}} \cdots \frac{k_{i_r}}{1+\epsilon k_{i_r}} \right\} \\
&= \sum_{1 \leq i_1 < \cdots < i_r \leq d-1} \left\{ \prod_{l=1}^{r} \frac{k_{i_l}}{\sqrt{1+k_{i_l}^2}} \prod_{t=1}^{d-1-r} \frac{1+\epsilon k_{j_t}}{\sqrt{1+k_{j_t}^2}} \right\}.
\end{aligned}$$

Now we have

$$\frac{1+\epsilon k_{j_t}}{\sqrt{1+k_{j_t}^2}} \to \frac{1}{\sqrt{1+k_{j_t}^2}} \text{ for } \epsilon \downarrow 0.$$

This is shown by distinguishing the two cases $k_{j_t} = \infty$ and $k_{j_t} < \infty$. These considerations are all valid for $\mathcal{H}^{d-1}$-almost every $(x, u) \in \mathcal{N}(K)$. The situation here is essentially the same as the one discussed in [15, p. 35]. Lemma 3.1 then completes the proof for the convergence of (24) to (25). □

4. Miscellanea

There seems to be little hope to obtain a complete description of those cases in which equality occurs in the right inequality of (2). However, in the plane and especially for polygons we can draw some conclusions from the above investigations which refine statements of [13].

For an equilateral triangle $T_0 \in \mathcal{K}_0^2$ it is shown in [13] that $n(T_0) = 6$. A more detailed statement can be found in the following proposition.

PROPOSITION 4.1. *For an arbitrary triangle* $T \in \mathcal{K}_0^2$ *the inequalities*

$$4 < n(T) \leq 6$$

hold. For a triangle T *the equality* $n(T) = 6$ *holds if and only if the internal angles of* T *do not exceed* $\frac{1}{2}\pi$. *In addition, for each* $r \in (4, 6]$ *there is a triangle* $T(r)$ *with* $n(T(r)) = r$.

We already know the general estimate $n(K) \leq 8$ for centrally symmetric $K \in \mathcal{K}_0^2$. In the class of all centrally symmetric convex polygons $K \in \mathcal{K}_0^2$ we shall characterize those polygons for which the upper bound is attained.

PROPOSITION 4.2. *Let* $K \in \mathcal{K}_0^2$ *be a centrally symmetric polygon with edges* $F_1, \ldots, F_{2m}$ *in cyclic order. Then the following two statements are equivalent.*

(a) $n(K) = 8$.

(b) *For each* $i \in \{1, \ldots, m\}$ *the convex hull* $\operatorname{conv}\{F_i \cup F_{m+i}\}$ *is a rectangle.*

Proof of Propositions 4.1 and 4.2. We start with a derivation of the inequality $I(K) \leq 8V(K)$ for $K \in \mathcal{K}_0^2$, $K = -K$, along the lines of the proofs for Proposition 3.3 and Theorem 1.1:

$$\begin{aligned}
I(K) &= \int_{\mathcal{N}(K)} \int_0^{\sigma(K,x,u)} \frac{|1 - \lambda k(K, x, u)|}{\sqrt{1 + k(K, x, u)^2}} \, d\mathcal{H}^1(\lambda) \, d\mathcal{H}^1(x, u) \\
&\overset{\circledast}{\leq} \int_{\mathcal{N}(K)} \int_0^{\sigma(K,x,u)} \frac{1 + \lambda k(K, x, u)}{\sqrt{1 + k(K, x, u)^2}} \, d\mathcal{H}^1(\lambda) \, d\mathcal{H}^1(x, u) \\
&= \int_{\mathcal{N}(K)} \sigma(K, x, u) \, d\Theta_1(K, x, u) \\
&\quad + \frac{1}{2} \int_{\mathcal{N}(K)} \sigma(K, x, u)^2 \, d\Theta_0(K, x, u)
\end{aligned}$$

$$= 2 \int_{\mathcal{N}(K)} \sigma(K, x, u) \, d\Theta_1(K, x, u)$$

$$\leq 2 \int_{\mathcal{N}(K)} \rho(DK, u) \, d\Theta_1(K, x, u)$$

$$\leq 8 \cdot \frac{1}{2} \int_{S^1} h(K, u) \, dS_1(K, u) = 8V(K).$$

Here, we used Theorem 1.3 for $d = 2$, $\rho(DK, u) = 2\rho(K, u) \leq 2h(K, u)$, and a very special case of [20, (5.1.18)]. In ⊛ equality occurs if and only if for $\mathcal{H}^1$-almost all $(x, u) \in \mathcal{N}(K)$ we either have $k(K, x, u) = 0$ or $k(K, x, u) = \infty$. Thus for a polygon equality holds in ⊛. It should be remarked that this is true for a typical convex body as we shall see in Theorem 4.5. Now, we observe that $I(K) = 8V(K)$ holds if and only if

$$\sigma(K, x, u) = 2\rho(K, u) = 2h(K, u)$$

is true for $\Theta_1(K, \cdot)$-almost all $(x, u) \in \mathcal{N}(K)$. For a polytope K as described in the assumptions of Proposition 4.2 this last condition may be paraphrased by saying that

$$\sigma(K, x, u_i) = 2\rho(K, u_i) = 2h(K, u_i) \tag{26}$$

must hold true for $i \in \{1, \ldots, 2m\}$ and for $\mathcal{H}^1$-almost all (hence for all) $x \in F_i$, if u_i is the exterior unit normal vector to F_i. A geometric interpretation of (26) yields the statement of Proposition 4.2.

In the special case of a triangle $T \in \mathcal{K}_0^2$ the above reasoning yields

$$I(T) = 2 \int_{\mathcal{N}(T)} \sigma(T, x, u) \, d\Theta_1(T, x, u)$$

$$= 2 \sum_{i=1}^{3} \int_{F_i} \sigma(T, x, u_i) \, d\mathcal{H}^1(x)$$

$$\leq 2 \sum_{i=1}^{3} V(T) = 6V(T),$$

if we denote by F_1, F_2, F_3 the edges of T with corresponding exterior unit normal vectors u_1, u_2, u_3. The inequality

$$\int_{F_i} \sigma(T, x, u_i) \, d\mathcal{H}^1(x) \leq V(T)$$

results from Fubini's theorem and the meaning of the function $\sigma(T, \cdot)$. The discussion of the equality case is obvious now.

Let us consider an isosceles triangle T_α whose baselength is equal to 1 and the adjacent internal angles of which are equal to $\alpha \in (0, \frac{\pi}{4}]$. An elementary geometric argument leads to

$$I(T_\alpha) = \left[2 + \frac{2}{\cos^2 \alpha}\right] V(T_\alpha).$$

This proves the last statement of Proposition 4.1. □

In the proof of Proposition 4.2 we announced a result on the generic boundary behaviour of convex bodies. A far reaching result of this type was discovered by Zamfirescu [23, Th. 1]. As a matter of fact the proof of our next theorem is so similar to the one given in [23] or in [20, Th. 2.6.2] that we can omit it here. Nevertheless, our theorem cannot be regarded as a corollary to Zamfirescu's result, since ϵ-smooth convex bodies merely form a meagre set in $\mathcal{K}_0^d$.

THEOREM 4.3. *Let $\epsilon > 0$ be given. For most convex bodies K in $\mathcal{K}_0^d$, at every boundary point $y \in$ bd K^ϵ, and for every tangent direction t to* bd K^ϵ *at y,*

$$\rho_i(K^\epsilon, y, t) = \epsilon \quad or \quad \rho_s(K^\epsilon, y, t) = \infty.$$

For a definition of these lower and upper radii of curvature see, e.g., [20, §2.5].

COROLLARY 4.4. *Let $\epsilon > 0$ be given. For most convex bodies K in $\mathcal{K}_0^d$ the normal curvature of K^ϵ at y in direction t is*

$$\kappa(K^\epsilon, y, t) = \frac{1}{\epsilon} \quad or \quad \kappa(K^\epsilon, y, t) = 0$$

for $\mathcal{H}^{d-1}$-almost all $y \in$ bd K^ϵ and every tangent direction t at y.

Proof. This follows immediately from Aleksandrov's [2] theorem on the almost everywhere existence of second differentials of a convex function together with Theorem 4.3. □

Finally, the definition of the generalized curvatures $k_i(K, x, u)$, $i \in \{1, \ldots, d-1\}$, leads to our next theorem.

THEOREM 4.5. *For most convex bodies K in $\mathcal{K}_0^d$, for $\mathcal{H}^{d-1}$-almost all $(x, u) \in \mathcal{N}(K)$, and for all $i \in \{1, \ldots, d-1\}$,*

$$k_i(K, x, u) = 0 \quad or \quad k_i(K, x, u) = \infty.$$

COROLLARY 4.6. *For most convex bodies K in $\mathcal{K}_0^d$*

$$I(K) = \sum_{r=0}^{d-1} \frac{1}{d} \binom{d}{r+1} \int_{\mathcal{N}(K)} \sigma(K, x, u)^{r+1} \, d\Theta_{d-r-1}(K, x, u).$$

Acknowledgement

The author wishes to thank Professor Rolf Schneider for his encouragement and support.

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