Curvature Relations and Affine Surface Area for a General Convex Body and its Polar*

Daniel Hug

Abstract. We investigate the relationship between generalized curvatures of an arbitrary convex body K and its polar body K^* in d-dimensional Euclidean space. For example, the generalized Gauß-Kronecker curvature of K is compared with the product of the generalized principal radii of curvature of K^*. This leads to a generalization of the classical statement saying that the product of the equiaffine support functions of K and K^* is equal to 1, provided K is sufficiently smooth and has positive Gauß-Kronecker curvature. Another consequence concerns the equality of the extended p-affine surface area of K and the q-affine surface area of K^*, if $pq = d^2$. In the special case of a smooth convex body and for $p = d$ this result is well known in centroaffine differential geometry.

1 Introduction

A substantial part of the research in affine differential geometry in Euclidean space $\mathbb{R}^d$ is devoted to the investigation of the centroaffinely invariant situation. First accounts of results in centroaffine differential geometry are contained in the books of Salkowski [15] and P. A. & A. P. Schirokow [16] as well as in the book of Laugwitz [9]. The recent paper by Oliker & Simon [14] describes a modern structural approach to the field, and in particular it emphasizes the relationship between Euclidean, equiaffine, and centroaffine geometry. See also the introductory chapter in [11].

It is an essential feature of centroaffine differential geometry that the centroaffine metric of a hypersurface is isometric to that of the polar (dual) hypersurface, cf. [15, §34], [9, p. 41], [14, Proposition 7.2.1, (7.16)]. As a consequence the centroaffine surface area of a hypersurface coincides with that of the polar surface. For the boundary of a sufficiently smooth convex body (nonempty compact convex set) with positive Gauß-Kronecker curvature this was mentioned by Gruber [5, Remark 1].

The present paper is mainly intended to provide generalizations in the setting of the geometry of convex bodies for results which originate from affine differential geometry. Hopefully, this approach also sheds new light on the classical situation. Although for a general convex

**AMS 1991 subject classification.* Primary 52A20, 52A40; Secondary 53A15.
Key words and phrases. Generalized Gauß-Kronecker curvature, generalized principal radii of curvature, polar body, affine surface area, equiaffine support function.

body there is no equiaffine metric in the classical sense, the notion of equiaffine surface area has recently been extended to include the case of an arbitrary convex body [10], [13], [19]. Subsequently, Lutwak [12] generalized this concept to the notion of the p-affine surface area $\mathcal{O}_p(K)$ of an arbitrary convex body K which contains the origin in its interior by setting

$$\mathcal{O}_p(K) = \inf\left\{ [dV_p(K,f)]^{\frac{d}{d+p}} \left[\int_{S^{d-1}} f^{-d}\,\mathrm{d}\mathcal{H}^{d-1}\right]^{\frac{p}{d+p}} \mid f \in C^+(S^{d-1})\right\},$$

where $C^+(S^{d-1})$ is the set of all positive continuous functions on the unit sphere S^{d-1} of $\mathbb{R}^d$, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, and

$$V_p(K,f) := \frac{1}{d}\int_{S^{d-1}} f(u)^p h(K,u)^{1-p}\,\mathrm{d}S_{d-1}(K,u).$$

In the last formula $S_{d-1}(K,\cdot)$ is the Euclidean surface area measure of the convex body K [18, Chapter 4], and $h(K,\cdot) = h_K$ denotes the support function of K. It should be observed that for $p = 1$ resp. $p = d$ we obtain the equiaffine resp. the centroaffine surface area. These ideas were further developed in [6]. There in particular it is proved that $\mathcal{O}_p(K)$, for every $p > 0$, can be represented alternatively as an integral over the boundary of K or as an integral over the unit sphere, i.e.,

$$\begin{aligned}\mathcal{O}_p(K) &= \int_{\mathrm{bd}\,K}\left\{\frac{H_{d-1}(K,x)}{\langle x,\sigma_K(x)\rangle^{(p-1)\frac{d}{p}}}\right\}^{\frac{p}{d+p}}\,\mathrm{d}\mathcal{H}^{d-1}(x)\\ &= \int_{S^{d-1}}\left\{\frac{D_{d-1}h(K,u)}{h(K,u)^{p-1}}\right\}^{\frac{d}{d+p}}\,\mathrm{d}\mathcal{H}^{d-1}(u). \qquad (1)\end{aligned}$$

Here and in the following we adopt the notation and terminology from [6]. For all notions of convexity, which are not explicitly defined in [6] or in the present work, [18] is our reference. Despite the fact that the concept of p-affine surface area is invariant with respect to the group of all volume preserving linear transformations, it turns out to be comfortable to work in a Euclidean space $\mathbb{R}^d$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Recall that for an arbitrary convex body K the generalized Gauß-Kronecker curvature, $H_{d-1}(K,x)$, and the product of the generalized principal radii of curvature, $D_{d-1}h(K,u)$, are defined for $\mathcal{H}^{d-1}$ almost all $x \in \mathrm{bd}\,K$ resp. for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, cf. [6]. The set of all convex bodies which contain the origin in their interiors is denoted by $\mathcal{K}^d_{00}$, and σ_K is the spherical image map of K. This map is defined on the set $\mathrm{reg}\,K$ of regular boundary points of K. We shall also repeatedly use the important concept of the second order differentiability of a convex function, see [18, Notes for §1.5] and [6] for references.

Now we describe our main results. In Section 3 it will be shown that for an arbitrary convex body $K \in \mathcal{K}^d_{00}$ and for each $p > 0$ the relation

$$\mathcal{O}_p(K) = \mathcal{O}_{d^2/p}(K^*)$$

holds true (Theorem 3.2). Here,

$$K^* := \{x \in \mathbb{R}^d \mid \langle x,y\rangle \le 1 \quad \text{for all } y \in K\}$$

denotes the polar body of K. Strictly speaking, the attempt to verify this general result has been the original motivation for our current research. In case $p = d^2/p = d$ and for a convex body K of class C^3_+ this exactly is the classical statement. In order to be able to prove this, we shall first have to establish a relationship between the generalized Gauß-Kronecker curvature of an arbitrary convex body $K \in \mathcal{K}^d_{00}$ and the product of the generalized principal radii of curvature of its polar body K^* at corresponding points (Theorem 2.2). In addition, relation (1) is required. The proof of Theorem 2.2 essentially uses a representation of the generalized Gauß-Kronecker curvature by means of a differentiation process with respect to curvature measures [1], [17].

To see another application of Theorem 2.2, first recall that in equiaffine differential geometry the equiaffine support function of a convex body K at $x \in \operatorname{bd} K$ is defined by

$$H_{d-1}(K,x)^{-\frac{1}{d+1}} \langle x, \sigma_K(x) \rangle.$$

It was shown by Salkowski [15, §39, (31) and (32)] that, for $K \in \mathcal{K}^d_{00}$ of class C^3_+ and $d = 3$, the relation

$$\frac{\langle x, \sigma_K(x) \rangle}{H_{d-1}(K,x)^{1/d+1}} \, \frac{\langle x^*, \sigma_{K^*}(x^*) \rangle}{H_{d-1}(K^*,x^*)^{1/d+1}} = 1 \tag{2}$$

holds true, if $x^* \in \operatorname{bd} K^*$ is the uniquely determined boundary point for which $\langle x, x^* \rangle = 1$. Kaltenbach [8, Satz 3.2] proved relation (2) in arbitrary dimensions, and he applied this result to the random approximation of smooth convex bodies. See also [4, Remark 3] and [14, (7.33)]. In the present paper we obtain a generalization of equation (2). In fact, equation (2) is proved for an arbitrary convex body K and corresponding normal boundary points $x \in \operatorname{bd} K$ respectively $x^* \in \operatorname{bd} K^*$ (Theorem 2.8).

2 Curvature relations

Our first lemma is just a simple observation. It will, however, be essential for describing a geometric quantity, the normal cone $\mathrm{N}(K,x)$ of K at the boundary point x, with the help of analytic properties of the support function $h(K^*,\cdot)$ of the polar body. In this context $F(K^*,u)$ denotes the support set of K^* with exterior unit normal vector u, $\partial h_{K^*}(u)$ is the subdifferential of h_{K^*} at u, and by $\nabla h_{K^*}(u)$ we denote an arbitrary element (subgradient) of the subdifferential $\partial h_{K^*}(u)$. If $\partial h_{K^*}(u)$ consists of one subgradient only, then h_{K^*} is differentiable at u and $\nabla h_{K^*}(u) = \operatorname{grad} h_{K^*}(u)$.

Lemma 2.1. *Let $K \in \mathcal{K}^d_{00}$ and $u \in S^{d-1}$. Then $h(K^*,u)^{-1}u \in \operatorname{bd} K$, and*

$$\begin{aligned} \mathrm{N}(K, h(K^*,u)^{-1}u) \cap S^{d-1} &= \left\{ \frac{\nabla h_{K^*}(u)}{\|\nabla h_{K^*}(u)\|} \mid \nabla h_{K^*}(u) \in \partial h_{K^*}(u) \right\} \\ &= \left\{ \|x\|^{-1}x \mid x \in F(K^*,u) \right\}. \end{aligned}$$

Proof. Let $v \in \mathrm{N}(K, h(K^*,u)^{-1}u) \cap S^{d-1}$, i.e., $\langle v, h(K^*,u)^{-1}u \rangle = h(K,v)$. Equivalently, $\langle h(K,v)^{-1}v, u \rangle = h(K^*,u)$, and hence $h(K,v)^{-1}v \in F(K^*,u)$. This shows $v \in \{\|x\|^{-1}x \mid x \in F(K^*,u)\}$.

Vice versa, choose $x \in F(K^*, u)$. Since $h(K, x) = \rho(K^*, x)^{-1} = 1$, we obtain

$$\left\langle \frac{x}{\|x\|}, h(K^*, u)^{-1} u \right\rangle = \frac{1}{\|x\|} \frac{\langle x, u \rangle}{h(K^*, u)} = \|x\|^{-1} = h\left(K, \|x\|^{-1} x\right).$$

This implies $\|x\|^{-1} x \in \mathrm{N}(K, h(K^*, u)^{-1} u) \cap S^{d-1}$. The remaining part of the lemma follows from [18, Theorem 1.7.4]. $\square$

In the sequel the set of all normal boundary points of a convex body K will be denoted by $\mathcal{M}(K)$ (cf. [6]).

Theorem 2.2. *Let $K \in \mathcal{K}^d_{00}$. Then, for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$, the support function h_{K^*} is second order differentiable at $\|x\|^{-1} x$ and $x \in \mathcal{M}(K)$. For each such $x \in \operatorname{bd} K$ we have*

$$H_{d-1}(K, x) = \left\langle \|x\|^{-1} x, \sigma_K(x) \right\rangle^{d+1} D_{d-1} h(K^*, \|x\|^{-1} x). \tag{3}$$

Before we can start with the proof of Theorem 2.2, some definitions are required which will be used throughout the present paper. A sequence of lemmas then will prepare the main part of the proof.

Fix $K \in \mathcal{K}^d_{00}$, and let $x_0 \in \mathcal{M}(K)$ be such that $h := h_{K^*}$ is second order differentiable at $u_0 := \|x_0\|^{-1} x_0$. It will be convenient to write

$$x^* := \langle x, \sigma_K(x) \rangle^{-1} \sigma_K(x) \in \operatorname{bd} K^*,$$

if $x \in \operatorname{reg} K$. This defines a mapping $^* : \operatorname{reg} K \to \operatorname{bd} K^*$. In addition, we shall use mappings

$$f : S^{d-1} \to \operatorname{bd} K, \quad u \mapsto h(u)^{-1} u,$$

and

$$\eta : \operatorname{bd} K \to S^{d-1}, \quad x \mapsto \|x\|^{-1} x.$$

These are bi-Lipschitz homeomorphisms with Lipschitz constants $\operatorname{Lip}(f)$ resp. $\operatorname{Lip}(\eta)$, which are inverse to each other. For $n \in \mathbb{N}$ let

$$\omega_n := \left\{ u \in S^{d-1} \mid \langle u, u_0 \rangle \geq 1 - (2n^2)^{-1} \right\}.$$

Let us choose $n_0 \in \mathbb{N}$ such that $\nabla h(u_0) + d^2 h(u_0)(u - u_0) \neq o$ for all $u \in \omega_n$ and $n \geq n_0$. If $\omega \subset \omega_{n_0}$, we can define

$$\begin{aligned} M_1(\omega) &:= \left\{ \|\nabla h(u)\|^{-1} \nabla h(u) \mid u \in \omega,\ \nabla h(u) \in \partial h(u) \right\} \\ &= \left\{ \|x\|^{-1} x \mid x \in F(K^*, u),\ u \in \omega \right\}, \end{aligned}$$

and

$$M_2(\omega) := \left\{ \frac{\nabla h(u_0) + d^2 h(u_0)(u - u_0)}{\|\nabla h(u_0) + d^2 h(u_0)(u - u_0)\|} \mid u \in \omega \right\}.$$

We also need the mappings

$$p : \operatorname{bd} K \to x_0 + T_{x_0}K, \quad q \mapsto p(x_0 + T_{x_0}K, q),$$

and

$$F : (x_0 + T_{x_0}K) \cap U \to \operatorname{bd} K, \quad x \mapsto x + z(x)e_d.$$

Here, p is the metric projection of the tangent space $x_0 + T_{x_0}K$ of K at x_0, $e_d := -\sigma_K(x_0)$, U is a suitably small neighbourhood of x_0, and z is the function which locally represents $\operatorname{bd} K$ at x_0 with respect to the tangent space $x_0 + T_{x_0}K$, i.e.,

$$z(x) = \min\{\lambda \geq 0 \mid x + \lambda e_d \in \operatorname{bd} K\}, \quad x \in (x_0 + T_{x_0}K) \cap U.$$

Note that p and F are bi-Lipschitz maps with Lipschitz constants $\operatorname{Lip}(p)$ resp. $\operatorname{Lip}(F)$, which locally are inverse to each other. For $\epsilon > 0$ and $A \subset \mathbb{R}^d$ let

$$U_\epsilon(A) := \left\{x \in \mathbb{R}^d \mid \inf\{\|x - a\| \mid a \in A\} \leq \epsilon\right\}.$$

The set $\beta_n := f(\omega_n) \subset \operatorname{bd} K$ represents a compact neighbourhood of x_0 relative to $\operatorname{bd} K$ for each $n \in \mathbb{N}$. We can assume that the sets $\beta_n' := p \circ f(\omega_n)$ and $U_{\epsilon_n}(\beta_n')$ are contained in U for $n \geq n_0$, if $(\epsilon_i)_{i\in\mathbb{N}}$ is a given positive sequence in $\mathbb{R}$ that converges to 0. Define

$$\|\beta_n'\| := \sup\{\|x - x_0\| \mid x \in \beta_n'\}.$$

Finally, the notion of a normal sequence of subsets of $\operatorname{bd} K$ with respect to a point $x_0 \in \operatorname{bd} K$ is defined in [17, §3].

Lemma 2.3. *The sequence $(\beta_n)_{n\geq n_0}$ is normal with respect to x_0.*

Proof. Let $u \in \omega_n$. Then $\|u - u_0\| \leq n^{-1}$, i.e.,

$$\|\beta_n'\| \leq \operatorname{Lip}(f)n^{-1} \to 0 \quad \text{for} \quad n \to \infty.$$

Now, choose a sequence $(\epsilon_i)_{i\in\mathbb{N}}$, $\epsilon_i > 0$, such that $\|\beta_i'\|^{-1}\epsilon_i \to 0$ for $i \to \infty$. This implies $n\epsilon_n \to 0$ for $n \to \infty$. We have to show that

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}\Big(U_{\epsilon_n}(\operatorname{bd} \beta_n') \cap (x_0 + T_{x_0}K)\Big)}{\mathcal{H}^{d-1}(\beta_n')} = 0.$$

Here and in the sequel "bd" is occasionally applied relative to $x_0 + T_{x_0}K$ or to S^{d-1}, as will be clear from the context.

First of all we have

$$\mathcal{H}^{d-1}(\beta_n') \geq (\operatorname{Lip}(\eta)\operatorname{Lip}(F))^{1-d}\mathcal{H}^{d-1}(\omega_n). \tag{4}$$

Next we prove the inclusion

$$U_{\epsilon_n}(\operatorname{bd} \beta_n') \cap (x_0 + T_{x_0}K) \subset p \circ f\left(U_{\epsilon_n \operatorname{Lip}(F)\operatorname{Lip}(\eta)}(\operatorname{bd} \omega_n) \cap S^{d-1}\right). \tag{5}$$

To see this, choose $x \in U_{\epsilon_n}(\mathrm{bd}\,\beta'_n) \cap (x_0 + T_{x_0}K)$. Then there is some $y \in \mathrm{bd}\,\beta'_n$ with $\|x - y\| \le \epsilon_n$. Since $\eta \circ F(y) \in \mathrm{bd}\,\omega_n$ and

$$\|\eta \circ F(x) - \eta \circ F(y)\| \le \mathrm{Lip}(\eta)\mathrm{Lip}(F)\epsilon_n,$$

it follows that

$$\eta \circ F(x) \in U_{\epsilon_n \mathrm{Lip}(F)\mathrm{Lip}(\eta)}(\mathrm{bd}\,\omega_n).$$

From (4) and (5) one deduces

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\mathcal{H}^{d-1}\Big(U_{\epsilon_n}(\mathrm{bd}\,\beta'_n) \cap (x_0 + T_{x_0}K)\Big)}{\mathcal{H}^{d-1}(\beta'_n)} \\
&\quad \le \limsup_{n\to\infty} \frac{(\mathrm{Lip}(p)\mathrm{Lip}(f))^{d-1}\mathcal{H}^{d-1}\Big(U_{\epsilon_n \mathrm{Lip}(F)\mathrm{Lip}(\eta)}(\mathrm{bd}\,\omega_n) \cap S^{d-1}\Big)}{(\mathrm{Lip}(\eta)\mathrm{Lip}(F))^{1-d}\mathcal{H}^{d-1}(\omega_n)} \\
&\quad \le \mathit{const}\, \limsup_{n\to\infty}(n\epsilon_n) = 0.
\end{aligned}$$

A detailed justification of the last inequality can be found in [7]. □

Lemma 2.4. *With the notations from above we have*

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(f(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} = \frac{\|\nabla h(u_0)\|}{h(u_0)^d}.$$

Proof. The function h is differentiable at u for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. For one such u let $u_1, \ldots, u_{d-1}$ be an orthonormal basis of $u^\perp$. Then one calculates for the partial derivatives $f_{|j}(u)$, $j = 1, \ldots, d-1$, with respect to $u_1, \ldots, u_{d-1}$,

$$f_{|j}(u) = \frac{u_j}{h(u)} - \frac{u}{h(u)^2}\langle \nabla h(u), u_j \rangle.$$

From this we conclude

$$\begin{aligned}
&\det\Big(\langle f_{|i}(u), f_{|j}(u)\rangle_{i,j=1}^{d-1}\Big) \\
&\quad = h(u)^{-2(d-1)} \left(1 + \sum_{i=1}^{d-1} \left\langle \frac{\nabla h(u)}{h(u)}, u_i \right\rangle^2\right) \\
&\quad = h(u)^{-2d} \left(\langle \nabla h(u), u\rangle^2 + \sum_{i=1}^{d-1} \langle \nabla h(u), u_i\rangle^2\right) \\
&\quad = \frac{\|\nabla h(u)\|^2}{h(u)^{2d}}.
\end{aligned} \tag{6}$$

A special case of the area/coarea formula [3, Theorem 3.2.22] applied to the Lipschitz map f then yields

$$\mathcal{H}^{d-1}(f(\omega_n)) = \int_{\omega_n} \frac{\|\nabla h(u)\|}{h(u)^d}\, \mathrm{d}\mathcal{H}^{d-1}(u).$$

Since h is second order differentiable at u_0, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{H}^{d-1}(f(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} - \frac{\|\nabla h(u_0)\|}{h(u_0)^d}\right| \\
&\quad \le \frac{1}{\mathcal{H}^{d-1}(\omega_n)} \int_{\omega_n} \left|\frac{\|\nabla h(u)\|}{h(u)^d} - \frac{\|\nabla h(u_0)\|}{h(u_0)^d}\right| \,\mathrm{d}\mathcal{H}^{d-1}(u) \\
&\quad \le \frac{1}{\mathcal{H}^{d-1}(\omega_n)} \int_{\omega_n} \left(\frac{\|\nabla h(u)\|}{h(u)^d h(u_0)^d}|h(u_0)^d - h(u)^d| + \right. \\
&\qquad\qquad \left. + \frac{1}{h(u_0)^d}\Big|\|\nabla h(u)\| - \|\nabla h(u_0)\|\Big|\right) \,\mathrm{d}\mathcal{H}^{d-1}(u).
\end{aligned}$$

Observe that

$$\|\nabla h(u)\| \le \|\nabla h(u_0)\| + \|d^2 h(u_0)\|\|u - u_0\| + R(\|u - u_0\|)\|u - u_0\|,$$

$$\Big|\|\nabla h(u)\| - \|\nabla h(u_0)\|\Big| \le (\|d^2 h(u_0)\| + R(\|u - u_0\|))\|u - u_0\|,$$

and

$$|h(u_0)^d - h(u)^d| \le \mathit{const}\,\mathrm{Lip}(h)\|u - u_0\|.$$

Therefore the lemma follows from

$$\begin{aligned}
&\left|\frac{\mathcal{H}^{d-1}(f(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} - \frac{\|\nabla h(u_0)\|}{h(u_0)^d}\right| \\
&\quad \le \frac{\mathit{const}}{\mathcal{H}^{d-1}(\omega_n)} \int_{\omega_n} \|u - u_0\| \,\mathrm{d}\mathcal{H}^{d-1}(u) \le \mathit{const}\, n^{-1}.
\end{aligned}$$

□

Lemma 2.5. *Using the notations from above, we also get*

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_2(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} = \frac{h(u_0)}{\|\nabla h(u_0)\|^d}\, D_{d-1} h(u_0).$$

Proof. Let us define the map g by

$$g : \omega_{n_0} \to \mathbb{R}^d, \quad u \mapsto \frac{\nabla h(u_0) + d^2 h(u_0)(u - u_0)}{\|\nabla h(u_0) + d^2 h(u_0)(u - u_0)\|}.$$

Again the area formula (for the smooth map g) yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_2(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} &= J_{d-1} g(u_0) \\
&= \frac{\det\left(\langle g_{|i}(u_0), f_{|j}(u_0)\rangle_{i,j=1}^{d-1}\right)}{\sqrt{\det\left(\langle f_{|i}(u_0), f_{|j}(u_0)\rangle_{i,j=1}^{d-1}\right)}}. \qquad (7)
\end{aligned}$$

In order to obtain the last equation we have used that

$$\operatorname{im}(Dg(u_0)) \subset (\nabla h(u_0))^{\perp} = \operatorname{im}(Df(u_0)).$$

Here, the partial derivatives $f_{|i}(u_0)$, $g_{|j}(u_0)$, $i,j = 1,\dots,d-1$, are calculated with respect to an orthonormal basis $u_1,\dots,u_{d-1} \in u_0^{\perp}$ of eigenvectors of the symmetric linear transformation $d^2h(u_0)|_{u_0^{\perp}} : u_0^{\perp} \to u_0^{\perp}$. The corresponding eigenvalues, the generalized principal radii of curvature of K^* at u_0 (cf. [6]), are denoted by $r_1,\dots,r_{d-1}$. In other words $D_{d-1}h(u_0) = r_1\cdots r_{d-1}$. An elementary calculation leads to

$$\begin{aligned} g_{|i}(u_0) &= \frac{d^2h(u_0)(u_i)}{\|\nabla h(u_0)\|} - \frac{\nabla h(u_0)}{\|\nabla h(u_0)\|^3}\langle \nabla h(u_0), d^2h(u_0)(u_i)\rangle \\ &= \frac{r_i}{\|\nabla h(u_0)\|}\left\{u_i - \frac{\nabla h(u_0)}{\|\nabla h(u_0)\|^2}\langle \nabla h(u_0), u_i\rangle\right\}, \end{aligned}$$

and hence

$$\begin{aligned} &\langle g_{|i}(u_0), f_{|j}(u_0)\rangle \\ &\quad = \delta_{ij}\,\frac{r_i}{h(u_0)\|\nabla h(u_0)\|} - r_i\,\frac{\langle \nabla h(u_0), u_i\rangle\langle \nabla h(u_0), u_j\rangle}{h(u_0)\|\nabla h(u_0)\|^3} + \\ &\qquad\qquad + r_i\,\frac{\langle \nabla h(u_0), u_0\rangle}{h(u_0)^2}\,\frac{\langle \nabla h(u_0), u_i\rangle\langle \nabla h(u_0), u_j\rangle}{\|\nabla h(u_0)\|^3} \\ &\quad = \delta_{ij}\,\frac{r_i}{h(u_0)\|\nabla h(u_0)\|}. \end{aligned} \tag{8}$$

Now the lemma follows from (6), (7), and (8). □

Lemma 2.6. *The inclusion* $\operatorname{bd} M_1(\omega_n) \subset M_1(\operatorname{bd}\omega_n)$ *holds for all* $n \geq n_0$.

Proof. Let $v \in \operatorname{bd} M_1(\omega_n)$. By definition of the boundary of a set there is a sequence $v_i \in M_1(\omega_n)$, $i \in \mathbb{N}$, such that $v_i \to v$ for $i \to \infty$. Thus for each $i \in \mathbb{N}$ there are $u_i \in \omega_n$ and $x_i \in F(K^*, u_i)$ with $\|x_i\|^{-1}x_i = v_i$. After suitable selection of subsequences and change of notation we can assume that $u_i \to u \in S^{d-1}$, $x_i \to x \in F(K^*, u)$, and $v = \|x\|^{-1}x$. Since ω_n is closed, we have $u \in \omega_n$.

Again by definition there is also a sequence $w_i \in S^{d-1} \setminus M_1(\omega_n)$, $i \in \mathbb{N}$, with $w_i \to v$ for $i \to \infty$. For each $i \in \mathbb{N}$ set $y_i := \rho(K^*, w_i)w_i \in \operatorname{bd} K^*$, and choose $\tilde{u}_i \in \mathrm{N}(K^*, y_i) \cap S^{d-1}$. Then $y_i \in F(K^*, \tilde{u}_i)$, $\|y_i\|^{-1}y_i = w_i$, and $\tilde{u}_i \notin \omega_n$ (since otherwise it follows that $w_i \in M_1(\omega_n)$). Taking subsequences we obtain $\tilde{u} \in (S^{d-1} \setminus \omega_n) \cup \operatorname{bd}\omega_n$ and $y \in F(K^*, \tilde{u})$ with $v = \|y\|^{-1}y$. But then $y = x$. The proof is finished, if $\tilde{u} \in \operatorname{bd}\omega_n$. In case that $\tilde{u} \in S^{d-1} \setminus \omega_n$, we have $u, \tilde{u} \in \mathrm{N}(K^*, x)$, $u \in \omega_n$, and $\tilde{u} \in S^{d-1} \setminus \omega_n$. Then, however, there also is some $u^* \in \mathrm{N}(K^*, x)$ which lies on the spherical arc joining u and $\tilde{u}$ such that $u^* \in \operatorname{bd}\omega_n$. This implies $v \in M_1(\operatorname{bd}\omega_n)$. □

Lemma 2.7. *Let $d^2h(u_0)|_{u_0^\perp}$ be regular. Then we obtain*

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(M_2(\omega_n))} = 1.$$

Proof. There is some $M > 0$ such that $\|\nabla h(u)\|^{-1} \le M$ for all $u \in S^{d-1}$ and $\nabla h(u) \in \partial h(u)$. Since h is second order differentiable at u_0, there also is an increasing function R such that $R(\|u - u_0\|) \to 0$ for $u \to u_0$ and

$$\|\nabla h(u) - \nabla h(u_0) - d^2h(u_0)(u - u_0)\| \le R(\|u - u_0\|)\|u - u_0\|,$$

for any subgradient choice $\nabla h(u) \in \partial h(u)$, $u \in S^{d-1}$, for the support function h, cf. [18, Notes for §1.5].

Set $\epsilon_n := 2MR(n^{-1})n^{-1}$ for $n \ge n_0$. The following estimate holds for $n \ge n_0$, $u \in \omega_n$, and $\nabla h(u) \in \partial h(u)$:

$$\begin{aligned}
&\left\| \frac{\nabla h(u)}{\|\nabla h(u)\|} - \frac{\nabla h(u_0) + d^2h(u_0)(u - u_0)}{\|\nabla h(u_0) + d^2h(u_0)(u - u_0)\|} \right\| \\
&\quad \le \left\| \frac{\nabla h(u) - \nabla h(u_0) - d^2h(u_0)(u - u_0)}{\|\nabla h(u)\|} \right\| + \\
&\qquad\qquad + \frac{\Big| \|\nabla h(u_0) + d^2h(u_0)(u - u_0)\| - \|\nabla h(u)\| \Big|}{\|\nabla h(u)\|} \\
&\quad \le 2MR(\|u - u_0\|)\|u - u_0\| \\
&\quad \le 2MR(n^{-1})n^{-1} = \epsilon_n.
\end{aligned}$$

This implies for $n \ge n_0$ and $\tilde{\omega} \subset \omega_n$

$$M_1(\tilde{\omega}) \subset U_{\epsilon_n}(M_2(\tilde{\omega})) \quad \text{and} \quad M_2(\tilde{\omega}) \subset U_{\epsilon_n}(M_1(\tilde{\omega})), \tag{9}$$

and hence in particular

$$M_2(\omega_n) \setminus M_1(\omega_n) \subset U_{\epsilon_n}(\operatorname{bd} M_1(\omega_n)), \tag{10}$$

as well as

$$M_1(\omega_n) \setminus M_2(\omega_n) \subset U_{\epsilon_n}(\operatorname{bd} M_2(\omega_n)). \tag{11}$$

Relation (9) with $\tilde{\omega} = \operatorname{bd}\omega_n$ and the fact that $g|_{\omega_n}$ is a diffeomorphism, if n is sufficiently large, lead to

$$M_1(\operatorname{bd}\omega_n) \subset U_{\epsilon_n}(M_2(\operatorname{bd}\omega_n)) = U_{\epsilon_n}(\operatorname{bd} M_2(\omega_n)). \tag{12}$$

Now, use (10), Lemma 2.6, and (12) to see that

$$M_2(\omega_n) \setminus M_1(\omega_n) \subset U_{2\epsilon_n}(\operatorname{bd} M_2(\omega_n)). \tag{13}$$

With the help of (11) and (13) it follows that for n large enough

$$\begin{aligned}
&\left|\frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(M_2(\omega_n))}-1\right| \\
&\qquad \le \frac{\mathcal{H}^{d-1}(M_1(\omega_n)\setminus M_2(\omega_n))}{\mathcal{H}^{d-1}(M_2(\omega_n))}+\frac{\mathcal{H}^{d-1}(M_2(\omega_n)\setminus M_1(\omega_n))}{\mathcal{H}^{d-1}(M_2(\omega_n))} \\
&\qquad \le 2\,\frac{\mathcal{H}^{d-1}\Big(U_{2\epsilon_n}(\operatorname{bd} M_2(\omega_n))\cap S^{d-1}\Big)}{\mathcal{H}^{d-1}(M_2(\omega_n))}. \qquad (14)
\end{aligned}$$

For the sake of clarity we write g_n for $g|_{\omega_n}$, $n \ge n_0$. Recall that g_n is a bi-Lipschitz homeomorphism for n sufficiently large. Hence, we can assume that for n_0 sufficiently large there is some $n_1 \in \mathbb{N}$ with $n_0 \ge n_1$ such that for $n \ge n_0$

$$U_{2\epsilon_n}(g_n(\operatorname{bd}\omega_n))\cap S^{d-1}\subset \operatorname{im}(g_{n_1})$$

and

$$U_{2\mathrm{Lip}(g_{n_1}^{-1})\epsilon_n}(\operatorname{bd}\omega_n)\cap S^{d-1}\subset \omega_{n_1}.$$

In addition, it can be assumed that g_{n_1} is a bi-Lipschitz map. Note that $\operatorname{bd} M_2(\omega_n) = g_n(\operatorname{bd}\omega_n)$ for $n \ge n_0$. We state that for $n \ge n_0$

$$U_{2\epsilon_n}(g_n(\operatorname{bd}\omega_n))\cap S^{d-1}\subset g_{n_1}\left(U_{2\mathrm{Lip}(g_{n_1}^{-1})\epsilon_n}(\operatorname{bd}\omega_n)\cap S^{d-1}\right). \qquad (15)$$

To see this, choose $x \in U_{2\epsilon_n}(g_n(\operatorname{bd}\omega_n))\cap S^{d-1}$. Then there is some $u \in \operatorname{bd}\omega_n$ such that $\|x-g_n(u)\| \le 2\epsilon_n$. Note that $x \in \operatorname{im}(g_{n_1})$, and $g_n(u) = g_{n_1}(u)$ for $u \in \omega_n$ and $n \ge n_0$. Therefore

$$\|g_{n_1}^{-1}(x)-u\| \le 2\mathrm{Lip}(g_{n_1}^{-1})\epsilon_n,$$

and this implies relation (15).

Finally, observe that for $n \ge n_0$

$$\mathcal{H}^{d-1}(M_2(\omega_n))^{-1} = \mathcal{H}^{d-1}(g_{n_1}(\omega_n))^{-1} \le (\mathrm{Lip}(g_{n_1}^{-1}))^{d-1}\mathcal{H}^{d-1}(\omega_n)^{-1}. \qquad (16)$$

Introducing (15) and (16) into (14), we end up with

$$\begin{aligned}
&\left|\frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(M_2(\omega_n))}-1\right| \\
&\qquad \le 2\,[\mathrm{Lip}(g_{n_1}^{-1})\mathrm{Lip}(g_{n_1})]^{d-1}\frac{\mathcal{H}^{d-1}\Big(U_{2\mathrm{Lip}(g_{n_1}^{-1})\epsilon_n}(\operatorname{bd}\omega_n)\cap S^{d-1}\Big)}{\mathcal{H}^{d-1}(\omega_n)} \\
&\qquad \le \mathit{const}\,(n\epsilon_n).
\end{aligned}$$

$\square$

Proof of Theorem 2.2. We have to distinguish two cases. First, let us assume that $d^2h(u_0)|_{u_0^\perp}$ is regular. Then we obtain successively from [17, Hilfssatz (3.6)], which is applicable because of Lemma 2.3, from [18, Theorem 4.2.5, (4.2.21) and (4.2.23)], Lemma 2.1, Lemma 2.4 and Lemma 2.7, and finally by an application of Lemma 2.5 that

$$\begin{aligned}
H_{d-1}(K,x_0) &= \lim_{n\to\infty} \frac{C_0(K,\beta_n)}{C_{d-1}(K,\beta_n)} \\
&= \lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(\sigma(K,f(\omega_n)))}{\mathcal{H}^{d-1}(f(\omega_n))} \\
&= \lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(f(\omega_n))} \\
&= \frac{h(u_0)^d}{\|\nabla h(u_0)\|} \lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_2(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} \\
&= \left(\frac{h(u_0)}{\|\nabla h(u_0)\|}\right)^{d+1} D_{d-1}h(u_0).
\end{aligned}$$

See [18] for a definition of the spherical image $\sigma(K,\cdot)$. Since $x_0 \in \mathcal{M}(K) \subset \operatorname{reg} K$, the associated point $x_0^* \in \operatorname{bd} K^*$ has already been defined. The definition of x_0^* implies $\langle x_0, x_0^*\rangle = 1$, and thus $u_0 = \|x_0\|^{-1}x_0 \in \mathrm{N}(K^*, x_0^*) \cap S^{d-1}$. Moreover, $\nabla h(u_0) = x_0^* = \langle x_0, \sigma_K(x_0)\rangle^{-1}\sigma_K(x_0)$, and hence

$$\|\nabla h(u_0)\|^{-1} h(u_0) = \langle \|x_0^*\|^{-1}x_0^*, u_0\rangle = \langle \sigma_K(x_0), \|x_0\|^{-1}x_0\rangle.$$

Now, assume that $d^2h(u_0)|_{u_0^\perp}$ is singular, i.e., $D_{d-1}h(u_0) = 0$. The proof is finished, if we can show that $H_{d-1}(K,x_0) = 0$. The preceding argument for the regular case tells us that it is sufficient to verify

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} = 0.$$

In the course of the proof of Lemma 2.7 it has been shown that $M_1(\omega_n) \subset U_{\epsilon_n}(M_2(\omega_n))$ for n sufficiently large. Note that we did not use the nondegeneracy of $d^2h(u_0)\,|_{u_0^\perp}$ in order to deduce this fact. Hence,

$$\frac{\mathcal{H}^{d-1}(M_1(\omega_n))}{\mathcal{H}^{d-1}(\omega_n)} \le \frac{\mathcal{H}^{d-1}\Big(U_{\epsilon_n}(M_2(\omega_n)) \cap S^{d-1}\Big)}{\mathcal{H}^{d-1}(\omega_n)}.$$

According to the assumption of the present case the set

$$M_n := \Big\{\nabla h(u_0) + d^2h(u_0)(u-u_0) \mid u \in \omega_n\Big\} \not\ni o \quad (\text{for } n \ge n_0)$$

is contained in a $(d-2)$-dimensional affine subspace, and

$$\operatorname{diam}(M_n) \le \|d^2h(u_0)\| \operatorname{diam}(\omega_n) \le const\, n^{-1}.$$

Thus we obtain

$$\lim_{n\to\infty} \frac{\mathcal{H}^{d-1}\Big(U_{\epsilon_n}(M_2(\omega_n)) \cap S^{d-1}\Big)}{\mathcal{H}^{d-1}(\omega_n)} = 0,$$

since $n\epsilon_n \to 0$ for $n \to \infty$.

The first statement of Theorem 2.2 is an immediate consequence of Aleksandrov's theorem on the second order differentiability of a convex function almost everywhere [1] and the fact that f is Lipschitzian. $\square$

Remark. Under the assumptions of Theorem 2.2 let us denote by $N^f : S^{d-1} \to S^{d-1}$ an arbitrary unit (normal) vector field which fulfills

$$N^f(u) \in \mathrm{N}(K, f(u)) \cap S^{d-1}.$$

The proof for Theorem 2.2 then implies that

$$H_{d-1}(K, x_0) = \frac{\det\left(\langle N^f_{|i}(u_0), f_{|j}(u_0)\rangle_{i,j=1}^{d-1}\right)}{\det\left(\langle f_{|i}(u_0), f_{|j}(u_0)\rangle_{i,j=1}^{d-1}\right)},$$

independent of the particular choice for N^f. Here, the derivatives are calculated with respect to an arbitrary orthonormal basis of $u_0^\perp$. The expression on the right side of this equation resembles the usual representation of the Gauß-Kronecker curvature of a hypersurface in classical differential geometry.

Theorem 2.8. *Let $K \in \mathcal{K}^d_{00}$. If $x_0 \in \mathcal{M}(K)$ and $x_0^* \in \mathcal{M}(K^*)$, then*

$$H_{d-1}(K, x_0) H_{d-1}(K^*, x_0^*) = (\|x_0\| \, \|x_0^*\|)^{-(d+1)}.$$

Proof. First of all it will be proved that $h := h_{K^*}$ is second order differentiable at $u_0 := \|x_0\|^{-1}x_0$. Note that h is differentiable at u_0, since $h = \rho(K, \cdot)^{-1}$ and $x_0 \in \operatorname{reg} K$. Again let U be a suitably small neighbourhood of x_0, and set $T_{x_0} := (x_0 + T_{x_0}K) \cap U$. The mappings f, η, p, F are also defined as in the course of the proof for Theorem 2.2. Note that the map

$$p \circ f : (p \circ f)^{-1}(T_{x_0}) \to x_0 + T_{x_0}K$$

is differentiable at u_0, since $x_0 \in \operatorname{reg} K$. A similar statement holds for $\eta \circ F$. Now, we say that N is a unit normal vector field choice for K at x_0 defined on T_{x_0}, if N fulfills

$$N : T_{x_0} \to S^{d-1}, \quad N(x) \in \mathrm{N}(K, F(x)) \cap S^{d-1} \quad \text{for all } x \in T_{x_0}.$$

If N is such a unit normal vector field choice, then

$$N_s := N \circ p \circ f : (p \circ f)^{-1}(T_{x_0}) \to S^{d-1}$$

is a unit normal vector field, which fulfills (cf. Lemma 2.1), for $u \in (p \circ f)^{-1}(T_{x_0})$,

$$N_s(u) = N \circ p \circ f(u) = \frac{\nabla h(u)}{\|\nabla h(u)\|}, \quad \nabla h(u) \in \partial h(u).$$

Vice versa, starting with any such unit normal vector field N_s, we obtain a unit normal vector field choice N for K at x_0 defined on T_{x_0} by setting $N := N_s \circ \eta \circ F$. Thus a remark in [2, §4, p. 318] implies that $x_0 \in \mathcal{M}(K)$ is equivalent to the uniform differentiability of all normed subgradient choices for $h|_{S^{d-1}}$ at u_0. Since $\nabla h(u) \in F(K^*, u) \subset \operatorname{bd} K^*$, we have

$$\nabla h(u) = \rho(K^*, \|\nabla h(u)\|^{-1} \nabla h(u)) \frac{\nabla h(u)}{\|\nabla h(u)\|}.$$

The radial function $\rho(K^*, \cdot) : S^{d-1} \to \mathbb{R}$ is differentiable at $\|x_0^*\|^{-1} x_0^*$, because $x_0^* \in \operatorname{reg} K^*$. We also have $\|\nabla h(u_0)\|^{-1} \nabla h(u_0) = \|x_0^*\|^{-1} x_0^*$, cf. the proof of Theorem 2.2. Hence, the family of the composed mappings

$$S^{d-1} \to \mathbb{R}, \quad u \mapsto \rho(K^*, \|\nabla h(u)\|^{-1} \nabla h(u)),$$

which is obtained for all subgradient choices for h, is uniformly differentiable at u_0. The rule for the differentiation of the product of two maps generalizes to the setting of uniform differentiability. This shows that the family of subgradient choices for h is uniformly differentiable, and thus h is second order differentiable at u_0. Observe that $x_0^* = \nabla h(u_0)$. Now, we obtain from [6, Lemma 2.5] that

$$H_{d-1}(K^*, x_0^*) D_{d-1} h(u_0) = 1. \tag{17}$$

In addition, Theorem 2.2 implies

$$H_{d-1}(K, x_0) = (\|x_0\| \|x_0^*\|)^{-(d+1)} \, D_{d-1} h(u_0). \tag{18}$$

Thus Theorem 2.8 follows from (17) and (18). □

Remarks.

1. Note that if K is a polytope, then there is no $x_0 \in \mathcal{M}(K)$ such that $x_0^* \in \mathcal{M}(K^*)$. In fact, this situation is generic, as can be seen from [18, Theorem 2.6.2] together with our Theorem 2.8.

2. The preceding proof shows that h_{K^*} is second order differentiable at $\|x\|^{-1} x$ and hence that equation (3) holds true, if we merely assume $x \in \mathcal{M}(K)$ and $x^* \in \operatorname{reg} K^*$. (This is, e.g., implied by $x \in \mathcal{M}(K)$ and $H_{d-1}(K, x) > 0$.) However, this is not a necessary condition. On the other hand, we do not know whether the assumption $x \in \mathcal{M}(K)$ is sufficient in general. Theorem 2.2 is also implied, if h_{K^*} is second order differentiable at $x \in \operatorname{bd} K$. In fact, this already implies $x \in \operatorname{reg} K$, and from the proof of Theorem 2.8 we hence infer that $x \in \mathcal{M}(K)$. Thus we have proved Corollary 2.9.

Corollary 2.9. *Let $K \in \mathcal{K}_{00}^d$. If h_{K^*} is second order differentiable at $x \in \operatorname{bd} K$, then $x \in \mathcal{M}(K)$, and equation* (3) *holds true.*

In a differential geometric setting the next two statements have been considered in [14, Proposition 8.4 and Theorem 9.1].

Corollary 2.10. *Let $K \in \mathcal{K}_{00}^d$. Then the inequality $H_{d-1}(K, x) H_{d-1}(K^*, x^*) \leq 1$ holds true for all $x \in \mathcal{M}(K)$ such that $x^* \in \mathcal{M}(K^*)$.*

Corollary 2.11. *Let $K \in \mathcal{K}^d_{00}$. If there is some positive constant c such that $x^* \in \mathcal{M}(K^*)$ and $H_{d-1}(K,x)H_{d-1}(K^*,x^*) = c$ for $\mathcal{H}^{d-1}$ almost all $x \in \mathcal{M}(K)$, then K is a centred ball.*

Proof. The assumptions of the corollary and Theorem 2.8 imply

$$H_{d-1}(K,x)H_{d-1}(K^*,x^*) = (\|x\|\,\|x^*\|)^{-(d+1)} = c,$$

for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$. From this we obtain $\langle \|x\|^{-1}x, \sigma_K(x)\rangle = c^{1/d+1}$, first for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$, and then, by continuity, this extends to all $x \in \operatorname{reg} K$. Consider a point $x_0 \in \operatorname{bd} K$ such that $\|x_0\| = \min\{\|x\| \mid x \in \operatorname{bd} K\}$. Hence, $x_0 \in \operatorname{reg} K$. This shows

$$1 = \langle \|x_0\|^{-1}x_0, \sigma_K(x_0)\rangle = c^{1/d+1},$$

i.e., $c = 1$. Let $x \in \operatorname{bd} K$. An easy variational argument and Fubini's theorem show that there is a Lipschitzian map $x : [0,1] \to \operatorname{bd} K$ with $x(0) = x_0$, $x(1) = x$ and such that $x(t) \in \operatorname{reg} K$ for $\mathcal{H}^1$ almost all $t \in [0,1]$. Consider the Lipschitzian mapping $f : [0,1] \to \mathbb{R}$, $t \mapsto \|x(t)\|^2$. Then, for $\mathcal{H}^1$ almost all $t \in [0,1]$,

$$\begin{aligned} f'(t) &= 2\langle x(t), x'(t)\rangle \\ &= 2\|x(t)\|\langle \sigma_K(x(t)), x'(t)\rangle = 0. \end{aligned}$$

This shows that f is constant, and hence K is a centred sphere. □

3 Affine surface area of polar pairs

The aim of this final section is to prove that dual (polar) pairs of convex bodies have equal affine surface area with respect to "dual" indices. This in particular underscores the special role of centroaffine surface area at the turning point of equal indices $p = d^2/p = d$. There is also a local version of our subsequent Theorem 3.2 in the sense of [6, Theorem 2.8], which, however, we shall not state explicitly.

The following lemma seems to be generally known, although we could not find a reference for a proof. For reasons of completeness we include the short argument.

Lemma 3.1. *Let $K \in \mathcal{K}^d_{00}$, and let $f : S^{d-1} \to \operatorname{bd} K$, $u \mapsto \rho(K,u)u$. Then f is differentiable for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, and for any such $u \in S^{d-1}$*

$$J_{d-1}f(u) = \langle u, \sigma_K(\rho(K,u)u)\rangle^{-1}\rho(K,u)^{d-1}.$$

Proof. The differentiability statement is clear, since $\rho_K^{-1} = h := h_{K^*}$ is Lipschitzian. Let the radial function $\rho_K = \rho(K,\cdot)$ be differentiable at $u \in S^{d-1}$. Equation (6) in the proof of Lemma 2.4 shows that

$$J_{d-1}f(u) = \frac{\|\nabla h(u)\|}{h(u)^d} = \frac{\|\nabla h(u)\|}{\langle \nabla h(u), u\rangle}\,\rho(K,u)^{d-1}.$$

This proves the lemma, since $\|\nabla h(u)\|^{-1}\nabla h(u) = \sigma_K(\rho(K,u)u)$. □

Theorem 3.2. *Let $K \in \mathcal{K}^d_{00}$ and $p > 0$. Then $\mathcal{O}_p(K) = \mathcal{O}_{d^2/p}(K^*)$.*

Proof. We only consider the case $p = d$. Apart from a more intricate notation, the argument is the same for general $p > 0$. Let us consider the bi-Lipschitz transformation

$$f : S^{d-1} \to \operatorname{bd} K, \quad u \mapsto h(K^*, u)^{-1} u = \rho(K, u) u.$$

According to Lemma 3.1 the Jacobian $J_{d-1} f(u)$ exists and is positive for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. Hence, the area formula [3, Theorem 3.2.22] yields for an arbitrary nonnegative map $g : S^{d-1} \to \mathbb{R}$, which is defined $\mathcal{H}^{d-1}$ almost everywhere on S^{d-1},

$$\int_{S^{d-1}} g(u) J_{d-1} f(u) \, \mathrm{d}\mathcal{H}^{d-1}(u) = \int_{\operatorname{bd} K} g\left(\frac{x}{\|x\|}\right) \mathrm{d}\mathcal{H}^{d-1}(x).$$

Choosing

$$g(u) := [J_{d-1} f(u)]^{-1} \left\{ \frac{D_{d-1} h(K^*, u)}{h(K^*, u)^{d-1}} \right\}^{\frac{1}{2}},$$

we get

$$\begin{aligned} \mathcal{O}_d(K^*) &= \int_{S^{d-1}} \left\{ \frac{D_{d-1} h(K^*, u)}{h(K^*, u)^{d-1}} \right\}^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{d-1}(u) \\ &= \int_{\operatorname{bd} K} \frac{\left\langle \frac{x}{\|x\|}, \sigma_K\left(\rho\left(K, \frac{x}{\|x\|}\right)\frac{x}{\|x\|}\right)\right\rangle}{\rho\left(K, \frac{x}{\|x\|}\right)^{d-1}} \left\{ \frac{D_{d-1} h\left(K^*, \frac{x}{\|x\|}\right)}{h\left(K^*, \frac{x}{\|x\|}\right)^{d-1}} \right\}^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{d-1}(x). \end{aligned}$$

Consequently, the theorem is proved (recall [6, Theorem 2.8]), if we can show that for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$

$$\frac{\left\langle \frac{x}{\|x\|}, \sigma_K(x)\right\rangle}{\rho\left(K, \frac{x}{\|x\|}\right)^{d-1}} \left\{ \frac{D_{d-1} h\left(K^*, \frac{x}{\|x\|}\right)}{h\left(K^*, \frac{x}{\|x\|}\right)^{d-1}} \right\}^{\frac{1}{2}} = \left\{ \frac{H_{d-1}(K, x)}{\langle x, \sigma_K(x)\rangle^{d-1}} \right\}^{\frac{1}{2}}.$$

In order to verify this relation, we can assume that $x \in \mathcal{M}(K)$ and also that h_{K^*} is second order differentiable at $\|x\|^{-1}x$. Recall $x^* = \langle x, \sigma_K(x)\rangle^{-1}\sigma_K(x) \in \operatorname{bd} K^*$. Hence, $\|x\|^{-1}x \in \mathrm{N}(K^*, x^*)$, $\|x^*\|^{-1}x^* \in \mathrm{N}(K, x)$, and $\nabla h(K^*, \|x\|^{-1}x) = x^*$. In addition, we have $\langle x, \sigma_K(x)\rangle = \|x^*\|^{-1}$, $\rho(K, \|x\|^{-1}x) = \|x\|$, and $h(K^*, \|x\|^{-1}x) = \|x\|^{-1}$. Thus the above relation, which has to be proved, is equivalent to

$$\frac{1}{\|x\|} \frac{1}{\|x^*\|} \frac{1}{\|x\|^{d-1}} \left\{ \frac{D_{d-1} h\left(K^*, \frac{x}{\|x\|}\right)}{\|x\|^{1-d}} \right\}^{\frac{1}{2}} = \left\{ \frac{H_{d-1}(K, x)}{\|x^*\|^{1-d}} \right\}^{\frac{1}{2}}.$$

However, after rearranging terms, this is the statement of Theorem 2.2. $\square$

References

1. A. D. Aleksandrov, Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it (in Russian), *Uchenye Zapiski Leningrad. Gos. Univ., Math. Ser.* **6** (1939), 3–35.

2. V. Bangert, Analytische Eigenschaften konvexer Funktionen auf Riemannschen Mannigfaltigkeiten, *J. reine angew. Math.* **307/308** (1979), 309–324.

3. H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

4. S. Glasauer and P. M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies III, Preprint (1995).

5. P. M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies I, *Forum Mathematicum* **5** (1993), 281–297.

6. D. Hug, Contributions to affine surface area, Preprint (1995).

7. D. Hug, *Geometrische Maße in der affinen Konvexgeometrie*, Dissertation, Freiburg, 1994.

8. F. J. Kaltenbach, *Asymptotisches Verhalten zufälliger konvexer Polyeder*, Dissertation, Freiburg, 1990.

9. D. Laugwitz, *Differentialgeometrie in Vektorräumen*, Vieweg, Braunschweig, 1965.

10. K. Leichtweiß, Zur Affinoberfläche konvexer Körper, *Manuscripta Math.* **56** (1986), 429–464.

11. An-Min Li, Udo Simon, and Guosong Zhao, *Global Affine Differential Geometry of Hypersurfaces*, Walter de Gruyter, Berlin, 1993.

12. E. Lutwak, The Brunn-Minkowski-Firey Theory II: Affine and Geominimal Surface Area, to appear in Advances in Math., Preprint (1995).

13. E. Lutwak, Extended affine surface area, *Adv. Math.* **85** (1991), 39–68.

14. V. Oliker and U. Simon, *Affine Geometry and Polar Hypersurfaces,* pp. 87–112, BI Verlag Mannheim (1992), B. Fuchssteiner & W. A. J. Luxemburg (eds): “ Analysis and Geometry ”.

15. E. Salkowski, *Affine Differentialgeometrie*, Walter de Gruyter & Co., Göschens Lehrbücherei, Band 22, 1934.

16. P. A. Schirokow and A. P. Schirokow, *Affine Differentialgeometrie*, Teubner Verlagsgesellschaft, Leipzig, 1962.

17. R. Schneider, Bestimmung konvexer Körper durch Krümmungsmaße, *Comment. Math. Helvetici* **54** (1979), 42–60.

18. R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1993.

19. C. Schütt and E. Werner, The convex floating body, *Math. Scand.* **66** (1990), 275–290.

D. Hug, Mathematisches Institut, Albert-Ludwigs-Universität, Albertstraße 23b, D-79104 Freiburg i. Br., Germany, e-mail: hug@sun1.mathematik.uni-freiburg.de

Eingegangen am 11. November 1995