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Equality of two representations of extended affine surface area

By

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1. Introduction and statement of the Theorem. In the last decade there has been growing interest in concepts and theorems related to the equiaffine surface area of a convex body. Whereas originally the notion of an equiaffinely invariant surface area was limited to the scope of affine differential geometry developed by Blaschke and his school, recent research is mainly devoted to determining and investigating corresponding expressions which are applicable to the boundary of an arbitrary convex body without imposing further smoothness or curvature assumptions. Interest in affine surface area is partly due to its connections with various geometric inequalities such as the Blaschke-Santaló inequality [9]. See Lutwak [7] for a survey. Some motivation can also be derived from its significance in the context of polyhedral approximation [10], [13], [2] or in the investigation of floating bodies [14], [15].

Recently, three substantially different definitions for the equiaffine surface area of an arbitrary convex body have been proposed by Lutwak [6], Leichtweiß [4], and Schütt and Werner [14]. It is a natural task to investigate how these definitions are related to each other.

Before we describe the main results, some definitions are required. In the following we work in Euclidean d-space, $\mathbb{R}^d$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let $d \geqq 2$ and denote by $\mathscr{K}^d$ the class of all nonempty compact, convex sets of $\mathbb{R}^d$. As usual, set $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$. We write $[o, x]$ for the closed line segment joining o and x. A star body in $\mathbb{R}^d$ is a nonempty compact set $L \subset \mathbb{R}^d$ satisfying $[o, x] \subset L$ for all $x \in L$ and such that the radial function $\varrho_L \colon S^{d-1} \to \mathbb{R}$, defined by

$$\varrho_L(u) := \max\{\lambda \geqq 0 \mid \lambda u \in L\}, \quad u \in S^{d-1},$$

is positive and continuous. Let $C^+(S^{d-1})$ be the set of all positive, continuous functions on S^{d-1}. Then for each $f \in C^+(S^{d-1})$ the body L, given by

$$L := \{t f(u) u \mid t \in [0, 1], \quad u \in S^{d-1}\},$$

is a star body with $\varrho_L = f$. The set of all star bodies in $\mathbb{R}^d$ is denoted by $\mathscr{S}_o^d$. For a star body $L \in \mathscr{S}_o^d$ the centre of mass $c(L)$ is given by

$$c(L) = V(L)^{-1} \frac{1}{d+1} \int_{S^{d-1}} \varrho_L(u)^{d+1} u \, d\mathscr{H}^{d-1}(u),$$

and the volume $V(L)$ can be expressed as

$$V(L)=\frac{1}{d}\int_{S^{d-1}}\varrho_L(u)^d\,d\mathscr{H}^{d-1}(u).$$

Here and in the following, $\mathscr{H}^r$ denotes r-dimensional Hausdorff measure. For the set of centred star bodies we write

$$\mathscr{S}_c^d:=\{L\in\mathscr{S}_o^d\,|\,c(L)=o\}.$$

Finally, we shall use the definition

$$\Omega(K,L):=\left(\int_{S^{d-1}}\varrho_L(u)^{-1}\,dS_{d-1}(K,u)\right)\left(\int_{S^{d-1}}\varrho_L(u)^d\,d\mathscr{H}^{d-1}(u)\right)^{\frac{1}{d}}$$

for $K\in\mathscr{K}^d$ and $L\in\mathscr{S}_o^d$. See [11, Chapter 4] for a definition and properties of the surface area measure $S_{d-1}(K,\cdot)$.

Now we are prepared to give Lutwak's definition of the extended equiaffine surface area $\Omega(K)$ of a convex body $K\in\mathscr{K}^d$:

(1) $$\Omega(K):=\inf\{\Omega(K,L)^{\frac{d}{d+1}}\,|\,L\in\mathscr{S}_c^d\}.$$

The motivation for this definition comes from earlier work by Petty [8] who has studied geominimal surface area as a connecting link between Minkowski geometry, relative differential geometry and affine differential geometry.

The representations of equiaffine surface area according to Leichtweiß respectively Schütt and Werner will not be discussed here. However, see [12] for a proof of equality of these two representations. A more direct approach can be found in [3]. Thus, in order to show the equality of all the different definitions of equiaffine surface area it only remains to prove that the definitions of Leichtweiß and Lutwak actually coincide. First progress in this direction is due to Leichtweiß who defined the following modification $\tilde{\Omega}(K)$ of Lutwak's original definition (1):

(2) $$\tilde{\Omega}(K):=\inf\{\Omega(K,L)^{\frac{d}{d+1}}\,|\,L\in\mathscr{S}_o^d\}.$$

Leichtweiß stated that $\tilde{\Omega}$ essentially has the same properties as Ω, and in addition he proved in [5] that $\tilde{\Omega}$ coincides with his definition of equiaffine surface area [4].

It is immediately clear that $\tilde{\Omega}(K)\leqq\Omega(K)$ for $K\in\mathscr{K}^d$. Moreover, it is known that equality holds for convex bodies with positive, continuous curvature functions. Leichtweiß conjectured that $\tilde{\Omega}(K)=\Omega(K)$ at least for centrally symmetric $K\in\mathscr{K}^d$. In the present note the general problem of comparing the various notions of equiaffine surface area is completely solved by the following Theorem.

Theorem. *For an arbitrary convex body $K\in\mathscr{K}^d$ the two notions of extended equiaffine surface area due to Lutwak respectively Leichtweiß coincide, i.e. $\Omega(K)=\tilde{\Omega}(K)$ for all $K\in\mathscr{K}^d$.*

2. Proof of the Theorem. Let us outline the main idea of the subsequent proof. It is sufficient to show that $\Omega(K)\leqq\tilde{\Omega}(K)$ for $K\in\mathscr{K}^d$. To this end for each $L\in\mathscr{S}_o^d$ we consider a set of star bodies $L_{\varepsilon,z}$ which depend on parameters $\varepsilon>0$ and $z\in\mathbb{R}^d\setminus\{o\}$. The star bodies L and $L_{\varepsilon,z}$ differ only in the direction of vectors which belong to a spherical ε-neighbourhood of $z/|z|$. For $\varepsilon \to 0$ the difference $V(L_{\varepsilon,z}) - V(L)$ can be made arbitrarily small. In addition, the limit $F(0,z)$ of the moment vectors $F(\varepsilon, z) := (d+1)\, V(L_{\varepsilon,z})\, c(L_{\varepsilon,z})$ for $\varepsilon \to 0$ exists. The equation $F(0,z) = o$ can be easily solved. But now, the homotopy invariance of the degree of a map tells us that $F(\varepsilon, z) = o$ also has solutions $z(\varepsilon)$, if $\varepsilon > 0$ is small enough. This yields $L_{\varepsilon, z(\varepsilon)} \in \mathcal{S}_c^d$ such that $\Omega(K, L_{\varepsilon, z(\varepsilon)}) \to \Omega(K, L)$ for $\varepsilon \to 0$.

Proof of the Theorem. For $\delta > 0$ and $L \in \mathcal{S}_o^d$ with $c(L) \neq o$ given we construct a star body $L^* \in \mathcal{S}_c^d$ such that $\Omega(K, L^*) \leqq (1+\delta)\, \Omega(K, L)$. In order to do so fix $\alpha \in C^0([0,\infty))$ with $\alpha \geqq 0$, $\alpha(0) = 1$ and $\alpha(t) = 0$ for $t \in [1, \infty)$. For $0 < \varepsilon \leqq 1$ and $z \neq o$ let

$$\varrho_{\varepsilon,z}(u) := \varrho_L(u) + |z|^{\frac{1}{d+1}} \varepsilon^{-\frac{d-1}{d+1}} \alpha\left(\frac{1}{\varepsilon}\left|\frac{z}{|z|} - u\right|\right), \quad u \in S^{d-1}.$$

Denote by $L_{\varepsilon,z}$ the star body determined by $\varrho_{\varepsilon,z}$. Then

$$\begin{aligned} V(L_{\varepsilon,z}) &= \frac{1}{d} \int_{S^{d-1}} \varrho_{\varepsilon,z}(u)^d \, d\mathcal{H}^{d-1}(u) \\ &= V(L) + \sum_{k=1}^{d} \frac{1}{d} \binom{d}{k} \int_{S^{d-1}} \varrho_L(u)^{d-k} |z|^{\frac{k}{d+1}} \varepsilon^{-k\frac{d-1}{d+1}} \\ &\quad \cdot \alpha\left(\frac{1}{\varepsilon}\left|\frac{z}{|z|} - u\right|\right)^k d\mathcal{H}^{d-1}(u) \leqq V(L) + C_1(|z|)\, \varepsilon^{\frac{d-1}{d+1}}, \end{aligned}$$

and choosing

$$\varepsilon \leqq \left(\frac{\delta\, V(L)}{C_1(|z|)}\right)^{\frac{d+1}{d-1}}$$

we obtain

$$V(L_{\varepsilon,z}) \leqq (1+\delta)\, V(L). \tag{3}$$

The expression $C_1(|z|)$ is bounded, if z varies in a compact region of $\mathbb{R}^d \setminus \{o\}$. This remark also applies to $C_2(|z|)$ and $C_3(|z|)$ which will be used in the following. It remains to show that for $\varepsilon \leqq \varepsilon_0$ there exists $z(\varepsilon)$ such that $c(L_{\varepsilon, z(\varepsilon)}) = o$. This, however, follows from the homotopy invariance of the degree of a map (see e.g. [1, Theorem 3.1]). Define the mapping $F: [0,1] \times \mathbb{R}^d \setminus \{o\} \to \mathbb{R}^d$ by

$$F(\varepsilon, z) := \begin{cases} \int_{S^{d-1}} \varrho_{\varepsilon,z}(u)^{d+1}\, u \, d\mathcal{H}^{d-1}(u) & \text{for} \quad \varepsilon \in (0,1] \\ V(L)(d+1)\, c(L) + \alpha_0 z & \text{for} \quad \varepsilon = 0 \end{cases},$$

where

$$\alpha_0 := \mathcal{H}^{d-2}(S^{d-2}) \int_0^1 \alpha(r)^{d+1} r^{d-2}\, dr \neq 0.$$

We claim that F is continuous. For continuity in $\varepsilon = 0$ note that

$$\begin{aligned} \varrho_{\varepsilon,z}(u)^{d+1} &= \varrho_L(u)^{d+1} + |z|\, \varepsilon^{-(d-1)} \alpha\left(\frac{1}{\varepsilon}\left|\frac{z}{|z|} - u\right|\right)^{d+1} \\ &\quad + C_2(|z|)\, O\left(\varepsilon^{-d\frac{d-1}{d+1}}\right) \alpha\left(\frac{1}{\varepsilon}\left|\frac{z}{|z|} - u\right|\right). \end{aligned}$$

Consequently

$$\int_{S^{d-1}} \varrho_{\varepsilon,z}(u)^{d+1}\, u\, d\mathscr{H}^{d-1}(u) = (d+1)\,V(L)\,c(L) + \alpha_\varepsilon z + C_3(|z|)\,O\left(\varepsilon^{\frac{d-1}{d+1}}\right),$$

where

$$\alpha_\varepsilon z = |z|\,\varepsilon^{-(d-1)} \int_{S^{d-1}} \alpha\left(\frac{1}{\varepsilon}\left|\frac{z}{|z|} - u\right|\right)^{d+1} u\, d\mathscr{H}^{d-1}(u) \to \alpha_0 \bar{z}$$

for $\varepsilon \to 0$ and $z \to \bar{z} \in \mathbb{R}^d \setminus \{o\}$. Hence

$$\lim_{(\varepsilon,z)\to(0,\bar{z})} F(\varepsilon, z) = F(0, \bar{z})$$

for each $\bar{z} \in \mathbb{R}^d \setminus \{o\}$. Now let $B := \{x \in \mathbb{R}^d \,|\, |x - z_0| < \frac{1}{2}|z_0|\}$ for

$$z_0 := -\frac{V(L)(d+1)}{\alpha_0} c(L),$$

and denote the boundary of B by ∂B. Clearly $o \notin F(0, \partial B)$ and $F(0, z) = o$ if and only if $z = z_0$. Thus

$$\deg(F(0, \cdot), B, o) = 1\,.$$

Since F is uniformly continuous on $[0,1] \times \partial B$ we may choose $\varepsilon_1 > 0$ such that $o \notin F(\varepsilon, \partial B)$ for $0 \leqq \varepsilon \leqq \varepsilon_1$. Thus

$$\deg(F(\varepsilon, \cdot), B, o) = \deg(F(0, \cdot), B, o) = 1\,,$$

and we see that there exists a point $z(\varepsilon) \in B$ such that $F(\varepsilon, z(\varepsilon)) = o$, i.e.

$$c(L_{\varepsilon, z(\varepsilon)}) = o\,. \tag{4}$$

The result now follows from (3) and (4) for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough, since the first factor in the representation of $\Omega(K, L_{\varepsilon, z(\varepsilon)})$ is smaller than the corresponding factor of $\Omega(K, L)$. □

Remark. An earlier variant of this proof for our Theorem avoided the use of the degree of a map by additional approximation arguments. In fact, it is sufficient to consider star bodies L for which ϱ_L is smooth. By adding to L a suitable counterpoise which in essence has the shape of a cone the centre of mass of this modified star body $\tilde{L}$ can be forced to lie arbitrarily close to the origin. Finally, $\tilde{L} - c(\tilde{L})$ is star-shaped at o, since the smoothness of ϱ_L implies that $L + t$ is still a star body at o for $|t|$ sufficiently small.

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