Manuscripta Mathematica Manuscript-Nr.
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Contributions to Affine Surface Area

Daniel Hug

Mathematisches Institut, Albert-Ludwigs-Universität, Albertstraße 23b, D-79104 Freiburg i. Br., Germany, E-mail: hug@sun1.mathematik.uni-freiburg.de

Received:

Representations of equiaffine surface area, due to Leichtweiß resp. Schütt & Werner, are generalized to p-affine surface area measures. We provide a direct proof which shows that these representations coincide. In addition, we establish two theorems which in particular characterize all those convex bodies geometrically for which the affine surface area is positive. The present approach also leads to proofs of the equiaffine isoperimetric inequality and the Blaschke-Santaló inequality, including the characterization of the case of equality.

Key words: Affine surface area, affine isoperimetric inequality, Blaschke-Santaló inequality, ellipsoids, generalized Gauß-Kronecker curvature.

Subject Classifications: Primary 52A20, 52A40; Secondary 53A15.

1. Introduction and definitions

In equiaffine differential geometry the equiaffine surface area of a smooth hypersurface with everywhere positive Gauß-Kronecker curvature is defined as the Riemannian volume with respect to the Berwald-Blaschke metric [4, §65], [21, §1.1]. If specialized to the boundary $\mathrm{bd}\,K$ of a convex body K (nonempty, compact, convex set) in Euclidean space $\mathbb{R}^d$, the equiaffine surface area $\mathcal{O}_a(K)$ of K can be calculated by

$$\mathcal{O}_a(K) = \int_{\mathrm{bd}\,K} H_{d-1}(K,x)^{\frac{1}{d+1}}\,\mathrm{d}\mathcal{H}^{d-1}(x). \tag{1}$$

Here, $H_{d-1}(K,x)$ is the Gauß-Kronecker curvature of $\mathrm{bd}\,K$ at x, and $\mathcal{H}^s$, $s \geq 0$, denotes the s-dimensional Hausdorff measure.

A major theme in recent investigations of affine surface area has been to establish notions and theorems for arbitrary convex bodies which were previously known in the smooth case. This development in particular led to the solution of various problems (upper semicontinuity [24], valuation property [37], random polyhedral approximation [38]), which in some cases had been unassailable even under additional smoothness assumptions. Moreover, new interrelations between old results were revealed. Progress in this direction is documented by the surveys of Leichtweiß [19], [20] and Lutwak [25], and the work of Lutwak [24], Schütt [37], Schütt & Werner [39], [40], [41], Werner [42], and Dolzmann & Hug [8]. For connections with affine invariant polyhedral approximation see also [35], [11], [38]. Applications of affine surface area for obtaining a priori estimates for PDEs have been studied by Lutwak & Oliker [26].

Equation (1) already opens the way for a definition of the equiaffine surface area of a general convex body K, since the notion of a generalized Gauß-Kronecker curvature $H_{d-1}(K,x)$ can still be defined for $\mathcal{H}^{d-1}$ almost all boundary points of K, cf. [16, pp. 440-446], [36, Notes for §1.5, §2.5], [34]. Henceforth, we work in Euclidean space $\mathbb{R}^d$, $d \geq 2$, with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. For all notions of convexity, which are not explicitly defined, we refer to [36]; measure geometric results are taken from [10]. Let $\mathcal{K}^d$ ($\mathcal{K}^d_0$ resp. $\mathcal{K}^d_{00}$) be the set of all convex bodies $K \subset \mathbb{R}^d$ (with $\operatorname{int} K \neq \emptyset$ resp. with $o \in \operatorname{int} K$). Also define $S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ and $B(x_0, r) := \{x \in \mathbb{R}^d \mid \|x - x_0\| \leq r\}$, if $x_0 \in \mathbb{R}^d$ and $r > 0$. Generalizing (1) we set, for $K \in \mathcal{K}^d_{00}$ and $p > 0$ ($K \in \mathcal{K}^d$ for $p = 1$), and for an arbitrary $\mathcal{H}^{d-1}$ measurable set $\beta \subset \mathbb{R}^d$,

$$\mathcal{O}_p(K,\beta) := \int_{\operatorname{bd} K \cap \beta} \left\{ \frac{H_{d-1}(K,x)}{\langle x, \sigma_K(x)\rangle^{(p-1)\frac{d}{p}}} \right\}^{\frac{p}{d+p}} \, \mathrm{d}\mathcal{H}^{d-1}(x). \tag{2}$$

Here, σ_K denotes the spherical image map of K. It is uniquely defined for regular boundary points $x \in \operatorname{reg} K$, i.e., for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$, if $K \in \mathcal{K}^d_0$. Obviously, for $p = 1$ and $\beta = \mathbb{R}^d$ the extended affine surface area, as defined in [39], is regained. For $p = d$ we obtain a general notion of centroaffine surface area, which is consistent with the one defined in centroaffine differential geometry.

Previous to the work by Schütt & Werner another definition was proposed by Leichtweiß [16], which can be generalized as well to yield a notion of p-affine surface area. In fact, we define, for $K \in \mathcal{K}^d_{00}$ and $p > 0$ ($K \in \mathcal{K}^d$ for $p = 1$), and for an arbitrary $\mathcal{H}^{d-1}$ measurable set $\omega \subset S^{d-1}$,

$$\widetilde{\mathcal{O}}_p(K,\omega) := \int_{\omega} \left\{ \frac{D_{d-1}h(K,u)}{h(K,u)^{p-1}} \right\}^{\frac{d}{d+p}} \, \mathrm{d}\mathcal{H}^{d-1}(u). \tag{3}$$

Here, $h(K,\cdot) = h_K$ denotes the support function of K, and $D_{d-1}h(K,u)$ is equal to the sum of the principal minors of order $(d-1)$ of the Hessian matrix $d^2h_K(u)$ which is defined for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, see, e.g., [16, p. 449] for the details. Our motivation for considering p-affine surface area originates from recent research by Lutwak [22] on the Brunn-Minkowski-Firey theory. There the notion of p-affine surface area is defined in analogy to the definition of extended affine surface area given in [24]. The representation in [22] can easily be localized (for all $p > 0$). Minor modifications of the proof for Theorem 1 in [18] then show that this localized definition of p-affine surface area coincides with the expression given in (3). This proves in particular that the integral in (3) is finite. The same holds true for definition (2), as can be seen, e.g., from our Theorem 2.8. Finally, it should be realized that the result of [8] immediately extends to the case of p-affine surface area (measures).

It has been shown by Schütt [37] that

$$\mathcal{O}_a(K) := \mathcal{O}_1(K, \mathbb{R}^d) = \widetilde{\mathcal{O}}_1(K, S^{d-1}) =: \widetilde{\mathcal{O}}_a(K).$$

However, Leichtweiß [19] has asked for a more direct approach to this equality than the one proposed by Schütt. In the present paper we provide such a direct route from representation (2) to (3), for general $p > 0$ and corresponding sets β resp. ω (Theorem 2.8). In addition, two theorems (Theorem 3.1 and Theorem 3.3) are derived, which in particular characterize those convex bodies for which the affine surface area does not vanish. One of these, Theorem 3.1, encompasses earlier partial results by Leichtweiß [17, Satz 1(f)], Bárány & Larman [3, Theorem 4], and Schütt & Werner [39, Corollary 2].

Finally, we present a direct proof of the equiaffine isoperimetric inequality for an arbitrary convex body by using Steiner symmetrization. The case of equality is also covered. It should be emphasized that we do not have to consider special cases such as the centrally symmetric or the two dimensional case. Eventually, this leads to a proof of the Blaschke-Santaló inequality, including a discussion of the case of equality.

The first complete treatment of the Blaschke-Santaló inequality was accomplished by Petty [31]. In fact, Petty simultaneously proved the equiaffine isoperimetric inequality for convex bodies which possess a curvature function. Later Meyer & Pajor [28] provided a direct proof of the Blaschke-Santaló inequality, and thus they considerably simplified Petty's original approach. At the same time they found an improved version of the Blaschke-Santaló inequality. Then, Leichtweiß [17, Satz 2] deduced the general equiaffine isoperimetric inequality from the Blaschke-Santaló inequality. Lutwak found a way to define the volume of the polar of a star body and thus obtained a version of the Blaschke-Santaló inequality for star bodies [24]. Up to now there are two proofs ([17] and [24]) of the general equiaffine isoperimetric inequality including the characterization of the case of equality. Both proofs make essential use of the Blaschke-Santaló inequality together with the corresponding characterization of the case of equality. In contrast to this approach we do not assume the validity of the Blaschke-Santaló inequality, but we rather deduce it from the equiaffine isoperimetric inequality. This finally leads to a characterization of the case of equality in the equiaffine isoperimetric inequality, which in turn yields the case of equality in the Blaschke-Santaló inequality.

Basically, the idea for our approach is due to Blaschke who considered smooth convex bodies in dimensions $d = 2$ and $d = 3$. Extensions to arbitrary dimensions were given by Santaló [33], Deicke [7], and Li, Simon & Zhao [21] still under restrictive smoothness assumptions. See [17] for an attempt to remove such restrictions, which, however, have hitherto been indispensible for this line of approach. The main idea of our proof for the equiaffine isoperimetric inequality is to establish a representation of the equiaffine surface area of a general convex body (Lemma 4.4) which allows us to show that the equiaffine surface area is not decreasing with respect to Steiner symmetrization. It should be noted that our method for dealing with the characterization of the case of equality is related to Petty's curvature-image conjecture. Whereas usually (see, e.g., [29], [41], [12]) additional geometric information is used to guarantee sufficient smoothness in order to be able to solve the corresponding Monge-Ampère equation [31, Lemma 8.4], we rather use such information to reduce a partial differential equation of second order to a first order equation.

2. Proof of equality

The main obstacle for a direct transformation from $\operatorname{bd} K$ to S^{d-1} via the spherical image map σ_K is the fact that $\sigma_K|_{\operatorname{reg} K}$ is not Lipschitzian in general. This is made clear by the example constructed in §6 of the classical paper by Busemann & Feller [6]. Therefore we further restrict the domain of σ_K to sets $(\operatorname{bd} K)_r$, $r > 0$ and $K \in \mathcal{K}_0^d$, defined by

$$(\operatorname{bd} K)_r := \left\{ x \in \operatorname{bd} K \mid \exists a \in \mathbb{R}^d : \, x \in B(a,r) \subset K \right\}.$$

The proof of our first lemma follows essentially by repeating the argument in [16, Hilfssatz 1].

Lemma 2.1. *Let $K \in \mathcal{K}_0^d$ and $r > 0$. Then $(\operatorname{bd} K)_r$ is a closed subset of $\operatorname{bd} K$, and the spherical image map $\sigma_K|_{(\operatorname{bd} K)_r}$ is Lipschitzian.*

The sets $(\operatorname{bd} K)_r$, $r > 0$, cover $\mathcal{H}^{d-1}$ almost all of $\operatorname{bd} K$. This observation is contained in Lemma 2.2, for which McMullen [27] has given a simple proof. Moreover note that

$$(\operatorname{bd} K)_+ := \bigcup_{r>0} (\operatorname{bd} K)_r = \bigcup_{n\in\mathbb{N}} (\operatorname{bd} K)_{\frac{1}{n}}.$$

Lemma 2.2. *For all $K \in \mathcal{K}_0^d$ the relation $\mathcal{H}^{d-1}(\operatorname{bd} K \setminus (\operatorname{bd} K)_+) = 0$ holds true.*

Next we calculate the approximate Jacobian of $\sigma_K|_{(\operatorname{bd} K)_r}$. This will be necessary for the ensuing application of Federer's area/coarea formula. But first let us agree on some terminology. If $x_0 \in \operatorname{reg} K$, then $\operatorname{bd} K$ can be (locally) represented at x_0 as the graph of a uniquely determined nonnegative, convex function f, defined on a neighbourhood D_f of x_0 relative to the tangent space $x_0 + T_{x_0}K$ of K at x_0 by

$$f(x) := \min\{\lambda \geq 0 \mid x - \lambda\sigma_K(x_0) \in \operatorname{bd} K\}, \quad x \in D_f,$$

cf. [16, p. 442] and [36, §2.5]. Hence, $f(x_0) = 0$ and $df(x_0) = o$. Recall that f is second order differentiable (s.o.d.) at $x \in D_f$ for $\mathcal{H}^{d-1}$ almost all $x \in D_f$ according to Aleksandrov's theorem [1] (see also [2], [9], [5]). In the following we write $\mathcal{M}(K)$ for the set of all $x_0 \in \operatorname{reg} K$ such that the function which locally represents $\operatorname{bd} K$ at x_0 is s.o.d. at x_0. As usual, the points of $\mathcal{M}(K)$ are called normal boundary points. Finally, for a nonempty closed convex set A and $x \in \mathbb{R}^d$ let $p(A, x)$ be the orthogonal projection of x onto A.

Lemma 2.3. *Let $K \in \mathcal{K}_0^d$ and $r > 0$. Then, for $\mathcal{H}^{d-1}$ almost all $x \in (\operatorname{bd} K)_r$, we have $\operatorname{ap}J_{d-1}\sigma_K(x) = H_{d-1}(K, x)$.*

Proof. The set $(\operatorname{bd} K)_r$ is closed and $(d-1)$-rectifiable, and $\mathcal{H}^{d-1}$ almost all $x \in (\operatorname{bd} K)_r$ are normal boundary points of K. For such a boundary point $x_0 \in (\operatorname{bd} K)_r$ the function f which locally represents $\operatorname{bd} K$ at x_0 is s.o.d. at x_0, and $H_{d-1}(K, x_0) = \det(d^2 f(x_0))$. Here, the mapping $d^2 f(x_0)$ is interpreted as a linear map from the linear space $T_{x_0}K$ to itself. For the proof it is sufficient to verify

$$(\mathcal{H}^{d-1} \llcorner (\operatorname{bd} K)_r, d-1)\text{ap}D\sigma_K(x_0) = d^2 f(x_0), \tag{4}$$

for $\mathcal{H}^{d-1}$ almost all normal boundary points $x_0 \in (\operatorname{bd} K)_r$, where f locally represents $\operatorname{bd} K$ at x_0. With regard to Federer's terminology [10, p. 253] we set $X := \mathbb{R}^d$, $Y := \mathbb{R}^d$, $\phi := \mathcal{H}^{d-1} \llcorner (\operatorname{bd} K)_r$, $m := d-1$, $a := x_0$, where $x_0 \in \mathcal{M}(K)$. In addition, excluding a set of $\mathcal{H}^{d-1}$ measure zero, we can assume

$$\operatorname{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner (\operatorname{bd} K)_r, x_0) = \operatorname{Tan}(\operatorname{bd} K, x_0) = T_{x_0}K.$$

Let $(-e_d)$ be the exterior unit normal vector of K at x_0, and let $(e_1, \ldots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$, i.e., $T_{x_0}K = \operatorname{lin}\{e_1, \ldots, e_{d-1}\}$. Define $\eta \in \mathbb{R}^d$ and the linear map $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\eta := -e_d, \quad \zeta(h) := d^2 f(x_0)(h - \langle h, e_d \rangle e_d).$$

In order to prove (4), we have to show, for an arbitrary $\epsilon > 0$,

$$\Theta^{d-1}\Big[(\mathcal{H}^{d-1} \llcorner (\operatorname{bd} K)_r) \llcorner \mathbb{R}^d \setminus \{x \in (\operatorname{bd} K)_r \mid \\ \|\sigma_K(x) - \eta - \zeta(x - x_0)\| \leq \epsilon \|x - x_0\|\}, x_0\Big] = 0.$$

See [10, 2.10.19] for a definition of this $(d-1)$-dimensional density. Obviously, it is sufficient to prove, for an arbitrary $\epsilon > 0$, that there is some $r(\epsilon) > 0$ such that for $0 < \delta \leq r(\epsilon)$

$$\{x \in (\operatorname{bd} K)_r \mid \|\sigma_K(x) - \eta - \zeta(x - x_0)\| > \epsilon \|x - x_0\|\} \cap B(x_0, \delta) = \emptyset.$$

Since f is s.o.d. at x_0, we get for $z \in D_f$

$$\|df(z) - df(x_0) - d^2 f(x_0)(z - x_0)\| \leq R(\|z - x_0\|)\|z - x_0\|$$

with

$$\lim_{z \to x_0} R(\|z - x_0\|) = 0.$$

Here, df denotes a subgradient choice for f (see [36, Notes for §1.5]). If we take $x \in (\operatorname{bd} K)_r$ and $z := p(x_0 + T_{x_0}K, x) \in D_f$, then f is differentiable at z, and

$$\sigma_K(x) = \frac{df(z) - e_d}{\|df(z) - e_d\|}.$$

Let $\epsilon \in (0, 1]$ be given and define

$$\tilde{\epsilon} := \epsilon[1 + (1 + \|d^2 f(x_0)\|)^2]^{-1}.$$

Now we can choose $r(\epsilon) \in (0, \tilde{\epsilon})$ such that $z \in D_f$ and $R(\|z - x_0\|) < \tilde{\epsilon}$, provided that $z = p(x_0 + T_{x_0}K, x)$ and $\|x - x_0\| \leq r(\epsilon)$. Let $x \in B(x_0, r(\epsilon)) \cap (\operatorname{bd} K)_r$ and $z := p(x_0 + T_{x_0}K, x)$. Then we obtain

$$
\begin{aligned}
&\|\sigma_K(x) - \eta - \zeta(x - x_0)\| \\
&\quad = \left\| \frac{df(z) - e_d}{\|df(z) - e_d\|} + e_d - d^2 f(x_0)(z - x_0) \right\| \\
&\quad \le \|df(z) - df(x_0) - d^2 f(x_0)(z - x_0)\| + \frac{\|df(z)\|^2}{1 + \|df(z) - e_d\|} \\
&\quad \le \tilde{\epsilon}\|z - x_0\| + \|df(z)\|^2 \\
&\quad \le \tilde{\epsilon}\|z - x_0\| + \left(\tilde{\epsilon}\|z - x_0\| + \|d^2 f(x_0)(z - x_0)\|\right)^2 \\
&\quad \le \tilde{\epsilon}\|z - x_0\| + \left(\tilde{\epsilon} + \|d^2 f(x_0)\|\right)^2 \|z - x_0\|^2 \\
&\quad \le \tilde{\epsilon}\left(1 + (1 + \|d^2 f(x_0)\|)^2 \|x - x_0\|/\tilde{\epsilon}\right) \|x - x_0\| \\
&\quad \le \epsilon\|x - x_0\|.
\end{aligned}
$$

Here, we have used $\tilde{\epsilon} \le 1$, $\|z - x_0\| \le \|x - x_0\|$, and $\|x - x_0\|/\tilde{\epsilon} \le r(\epsilon)/\tilde{\epsilon} \le 1$. This finishes the proof of Lemma 2.3. □

In a differential geometric context our next lemma is well known. The following more general statement, which will be needed for the proof of Lemma 2.5, has been proved by Noll [30, Corollary 4.2].

Lemma 2.4. *Let $K \in \mathcal{K}_0^d$, $x_0 \in \mathcal{M}(K)$, $u_0 := \sigma_K(x_0)$, $\epsilon > 0$, and let f denote the function representing* $\operatorname{bd} K$ *at x_0. Then $x_0 + \epsilon u_0 \in \mathcal{M}(K^\epsilon)$, and if the eigenvalues of $d^2 f(x_0)$ are denoted by $k_1(x_0), \dots, k_{d-1}(x_0)$, the eigenvalues $k_1^\epsilon(x_0 + \epsilon u_0), \dots, k_{d-1}^\epsilon(x_0 + \epsilon u_0)$ of the function f^ϵ representing* $\operatorname{bd} K^\epsilon$ *at $x_0 + \epsilon u_0$ can be calculated by*

$$
k_i^\epsilon(x_0 + \epsilon u_0) = \frac{k_i(x_0)}{1 + \epsilon k_i(x_0)}, \quad i = 1, \dots, d - 1,
$$

if the ordering is chosen properly.

Lemma 2.5. *Let $K \in \mathcal{K}_0^d$, $x_0 \in \mathcal{M}(K)$, and let h_K be second order differentiable at $u_0 := \sigma_K(x_0)$. Then $H_{d-1}(K, x_0) D_{d-1} h(K, u_0) = 1$.*

Proof. For any $\epsilon > 0$ the parallel body K^ϵ is smooth, $x_0 + \epsilon u_0 \in \mathcal{M}(K^\epsilon)$ according to Lemma 2.4, and $h(K^\epsilon, \cdot)$ is s.o.d. at u_0. In this situation Leichtweiß [16, pp. 447-449] showed that $H_{d-1}(K^\epsilon, x_0 + \epsilon u_0) D_{d-1} h(K^\epsilon, u_0) = 1$. As both factors depend continuously on ϵ, the statement of the lemma follows for $\epsilon \to 0$. □

Remarks.

1. From Lemma 2.5 we learn, e.g., that h_K is definitely not second order differentiable at $u_0 := \sigma_K(x_0)$, if $x_0 \in \mathcal{M}(K)$ and $H_{d-1}(K, x_0) = 0$. This situation can occur, even if $\operatorname{bd} K$ is a C^∞ submanifold and h_K is differentiable (i.e., K is strictly convex).
2. The eigenvalues $k_1(x_0), \dots, k_{d-1}(x_0)$, appearing in Lemma 2.4, are called the generalized principal curvatures of K at x_0. More generally, the following can be proved by similar arguments.

   Let $K \in \mathcal{K}_0^d$, $x_0 \in \mathcal{M}(K)$, and let h_K be second order differentiable at $u_0 := \sigma_K(x_0)$. If the generalized principal curvatures of K at x_0 are denoted by

$k_1(x_0), \dots, k_{d-1}(x_0)$, and if the eigenvalues of $d^2h_K(u_0)|_{u_0^\perp} : u_0^\perp \to u_0^\perp$ are denoted by $r_1(u_0), \dots, r_{d-1}(u_0)$, then

$$k_i(x_0) = r_i(u_0)^{-1} \in (0, \infty), \quad i = 1, \dots, d-1,$$

if the ordering is chosen properly. In particular,

$$d\sigma_K(x_0) \circ \left(d^2h_K(u_0)|_{u_0^\perp} \right) = \mathrm{id}_{u_0^\perp},$$

if $d\sigma_K(x_0) = d^2f(x_0)$ is suitably interpreted as a linear map of the vector space $u_0^\perp$.

Note that here and subsequently we do not strictly distinguish between bilinear forms and corresponding linear maps. The quantities $r_1(u_0), \dots, r_{d-1}(u_0)$ are called the generalized principal radii of curvature of K at u_0, cf. [36, (2.5.26)] for the C^2_+-case. The preceding remark also yields generalizations of the relations (2.5.27) and (2.5.28) from [36] in the sense of our next lemma. However, Lemma 2.6 will be sufficient for the present purpose. If h_K is differentiable at $u \in \mathbb{R}^d$, we set $\nabla h_K(u) := \operatorname{grad} h_K(u)$.

Lemma 2.6. *Let $K \in \mathcal{K}^d_0$ and $r > 0$. Then for $\mathcal{H}^{d-1}$ almost all $u \in \sigma_K((\mathrm{bd}\, K)_r)$ the mapping h_K is second order differentiable at u, $x := \nabla h_K(u) \in \mathcal{M}(K)$, and $H_{d-1}(K, x) D_{d-1} h(K, u) = 1$.*

Proof. In view of Lemma 2.5 it is sufficient to show that

$$\mathcal{H}^{d-1}(\sigma_K((\mathrm{bd}\, K)_r) \setminus \mathcal{S}_r) = 0,$$

if we define

$$\mathcal{S}_r := \{u \in \sigma_K((\mathrm{bd}\, K)_r) \mid h_K \text{ is s.o.d. at } u \text{ and } \nabla h_K(u) \in \mathcal{M}(K)\}.$$

We observe that $\sigma_K((\mathrm{bd}\, K)_r) \setminus \mathcal{S}_r \subset \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \{u \in S^{d-1} \mid h_K \text{ is not s.o.d. at } u\}$$

and

$$\mathcal{C}_2 := \{u \in S^{d-1} \mid h_K \text{ is s.o.d. at } u \text{ and } \nabla h_K(u) \in (\mathrm{bd}\, K)_r \setminus \mathcal{M}(K)\}.$$

But according to Aleksandrov's theorem [1], $\mathcal{H}^{d-1}(\mathcal{C}_1) = 0$ and

$$\mathcal{H}^{d-1}(\{x \in (\mathrm{bd}\, K)_r \mid x \notin \mathcal{M}(K)\}) = 0.$$

Since $\mathcal{C}_2 \subset \sigma_K(\{x \in (\mathrm{bd}\, K)_r \mid x \notin \mathcal{M}(K)\})$ and because $\sigma_K|_{(\mathrm{bd}\, K)_r}$ is Lipschitzian, we also have $\mathcal{H}^{d-1}(\mathcal{C}_2) = 0$. □

The following lemma describes what it means geometrically that all generalized principal radii of curvature of K at u_0 are positive. A dual version of Lemma 2.7 is implicitly contained in the proof of Theorem 3.1. For later reference this will be stated as Corollary 3.2. It is probably worth mentioning that there is some $R > 0$ such that $K \subset B(\nabla h_K(u_0) - Ru_0, R)$, if $K \in \mathcal{K}^d$ and h_K is second order differentiable at $u_0 \in S^{d-1}$. This can be proved similarly to Lemma 2.7, $(b) \Rightarrow (a)$.

Lemma 2.7. *Let $K \in \mathcal{K}^d$, and let h_K be second order differentiable at $u_0 \in S^{d-1}$. Then the following two conditions are equivalent.*

(a) *There is some $r > 0$ such that $B(\nabla h_K(u_0) - ru_0, r) \subset K$.*
(b) $D_{d-1}h(K, u_0) > 0$.

Proof. We set $h := h_K$ for short. Since h is s.o.d. at u_0, there is a function R such that $R(\|u - u_0\|) \to 0$ for $u \to u_0$ and

$$\begin{aligned}&|h(u) - h(u_0) - dh(u_0)(u - u_0) - \frac{1}{2}d^2h(u_0)(u - u_0, u - u_0)|\\&\quad\le R(\|u - u_0\|)\|u - u_0\|^2.\end{aligned}$$

$(a) \Rightarrow (b)$: Define $x_0 := \nabla h(u_0) \in \operatorname{bd} K$. By assumption $x_0 \in B(x_0 - ru_0, r) \subset K$. We may assume $o = x_0 - ru_0$. Hence, $h(u) \ge r$ for all $u \in S^{d-1}$. Suppose (b) is false, i.e., $d^2h(u_0)|_{(u_0^\perp \times u_0^\perp)}$ is not positive definite. Then there is some $v \in S^{d-1} \cap u_0^\perp$ with $d^2h(u_0)(v, v) = 0$. So we can find a sequence $(u_n)_{n\in\mathbb{N}} \subset S^{d-1} \setminus \{u_0\}$ such that $u_n \to u_0$ for $n \to \infty$ and $d^2h(u_0)(u_n - u_0, u_n - u_0) = 0$ for all $n \in \mathbb{N}$. According to our choice of the position of the origin we have $h(u_0) = r$ and $dh(u_0)(u_n - u_0) = -r(1 - \langle u_n, u_0\rangle)$. Therefore we obtain

$$\begin{aligned}r(1 - \langle u_n, u_0\rangle) &\le h(u_n) - r - dh(u_0)(u_n - u_0)\\&\le R(\|u_n - u_0\|)\|u_n - u_0\|^2\\&= 2R(\|u_n - u_0\|)(1 - \langle u_n, u_0\rangle).\end{aligned}$$

This implies $R(\|u_n - u_0\|) \ge r/2$ for all $n \in \mathbb{N}$ in contradiction to $R(\|u_n - u_0\|) \to 0$ for $n \to \infty$.

$(b) \Rightarrow (a)$: According to the assumption there is some $r_1 > 0$ such that $d^2h(u_0)(v, v) \ge 4r_1$ for all $v \in S^{d-1} \cap u_0^\perp$. We may assume that $x_0 := \nabla h(u_0) = o$. Hence, $h(u_0) = dh(u_0)(u - u_0) = 0$ for all $u \in S^{d-1}$. Thus

$$\begin{aligned}h(u) &\ge \frac{1}{2}d^2h(u_0)(u - u_0, u - u_0) - R(\|u - u_0\|)\|u - u_0\|^2\\&\ge 2r_1\|p(u_0^\perp, u - u_0)\|^2 - R(\|u - u_0\|)\|u - u_0\|^2\\&\ge (r_1 - R(\|u - u_0\|))\,\|u - u_0\|^2\\&\ge r_2\|u - u_0\|^2, \qquad (5)\end{aligned}$$

for some $r_2 > 0$ and all $u \in U(u_0)$, where $U(u_0)$ is a sufficiently small, open, spherical neighbourhood of u_0. In deducing (5), we have used the elementary estimate

$$\|p(u_0^\perp, u - u_0)\|^2 \ge \frac{1}{2}\|u - u_0\|^2.$$

Observe that $h(u) \ge 0$ for all $u \in S^{d-1}$, since $o \in K$. Next we show that $h(u) > 0$ for all $u \in S^{d-1} \setminus \{u_0\}$. This is proved by contradiction. Assume $h(u_1) = 0$ for some $u_1 \in S^{d-1} \setminus \{u_0\}$. Let $\mathrm{N}(K, x)$, $K \in \mathcal{K}^d$ and $x \in \operatorname{bd} K$, be the normal cone of K at x. Hence, $u_1 \in \mathrm{N}(K, x_0)$, and this implies $u_0 + \lambda u_1 \in \mathrm{N}(K, x_0)$ for all $\lambda \ge 0$. Thus we get $h(u_0 + \lambda u_1) = 0$ for all $\lambda \ge 0$. If $\lambda > 0$ is sufficiently small, $\|u_0 + \lambda u_1\|^{-1}(u_0 + \lambda u_1) \in U(u_0)$. This, however, contradicts (5).

Since $S^{d-1} \setminus U(u_0)$ is compact, there is a positive constant $r_3 > 0$ such that

$$h(u) \geq 4r_3 \geq r_3 \|u - u_0\|^2, \quad u \in S^{d-1} \setminus U(u_0).$$

Let $r := 2 \min\{r_2, r_3\}$. Now, for all $u \in S^{d-1}$,

$$h(K, u) \geq \frac{1}{2}\, r\, \|u - u_0\|^2 = h(B(x_0 - r u_0, r), u),$$

and this proves (a). □

Theorem 2.8 contains the promised transformation formula. The method, which we employ to prove this theorem, will also turn out to be useful for characterizing absolute continuity of Euclidean surface area and curvature measures. This will be investigated in a subsequent paper. As regards notation used for stating Theorem 2.8, see [36] for a definition and properties of the spherical image $\sigma(K, \cdot)$ and the reverse spherical image $\tau(K, \cdot)$. If X is a topological space, $\mathfrak{B}(X)$ denotes the σ-algebra of Borel sets of X [10, p. 60].

Theorem 2.8. *For an arbitrary $K \in \mathcal{K}^d_{00}$ and $p > 0$ ($K \in \mathcal{K}^d$, if $p = 1$) we have $\mathcal{O}_p(K, \beta) = \widetilde{\mathcal{O}}_p(K, \sigma(K, \beta))$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $\widetilde{\mathcal{O}}_p(K, \omega) = \mathcal{O}_p(K, \tau(K, \omega))$, $\omega \in \mathfrak{B}(S^{d-1})$.*

Proof. In case $\dim K \leq d - 1$ and $p = 1$ both integrals vanish. Thus we may assume $K \in \mathcal{K}^d_{00}$. If $\beta \in \mathfrak{B}(\mathbb{R}^d)$, then $\sigma(K, \beta)$ is $\mathcal{H}^{d-1}$ measurable [36, Lemma 2.2.10]. A similar statement holds for the reverse spherical image $\tau(K, \omega)$. Note that [10, Theorem 3.2.22] is applicable to functions with values in $\overline{\mathbb{R}}^+$. In this context the product $g(x)\mathrm{ap}J_{d-1} f(x)$ has to be interpreted as 0, if $\mathrm{ap}J_{d-1} f(x) = 0$ and $g(x) = \infty$. Lemma 2.1 ensures that Federer's coarea formula can be applied to $\sigma_K|_{(\mathrm{bd}\, K)_r}$, for an arbitrary $r > 0$. Thus we obtain for $\beta \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}
&\int_{\beta \cap (\mathrm{bd}\, K)_r} H_{d-1}(K, x)^{\frac{1}{d+1}} \, \mathrm{d}\mathcal{H}^{d-1}(x) \\
&\quad = \int_{\beta \cap (\mathrm{bd}\, K)_r} H_{d-1}(K, x)^{-\frac{d}{d+1}} \, \mathrm{ap}J_{d-1}\sigma_K(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) \\
&\quad = \int_{S^{d-1}} \int_{\sigma_K^{-1}(\{u\}) \cap (\mathrm{bd}\, K)_r} \mathbf{1}_\beta(x) H_{d-1}(K, x)^{-\frac{d}{d+1}} \, \mathrm{d}\mathcal{H}^0(x) \, \mathrm{d}\mathcal{H}^{d-1}(u) \\
&\quad = \int_{\sigma_K((\mathrm{bd}\, K)_r)} \mathbf{1}_\beta \circ \nabla h_K(u) H_{d-1}(K, \nabla h_K(u))^{-\frac{d}{d+1}} \, \mathrm{d}\mathcal{H}^{d-1}(u) \\
&\quad = \int_{\sigma_K((\mathrm{bd}\, K)_r)} \mathbf{1}_{\sigma(K, \beta)}(u) D_{d-1} h(K, u)^{\frac{d}{d+1}} \, \mathrm{d}\mathcal{H}^{d-1}(u),
\end{aligned}$$

where we have used Lemma 2.3, Lemma 2.6 and the fact that for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ the equality $\mathbf{1}_\beta \circ \nabla h_K(u) = \mathbf{1}_{\sigma(K, \beta)}(u)$ holds true. Furthermore recall [36, Corollary 1.7.3] for $\nabla h_K(u) = \mathrm{grad}\, h_K(u)$, whenever h_K is differentiable at u (and this is the case for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$). Now Lemma 2.2 and Lebesgue's increasing convergence theorem yield

$$\mathcal{O}_a(K, \beta) = \int_{\sigma_K((\mathrm{bd}\, K)_+)} \mathbf{1}_{\sigma(K, \beta)}(u) D_{d-1} h(K, u)^{\frac{d}{d+1}} \, \mathrm{d}\mathcal{H}^{d-1}(u).$$

Hence, the first statement of Theorem 2.8 (for $p = 1$) is implied by Lemma 2.7. The second statement follows, if we observe that excluding a set of measure zero $D_{d-1}h(K,u) > 0 \Rightarrow \mathbf{1}_{\tau(K,\omega)} \circ \nabla h_K(u) = \mathbf{1}_\omega(u)$. For a proof of this fact Lemma 2.7 can be used. In the case of an arbitrary $p > 0$ the theorem follows in the same way. □

3. Properties of affine surface area

It has recently been proved that the equiaffine surface area is a valuation [37] and upper semicontinuous [24], [18]. In the previous sections we investigated various definitions of the p-affine surface area of an arbitrary convex body, and in particular we proved the coincidence of these definitions. From this it is easy to see that the p-affine surface area is a valuation and upper semicontinuous. There is also a natural definition of mixed affine surface area (different from the one proposed in [23]) and a number of geometric inequalities connected with this notion (cf. [14]), which, however, we shall not pursue presently.

The set of furthest points of a convex body $K \in \mathcal{K}_0^d$ is defined by

$$\exp^* K := \{x \in \operatorname{bd} K \mid \exists u \in \mathrm{N}(K,x) \cap S^{d-1}\, \exists R > 0 : K \subset B(x - Ru, R)\},$$

and, for $u \in \mathbb{R}^d \setminus \{o\}$ and $t \in \mathbb{R}$, we set $H_{u,t} := \{x \in \mathbb{R}^d \mid \langle x, u\rangle = t\}$ and $H^+_{u,t} := \{x \in \mathbb{R}^d \mid \langle x, u\rangle \geq t\}$.

Theorem 3.1. *Let $K \in \mathcal{K}_{00}^d$ and $p > 0$ ($K \in \mathcal{K}^d$, if $p = 1$), and let $\beta \subset \mathbb{R}^d$ be $\mathcal{H}^{d-1}$ measurable. Then $\mathcal{O}_p(K,\beta) > 0$ if and only if $\mathcal{H}^{d-1}(\exp^* K \cap \beta) > 0$.*

Proof. First we assume $\mathcal{H}^{d-1}(\exp^* K \cap \beta) > 0$. If we can show $H_{d-1}(K,x) > 0$ for $\mathcal{H}^{d-1}$ almost all $x \in \exp^* K \cap \beta$, we obtain

$$\mathcal{O}_p(K,\beta) \geq \mathcal{O}_p(K, \exp^* K \cap \beta) > 0.$$

In order to prove this we can assume that $x \in \exp^* K \cap \beta \cap \mathcal{M}(K)$. Let us denote by $\Delta(K,x,\delta)$, $\delta > 0$ small enough, the uniquely determined number $t > 0$ such that for $u := \sigma_K(x)$

$$V\left(K \cap H^+_{u,\, h(K,u)-t}\right) = \delta.$$

Leichtweiß [16, Hilfssatz 2] showed that

$$H_{d-1}(K,x)^{\frac{1}{d+1}} = c_d \lim_{\delta \to 0} \frac{\Delta(K,x,\delta)}{\delta^{2/d+1}},$$

where c_d is a suitable positive constant. Especially for a ball of radius $\rho > 0$

$$\rho^{-\frac{d-1}{d+1}} = c_d \lim_{\delta \to 0} \frac{\Delta(B(x-\rho u,\rho),x,\delta)}{\delta^{2/d+1}}.$$

Since $x \in \exp^* K \cap \operatorname{reg} K$, there is some $\rho > 0$ such that $K \subset B(x - \rho u, \rho)$. This in turn implies $\Delta(B(x-\rho u,\rho),x,\delta) \leq \Delta(K,x,\delta)$. Thus we have

$$H_{d-1}(K,x)^{\frac{1}{d+1}} \geq \rho^{-\frac{d-1}{d+1}} > 0.$$

For the converse it is sufficient to prove that if $x_0 \in \mathcal{M}(K)$ and $H_{d-1}(K,x_0) > 0$, then $x_0 \in \exp^* K$. We may assume $x_0 = o$. Let $e_d := -\sigma_K(x_0)$, and let f locally represent $\operatorname{bd} K$ at x_0. Since $x_0 \in \mathcal{M}(K)$, we obtain for $x \in B(x_0, r_1) \cap e_d^\perp$, $r_1 > 0$ sufficiently small,

$$|f(x) - \frac{1}{2} d^2 f(x_0)(x,x)| \leq R(\|x\|)\|x\|^2,$$

where $R(\|x\|) \to 0$ for $x \to x_0$. From $H_{d-1}(K,x_0) = \det\big(d^2 f(x_0)\big) > 0$ we conclude that there is a constant $c > 0$ such that for all $x \in B(x_0, r_2) \cap e_d^\perp$, $r_2 \in (0,c)$ sufficiently small,

$$f(x) \geq \frac{1}{2} d^2 f(x_0)(x,x) - R(\|x\|)\|x\|^2 \geq c^{-1}\|x\|^2.$$

Now it is easily checked that

$$c^{-1}\|x\|^2 \geq c - \sqrt{c^2 - \|x\|^2}, \quad x \in B(x_0, r_2) \cap e_d^\perp.$$

In other words there is a neighbourhood U of x_0 such that $U \cap \operatorname{bd} K$ is contained in $B(x_0 + c e_d, c)$. This shows $x_0 \in \exp^* K$. □

It is known that for $K \in \mathcal{K}_0^d$ and $x_0 \in \mathcal{M}(K)$ there is some $r > 0$ such that $x_0 \in (\operatorname{bd} K)_r$, i.e., $\mathcal{M}(K) \subset (\operatorname{bd} K)_+$ [36, Notes for §2.5]. Corollary 3.2 provides an analytical description of the set $\exp^* K \cap \mathcal{M}(K)$, which is dual to Lemma 2.7.

Corollary 3.2. *Let $K \in \mathcal{K}^d$ and $x_0 \in \mathcal{M}(K)$. Then the following two conditions are equivalent.*

(a) $x_0 \in \exp^* K$.
(b) $H_{d-1}(K,x_0) > 0$.

The following theorem is a spherical counterpart to Theorem 3.1. It is an immediate consequence of Lemma 2.7. In analogy to $\exp^* K$ the set $\operatorname{expn}^* K$ of directions of nearest (boundary) points is defined by

$$\operatorname{expn}^* K := \{u \in S^{d-1} \mid \exists x \in F(K,u)\, \exists r > 0 : B(x - ru, r) \subset K\}.$$

Theorem 3.3. *Let $K \in \mathcal{K}_{00}^d$ and $p > 0$ ($K \in \mathcal{K}^d$, if $p = 1$), and let $\omega \subset S^{d-1}$ be $\mathcal{H}^{d-1}$ measurable. Then $\widetilde{\mathcal{O}}_p(K,\omega) > 0$ if and only if $\mathcal{H}^{d-1}(\operatorname{expn}^* K \cap \omega) > 0$.*

Lemma 3.4 generalizes a statement by Petty for convex bodies of class C_+^2. We shall use it to establish invariance properties of affine surface area measures, but it seems to be helpful in other contexts, too. For the proof let us define, for $L \in \mathcal{K}_0^d$, $u \in S^{d-1}$ and $t \geq 0$, the sets $L(u,t)$ and $L^+(u,t)$ by

$$L^{(+)}(u,t) := L \cap H^{(+)}_{u,\, h(K,u)-t},$$

and let λ^d denote d-dimensional Lebesgue measure. We write $\alpha(d-1)$ for the volume of the $(d-1)$-dimensional unit ball.

Lemma 3.4. *Let $K \in \mathcal{K}_0^d$, $x_0 \in \mathcal{M}(K)$, and $u_0 := \sigma_K(x_0)$. Then*

$$H_{d-1}(K, x_0) = const(d) \lim_{t \to 0} \frac{\lambda^d(K^+(u_0,t))^{d-1}}{\mathcal{H}^{d-1}(K(u_0,t))^{d+1}},$$

where

$$const(d) := (d+1)^{d-1}\alpha(d-1)^2.$$

Proof. We only sketch the proof for the case $H_{d-1}(K, x_0) = 0$, since similar arguments have been used in [16, Hilfssatz 2] for the remaining case. Details can be found in [14]. Let $\mathcal{S}$ be the Schwarz symmetrization [15, §19] of K with respect to the line $\mathbb{R}u_0$. Then by construction resp. by Fubini's theorem

$$\mathcal{H}^{d-1}(\mathcal{S}(u_0,t)) = \mathcal{H}^{d-1}(K(u_0,t)) \quad \text{and} \quad \lambda^d(\mathcal{S}^+(u_0,t)) = \lambda^d(K^+(u_0,t)).$$

Write $r(t) \geq 0$, $0 \leq t \leq h(K,u_0) + h(K,-u_0)$, for the radius of the $(d-1)$-dimensional ball $\mathcal{S}(u_0,t)$. The function r is concave and continuous. If $r(0) > 0$, the lemma follows from

$$\lim_{h \to 0} \frac{\left\{ \int_0^h r(t)^{d-1}\, \mathrm{d}t \right\}^{d-1}}{r(h)^{d^2-1}} = 0.$$

Now, assume $r(0) = 0$. Then, for $h > 0$ sufficiently small, the function r is strictly increasing on $[0,h]$. Thus

$$\begin{aligned} \frac{\left\{ \int_0^h \mathcal{H}^{d-1}(K(u_0,t))\, \mathrm{d}t \right\}^{d-1}}{\mathcal{H}^{d-1}(K(u_0,h))^{d+1}} &\leq h^{d-1} \frac{\mathcal{H}^{d-1}(K(u_0,h))^{d-1}}{\mathcal{H}^{d-1}(K(u_0,h))^{d+1}} \\ &= 2^{1-d}\, \mathcal{H}^{d-1}\left(\frac{1}{\sqrt{2h}} K(u_0,h) \right)^{-2}. \end{aligned}$$

Since $H_{d-1}(K, x_0) = 0$, the right side converges to zero for $h \downarrow 0$. □

Corollary 3.5. *Let $K \in \mathcal{K}_0^d$, $x_0 \in \mathcal{M}(K)$, and let α be a regular affine transformation with $\alpha(x) = \varphi(x) + b$, $\varphi \in \mathrm{GL}(\mathbb{R}^d)$ and $b \in \mathbb{R}^d$, for all $x \in \mathbb{R}^d$. Then $\alpha(x_0) \in \mathcal{M}(\alpha(K))$, and*

$$H_{d-1}(\alpha(K), \alpha(x_0)) = \frac{|\det \varphi|^{d-1}}{\sqrt{\det\left(\langle \varphi(e_i), \varphi(e_j) \rangle_{i,j=1}^{d-1} \right)}^{\,d+1}} H_{d-1}(K, x_0),$$

where $(e_1, \ldots, e_{d-1})$ is an orthonormal basis of $T_{x_0}K$.

Proof. In the course of the proof we write αx instead of $\alpha(x)$, etc. The first statement follows from [16, p. 444, (39) and (40)], [2, Section 4, Bemerkung 2], and from

$$u \in \mathrm{N}(K,x) \Leftrightarrow \frac{\varphi^{-t}u}{\|\varphi^{-t}u\|} \in \mathrm{N}(\alpha K, \alpha x),$$

for all $x \in \operatorname{bd} K$. The second statement then is implied by Lemma 3.4, since for $u_0 := \sigma_K(x_0)$

$$
\begin{aligned}
|\det\varphi|\lambda^d(K^+(u_0,s)) &= \lambda^d(\alpha K^+(u_0,s)) \\
&= \lambda^d\left(\alpha K \cap H^+_{\frac{\varphi^{-t}u_0}{\|\varphi^{-t}u_0\|},\, h\left(\alpha K,\, \frac{\varphi^{-t}u_0}{\|\varphi^{-t}u_0\|}\right)-s_\alpha}\right) \\
&= \lambda^d\left(\alpha K \cap H^+_{\sigma_{\alpha K}(\alpha x_0),\, h(\alpha K, \sigma_{\alpha K}(\alpha x_0))-s_\alpha}\right),
\end{aligned}
$$

where $s_\alpha = \|\varphi^{-t}u_0\|^{-1}s$, and

$$
\mathcal{H}^{d-1}(\alpha K(u_0,s)) = \sqrt{\det\left(\langle \varphi e_i, \varphi e_j\rangle_{i,j=1}^{d-1}\right)}\,\mathcal{H}^{d-1}(K \cap H_{u_0,\, h(K,u_0)-s}).
$$

□

In the case of a hypersurface of class C^2 with nonvanishing Gauß-Kronecker curvature the preceding result is due to Guggenheimer [13].

Theorem 3.6. *Let $K \in \mathcal{K}^d_{00}$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, $\varphi \in \mathrm{GL}(\mathbb{R}^d)$, and $p > 0$. Then we obtain*

$$
\mathcal{O}_p(\varphi(K), \varphi(\beta)) = |\det\varphi|^{\frac{d-p}{d+p}}\,\mathcal{O}_p(K,\beta).
$$

In addition, $\mathcal{O}_1$ is invariant with respect to translations.

Proof. The proof immediately follows from Corollary 3.5 and from the relation

$$
\langle \varphi(x), \sigma_{\varphi(K)}(\varphi(x))\rangle = \frac{|\det\varphi|}{\mathrm{ap}J_{d-1}\varphi(x)}\,\langle x, \sigma_K(x)\rangle,
$$

which holds for $\varphi \in \mathrm{GL}(\mathbb{R}^d)$ and $x \in \operatorname{reg} K$. □

The following result is contained in a more elaborate statement in [14, Theorem 3.3.5]. However, Lemma 3.7 will be sufficient for an application in Section 4. In the sequel let K^* denote the polar body of $K \in \mathcal{K}^d_{00}$ with respect to o.

Lemma 3.7. *Let $K \in \mathcal{K}^d_{00}$. Then $\mathcal{O}_a(K)^{d+1} \le d^{d+1}V(K)^dV(K^*)$. In the case of equality there is a positive constant λ such that for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$ the relation $H_{d-1}(K,x) = \lambda\,\langle x, \sigma_K(x)\rangle^{d+1}$ holds.*

Proof. An application of Hölder's inequality yields

$$
\begin{aligned}
\mathcal{O}_a(K) &= \int_{\operatorname{bd} K}\left\{\sqrt{\frac{H_{d-1}(K,x)}{\langle x, \sigma_K(x)\rangle^{d-1}}}\right\}^{\frac{2}{d+1}}\langle x, \sigma_K(x)\rangle^{\frac{d-1}{d+1}}\,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&\le \mathcal{O}_d(K, \mathbb{R}^d)^{\frac{2}{d+1}}\,(dV(K))^{\frac{d-1}{d+1}}.
\end{aligned}
$$

Similarly one estimates after an application of Theorem 2.8 for $p = d$

$$\begin{aligned}
\mathcal{O}_d(K,\mathbb{R}^d) &= \int_{S^{d-1}} (h_K D_{d-1} h_K)^{\frac{1}{2}} (h_K)^{-\frac{d}{2}} \,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \left(\int_{S^{d-1}} h_K D_{d-1} h_K \,\mathrm{d}\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \left(\int_{S^{d-1}} (h_K)^{-d} \,\mathrm{d}\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \\
&\le \left(\int_{S^{d-1}} h(K,u) \,\mathrm{d}S_{d-1}(K,u) \right)^{\frac{1}{2}} (dV(K^*))^{\frac{1}{2}} \\
&= d\sqrt{V(K)V(K^*)}.
\end{aligned}$$

The statement on the case of equality is implied by the condition for equality in Hölder's inequality. □

Remark. By similar arguments the following more general result can be proved. Let $K \in \mathcal{K}^d_{00}$. Then

$$\mathcal{O}_p(K,\mathbb{R}^d) \le [dV(K)]^{\frac{d}{d+p}} [dV(K^*)]^{\frac{p}{d+p}}.$$

Equality holds if and only if K has a positive, continuous curvature function f_K and there is a positive constant $\lambda > 0$ such that for all $u \in S^{d-1}$

$$f_K(u) = \lambda\, h(K,u)^{-(d+1)}.$$

See also [22, Prop. (4.6) and Prop. (4.7)] for another proof of this inequality and for a statement on the case of equality in a more restricted situation.

4. On the affine isoperimetric inequality

In this section we provide proofs for two central inequalities of affine convex geometry. The first two well known lemmas are included for the reader's convenience. Lemma 4.1 can be found in [32, Theorem G, p. 205], and Lemma 4.2 follows from Brunn's classical characterization of ellipsoids. Let $\mathcal{M}_n(\mathbb{R})$, $n \ge 1$, be the set of real $n \times n$ matrices.

Lemma 4.1. *Let $A, B \in \mathcal{M}_{d-1}(\mathbb{R})$ be symmetric and positive semidefinite. Then*

$$2\left(\det\left[\frac{1}{2}(A+B)\right]\right)^{\frac{1}{d+1}} \ge (\det A)^{\frac{1}{d+1}} + (\det B)^{\frac{1}{d+1}}.$$

If, in addition, B is positive definite, equality holds if and only if $A = B$.

For $K \in \mathcal{K}^d_0$ and $u \in S^{d-1}$ we define by $M(K,u)$ the set of the midpoints of all line segments $K \cap L$ where L varies over all lines in $\mathbb{R}^d$ of direction u that meet $\operatorname{int} K$.

Lemma 4.2. *Let $K \in \mathcal{K}^d_0$, and let S^* be a dense subset of S^{d-1}. Then K is an ellipsoid if and only if for each $u \in S^*$ the set $M(K,u)$ is contained in a hyperplane.*

Lemma 4.3. *Let $U \subset \mathbb{R}^{d-1}$, $o \in U$, be open and convex. Let $f : U \to \mathbb{R}$ be locally Lipschitzian and differentiable at o. If $\langle x, \nabla f(x)\rangle = f(x)$ for $\mathcal{H}^{d-1}$ almost all $x \in U$ such that f is differentiable at x, then $f(x) = \langle v, x\rangle$ for all $x \in U$ and some suitable $v \in \mathbb{R}^{d-1}$.*

Proof. For $u \in S^{d-2}$ let $t(u) := \sup\{t \geq 0 \mid tu \in U\}$. Let S^* be the set of all $u \in S^{d-2}$ such that for $\mathcal{H}^1$ almost all $t \in [0, t(u))$ the function f is differentiable at tu and $\langle tu, \nabla f(tu)\rangle = f(tu)$. By Rademacher's theorem $\mathcal{H}^{d-2}(S^{d-2} \setminus S^*) = 0$. Fix $u \in S^*$ and consider $h_u : [0, t(u)) \to \mathbb{R}, \quad t \mapsto f(tu)$. Then the equation $h_u'(t) = \langle u, \nabla f(tu)\rangle$ holds for $\mathcal{H}^1$ almost all $t \in [0, t(u))$. The function $\varphi_u(t) := t^{-1} h_u(t)$ is defined for $t \in (0, t(u))$, and for $\mathcal{H}^1$ almost all $t \in (0, t(u))$ we obtain $\varphi_u'(t) = 0$. Since φ_u is locally Lipschitzian on $(0, t(u))$, we get $\varphi_u(t) = c(u)$ for all $t \in (0, t(u))$. Hence, $f(tu) = h_u(t) = c(u)t$ for all $t \in [0, t(u))$ which implies that $Df(o)(tu) = f(tu)$ first for $u \in S^*$ then by continuity for all $u \in S^{d-2}$. This shows for $x \in U$ and $v := \nabla f(o) \in \mathbb{R}^{d-1}$ that $f(x) = Df(o)(x) = \langle \nabla f(o), x\rangle = \langle v, x\rangle$. □

Before we can proceed to Lemma 4.4, which generalizes a representation in [16, p. 457], some definitions are required. Let $K \in \mathcal{K}_0^d$ and $u \in S^{d-1}$. Then $K^\circ(u) := \operatorname{relint} p(H_{u,0}, K)$, and $f_u^-, f_u^+ : K^\circ(u) \to \mathbb{R}$ are defined by

$$f_u^-(x) := \min\{\lambda \in \mathbb{R} \mid x + \lambda u \in K\}, \quad x \in K^\circ(u),$$

$$f_u^+(x) := \max\{\lambda \in \mathbb{R} \mid x + \lambda u \in K\}, \quad x \in K^\circ(u).$$

Finally, we set $K^-(u) := \operatorname{graph}(f_u^-)$ and $K^+(u) := \operatorname{graph}(f_u^+)$.

Lemma 4.4. *Let $K \in \mathcal{K}_0^d$ and $u \in S^{d-1}$. Then we have*

$$\mathcal{O}_a(K) = \int_{K^\circ(u)} \left\{ \left[\det\left(d^2 f_u^-(x)\right)\right]^{\frac{1}{d+1}} + \left[\det\left(d^2(-f_u^+)(x)\right)\right]^{\frac{1}{d+1}} \right\} \mathrm{d}\mathcal{H}^{d-1}(x).$$

Proof. Since u is fixed in the proof, we can omit the index u of the functions f_u^-, f_u^+. Obviously, we have $\operatorname{bd} K = K^-(u) \,\dot\cup\, K^+(u) \,\dot\cup\, (\operatorname{bd} K \cap Z(u))$, where $Z(u) := \operatorname{relbd} K^\circ(u) + \mathbb{R}u$. From generalized cylindrical coordinates it is easy to see that

$$\int_{\operatorname{bd} K \cap Z(u)} H_{d-1}(K, y)^{\frac{1}{d+1}} \,\mathrm{d}\mathcal{H}^{d-1}(y) = 0.$$

Thus we obtain

$$\mathcal{O}_a(K) = \int_{K^-(u)} H_{d-1}(K, y)^{\frac{1}{d+1}} \,\mathrm{d}\mathcal{H}^{d-1}(y) + \int_{K^+(u)} H_{d-1}(K, y)^{\frac{1}{d+1}} \,\mathrm{d}\mathcal{H}^{d-1}(y).$$

The injective mapping $F^- : K^\circ(u) \to \mathbb{R}^d$, $x \mapsto x + f^-(x)u$, is locally Lipschitzian, and

$$g : K^\circ(u) \to \overline{\mathbb{R}}^+, \quad x \mapsto H_{d-1}(K, F^-(x))^{\frac{1}{d+1}},$$

is $\mathcal{H}^{d-1} \llcorner K^\circ(u)$ measurable. Note that for $\mathcal{H}^{d-1}$ almost all $x \in K^\circ(u)$

$$J_{d-1}F^-(x) = \sqrt{1 + \|\nabla f^-(x)\|^2}.$$

An immediate extension of Federer's area formula [10, Theorem 3.2.5] to locally Lipschitzian maps yields now

$$\int_{K^\circ(u)} H_{d-1}(K, F^-(x))^{\frac{1}{d+1}} \sqrt{1+\|\nabla f^-(x)\|^2}\, \mathrm{d}\mathcal{H}^{d-1}(x)$$
$$= \int_{K^-(u)} H_{d-1}(K,y)^{\frac{1}{d+1}}\, \mathrm{d}\mathcal{H}^{d-1}(y).$$

According to [16, p. 446, (49)] for $\mathcal{H}^{d-1}$ almost all $x \in K^\circ(u)$

$$H_{d-1}(K, F^-(x))^{\frac{1}{d+1}} = \frac{\left[\det\left(d^2 f^-(x)\right)\right]^{\frac{1}{d+1}}}{\sqrt{1+\|\nabla f^-(x)\|^2}}. \tag{6}$$

Thus

$$\int_{K^\circ(u)} \left[\det\left(d^2 f^-(x)\right)\right]^{\frac{1}{d+1}}\, \mathrm{d}\mathcal{H}^{d-1}(x)$$
$$= \int_{K^-(u)} H_{d-1}(K,y)^{\frac{1}{d+1}}\, \mathrm{d}\mathcal{H}^{d-1}(y).$$

A similar argument applied to f^+ concludes the proof of Lemma 4.4. □

The next lemma expresses the fact that equiaffine surface area is not decreased by Steiner symmetrization (see, e.g., [15, §18] for a definition).

Lemma 4.5. *Let $K \in \mathcal{K}_0^d$ and $u \in S^{d-1}$. Then $\mathcal{O}_a(K) \le \mathcal{O}_a(S_u K)$, where $S_u K$ denotes the Steiner symmetral of K with respect to the hyperplane $H_{u,0}$.*

Proof. Let f_u^- and f_u^+ be defined as before Lemma 4.4. Then we get

$$\mathcal{O}_a(S_u K) = 2\int_{K^\circ(u)} \left[\det\left(d^2\left[\frac{1}{2}(f_u^- - f_u^+)\right](x)\right)\right]^{\frac{1}{d+1}}\, \mathrm{d}\mathcal{H}^{d-1}(x).$$

Now apply Lemma 4.1, and the proof is finished. □

Theorem 4.6. *For $K \in \mathcal{K}^d$ the general equiaffine isoperimetric inequality $\mathcal{O}_a(K)^{d+1} \le d^{d+1}\alpha(d)^2 V(K)^{d-1}$ holds true.*

Proof. Everything is clear, if $\operatorname{int} K = \emptyset$. Thus we can assume $K \in \mathcal{K}_0^d$. Choose a sequence $(K_n)_{n\in\mathbb{N}} \subset \mathcal{K}_0^d$ such that $\lim_{n\to\infty} K_n = B(o,r)$ and such that K_n is obtained from K by repeated Steiner symmetrization [15, Korollar, p. 226]. Lemma 4.5 shows that $\mathcal{O}_a(K) \le \mathcal{O}_a(K_n)$ for all $n \in \mathbb{N}$. The upper semicontinuity of affine surface area hence implies

$$\begin{aligned}\mathcal{O}_a(K) &\le \limsup_{n\to\infty} \mathcal{O}_a(K_n) \le \mathcal{O}_a\left(\lim_{n\to\infty} K_n\right)\\ &= \mathcal{O}_a(B(o,r)) = \left(\frac{V(K)}{\alpha(d)}\right)^{\frac{d-1}{d+1}} d\alpha(d),\end{aligned}$$

which was to be proved. □

Using Minkowski's existence theorem, Minkowski's inequality and Theorem 4.6 we obtain Theorem 4.7. Moreover, if equality in Theorem 4.6 holds only for ellipsoids, the same is true for Theorem 4.7. A short proof of these two statements is reproduced in [36, pp. 420-421] or [14, Satz 1.1.1].

Theorem 4.7. *For $K \in \mathcal{K}_0^d$ with $s(K) = o$ the Blaschke-Santaló inequality $V(K)V(K^*) \le \alpha(d)^2$ holds true.*

Theorem 4.8. *If $K \in \mathcal{K}_0^d$ and $\mathcal{O}_a(K)^{d+1} = d^{d+1}\alpha(d)^2 V(K)^{d-1}$, then K is an ellipsoid.*

Proof. We may assume $s(K) = o$. From Lemma 3.7, Theorem 4.7 and from the assumption of the theorem we infer that there is a constant $\lambda > 0$ such that for $\mathcal{H}^{d-1}$ almost all $y \in \operatorname{bd} K$

$$H_{d-1}(K, y) = \lambda \langle y, \sigma_K(y) \rangle^{d+1}. \tag{7}$$

We also know that $\mathcal{O}_a(K) = \mathcal{O}_a(S_u K)$ for each $u \in S^{d-1}$. Let S^* denote the set of all $u \in S^{d-1}$ such that the functions f_u^-, f_u^+, defined before Lemma 4.4, are differentiable at o. Since $u \in S^*$ if and only if the radial function $\rho(K, \cdot)$ of K is differentiable at $\pm u$, we have $\mathcal{H}^{d-1}(S^{d-1} \setminus S^*) = 0$. Choose $u \in S^*$, and again omit the index u of the functions f_u^-, f_u^+ for the moment. From (6), (7), from the proof of Lemma 4.5, and from Lemma 4.1 it follows that $d^2 f^-(x) = d^2 f^+(x)$, and hence in particular

$$\det\left(d^2 f^-(x)\right) = \det\left(d^2(-f^+)(x)\right), \tag{8}$$

for $\mathcal{H}^{d-1}$ almost all $x \in K^\circ(u)$. In addition,

$$\begin{aligned} H_{d-1}(K, F^-(x)) &= \lambda \langle F^-(x), \sigma_K(F^-(x)) \rangle^{d+1} \\ &= \lambda \left\langle x + f^-(x)u, \frac{\nabla f^-(x) - u}{\sqrt{1 + \|\nabla f^-(x)\|^2}} \right\rangle^{d+1}, \end{aligned}$$

which together with (6) implies

$$\det\left(d^2 f^-(x)\right) = \lambda \langle x + f^-(x)u, \nabla f^-(x) - u \rangle^{d+1}. \tag{9}$$

A similar argument leads to

$$\det\left(d^2(-f^+)(x)\right) = \lambda \langle x + f^+(x)u, \nabla(-f^+)(x) + u \rangle^{d+1}. \tag{10}$$

Thus (8), (9) and (10) yield for $\mathcal{H}^{d-1}$ almost all $x \in K^\circ(u)$

$$\langle x, \nabla(f^- + f^+)(x) \rangle = (f^- + f^+)(x).$$

From Lemma 4.3 we obtain that $f_u^- + f_u^+$ is linear for each $u \in S^*$. Hence, the statement of our theorem follows by an application of Lemma 4.2. □

Theorem 4.9. *If $K \in \mathcal{K}_0^d$, $s(K) = o$, and $V(K)V(K^*) = \alpha(d)^2$, then K is an ellipsoid.*

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