Generalized Curvature Measures and Singularities of Sets with Positive Reach*

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Abstract. For a convex body K in Euclidean space $\mathbb{R}^d$ ($d \geq 2$) and for $r \in \{0, \ldots, d-1\}$, let $\Sigma^r(K)$ be the set of r-singular boundary points of K. It is known that $\Sigma^r(K)$ is countably r-rectifiable and hence has σ-finite r-dimensional Hausdorff measure. We obtain a quantitative improvement of this result, taking into account the strength of the singularities. Denoting by $\Sigma^r(K, \tau)$ the set of those r-singular boundary points of K at which the spherical image has $(d-1-r)$-dimensional Hausdorff measure at least $\tau > 0$, we establish a finite upper bound for the r-dimensional Hausdorff measure of $\Sigma^r(K, \tau)$. This estimate is deduced from an identity that connects Hausdorff measures of spherical images of singularities to the generalized curvature measure of the convex body K. The latter relation is, in fact, proved for the class of sets with positive reach. For convex bodies, similar results as for singular boundary points are obtained for singular normal vectors. We also consider the disintegration of generalized curvature measures with respect to projections onto the components of the product space $\mathbb{R}^d \times S^{d-1}$.

1 Introduction

For a convex body K (nonempty compact convex set) in Euclidean space $\mathbb{R}^d$, $d \geq 2$, and for a number $r \in \{0, \ldots, d-1\}$ the set of r-singular boundary points of K is defined by

$$\Sigma^r(K) := \{x \in \operatorname{bd} K \mid \dim (N(K,x) \cap S^{d-1}) \geq d-1-r\}.$$

Here, $N(K,x)$ is the normal cone of K at x and S^{d-1} is the unit sphere. It is known that $\Sigma^r(K)$ is a countably r-rectifiable set, see [4] and [24, §2.2], and hence $\Sigma^r(K)$ has σ-finite r-dimensional Hausdorff measure. However, examples show that the r-dimensional Hausdorff measure of $\Sigma^r(K)$ can be infinite.

**AMS 1991 subject classification.* Primary 52A20; Secondary 53C65, 52A22, 28A50, 49Q15. *Key words and phrases.* Singular boundary points, singular normal vectors, generalized curvature measures, sets of positive reach, convex bodies, disintegration of measures.

In the present paper, we obtain a quantitative improvement of the result that $\Sigma^r(K)$ can be written as a countable union of sets which have finite r-dimensional Hausdorff measure, by taking into account the strength of the singularities. For that purpose we introduce the set $\Sigma^r(K,\tau)$ of those r-singular boundary points of K at which the spherical image has $(d-1-r)$-dimensional Hausdorff measure at least $\tau > 0$, i.e.,

$$\Sigma^r(K,\tau) := \{x \in \mathrm{bd}\, K \mid \mathcal{H}^{d-1-r}(N(K,x) \cap S^{d-1}) \geq \tau\},$$

where $\mathcal{H}^s$, for $s \geq 0$, denotes s-dimensional Hausdorff measure.

In Section 3 it will be shown that

$$\mathcal{H}^r(\Sigma^r(K,\tau)) \leq \frac{d}{\tau}\binom{d-1}{r} W_{d-r}(K). \tag{1}$$

The quantities $W_0(K), \ldots, W_d(K)$ are Minkowski's quermassintegrals of the convex body K. Note that the previous inequality is sharp, since equality can occur for example for regular polytopes and certain values of τ. For the proof of this estimate we study the integral

$$\int_{\Sigma^r(K)} \mathcal{H}^{d-1-r}(N(K,x) \cap S^{d-1})\, \mathrm{d}\mathcal{H}^r(x). \tag{2}$$

Intuitively speaking, for $\mathcal{H}^r$ almost all r-singular boundary points $x \in \Sigma^r(K)$ the contribution of the point x to the integral depends on the $(d-r)$-dimensional volume of the intersection of the normal cone $N(K,x)$ with the unit ball $B(o,1)$. We will prove that the integral in (2), which may be viewed as a suitably weighted r-dimensional Hausdorff measure of the set of r-singular boundary points, can be estimated from above by $d\binom{d-1}{r}W_{d-r}(K)$. From this, inequality (1) can immediately be deduced.

More generally, if $\Theta_0(K,\cdot), \ldots, \Theta_{d-1}(K,\cdot)$ are the generalized curvature measures of the convex body K, see [24, §4], if $r \in \{0, \ldots, d-1\}$, and if η is a Borel subset of $\mathbb{R}^d \times S^{d-1}$, it will be proved in Section 3, Theorem 3.2, that

$$\int_{\Sigma^r(K)} \mathcal{H}^{d-1-r}(N(K,x) \cap \eta_x)\, \mathrm{d}\mathcal{H}^r(x) = \binom{d-1}{r} \Theta_r(K, (\Sigma^r(K) \times S^{d-1}) \cap \eta). \tag{3}$$

The set $\eta_x := \{u \in S^{d-1} \mid (x,u) \in \eta\}$ is defined as the x-section of the set η. In particular, if η is specialized to $\eta = \mathbb{R}^d \times S^{d-1}$, equation (3) implies the asserted upper bound for the integral in (2).

For $d = 3$ and $r = 1$, inequality (1) and a special case of equation (3) have been treated by Colesanti & Pucci [9]. They used completely different methods which, however, seem to be restricted to the three-dimensional situation. In [9] one can also find instructive examples which exhibit the intricate structure of the set $\Sigma^1(K) \setminus \Sigma^0(K)$ of ridge points of order one in $\mathbb{R}^3$. In addition, Theorem 3.2 represents a substantial generalization of Theorem 3.3 (ii) in a paper of Anzellotti & Ossanna [5]. This observation may have further consequences for the variational problem which is studied in §5 of their paper [5].

Another way to interpret equation (3) is to say that it provides an explicit description of the generalized curvature measure $\Theta_r(K,\cdot)$ on the Borel subsets of $\Sigma^r(K) \times S^{d-1}$. This point of view leads to a new result concerning the measure $\Theta_r(K,\cdot)$ if $r = d-2$. In fact, recall that the generalized curvature measures $\Theta_r(K,\cdot)$, for a convex body K and $r \in \{0,\ldots,d-1\}$, are defined on the σ-algebra $\mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ of Borel sets of $\mathbb{R}^d \times S^{d-1}$. Let $\pi_1 : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^d$, $(x,u) \mapsto x$, be the projection map onto the first component, and define Federer's curvature measure of order r for K, $C_r(K,\cdot) := \Theta_r(K,\cdot) \circ \pi_1^{-1}$, as the image measure of $\Theta_r(K,\cdot)$ under π_1. General results of abstract measure theory then yield that the measure $\Theta_r(K,\eta)$ can be calculated by integrating the characteristic function $\mathbf{1}_\eta$ of the set $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ according to

$$\Theta_r(K,\eta) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{1}_\eta(x,u)\, \mathrm{d}\lambda_x(u)\, \mathrm{d}C_r(K,x).$$

The measures λ_x, for $x \in \mathbb{R}^d$, are suitable probability measures on the Borel sets of S^{d-1}, which are uniquely determined up to a set of $C_r(K,\cdot)$ measure zero. Note that the measures λ_x may depend on the index $r \in \{0,\ldots,d-1\}$. The pair $(C_r(K,\cdot),\lambda_x)$ is called a disintegration or stratification of the measure $\Theta_r(K,\cdot)$. For general reference see [10, I, §9, and the Appendix], [17, Satz 5.3.21], [26], [3] and especially [11].

The general measure theoretic results, however, do not provide any specific information about the "conditional probability measures" (fibre measures) λ_x. In the present situation we obtain an explicit representation for λ_x in the disintegration $(C_{d-2}(K,\cdot),\lambda_x)$ of $\Theta_{d-2}(K,\cdot)$, if K is a convex body with nonempty interior. The precise statement is contained in Theorem 3.10. Again Theorem 3.2 is crucial for the proof of this result, which has recently been conjectured by J. Sangwine-Yager [23]. In addition, Theorem 3.10 can also be viewed as a contribution to Theorem 1 in an article by E. Ossanna [22] on the comparison between the generalized mean curvature of Allard and Federer's mean curvature measure, since in this paper the measures λ_x have not yet been determined.

Up to this point we focused on singular boundary points of convex bodies. This is the subject of Section 3. The main results stated there even hold true for the more general class of sets with positive reach. Moreover, there also exists a dual theory for singular normal vectors. This duality is for example suggested by the symmetry of the definition of the generalized curvature measures with respect to boundary points and normal vectors. The corresponding results will be stated for convex bodies in Section 4, although some of the statements remain true, with almost identical proofs, for nonempty closed convex sets $K \subset \mathbb{R}^d$ with

$K \neq \mathbb{R}^d$. Despite the strong analogy to the results and methods of Section 3, independent proofs are required.

Finally, it should be observed that some results on semi-convex functions, which are related to the present work, are contained in [2]. The connection between semi-convex functions and sets of positive reach has been depicted by Bangert [7] and Fu [14]. In the present context we should also mention recent research on the C^2-rectifiability of the set of singular points of convex functions and convex bodies, see [29], [1], [6], [9], [5], [16].

2 Definitions and technical background

In the present paper, singularities of sets with positive reach and especially singularities of convex bodies are studied. The ambient space is the Euclidean space $\mathbb{R}^d$, $d \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. More generally, however, some of the results of Section 3 remain true in space forms for sets with the unique footpoint property, cf. [7], [20].

For a nonempty set $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ let $d_A(x) := \inf\{\|x-a\| \mid a \in A\}$ be the distance of x to A and define

$$A^\epsilon := \{x \in \mathbb{R}^d \mid d_A(x) \leq \epsilon\}.$$

If $\emptyset \neq X \subset \mathbb{R}^d$ and $X \neq \mathbb{R}^d$, then the supremum of all $\epsilon \geq 0$ such that

$$\operatorname{card}\{x \in X \mid \|y - x\| = d_X(y)\} = 1$$

holds true for all $y \in X^\epsilon$, is denoted by $\operatorname{reach}(X)$. The set X is said to have positive reach if $\operatorname{reach}(X) > 0$. Any such set is closed. Note in particular that any nonempty closed convex set X has $\operatorname{reach}(X) = \infty$. A nonempty compact convex set is called a convex body. Thus the set of all convex bodies in $\mathbb{R}^d$, which is denoted by $\mathcal{K}^d$, is an important subclass of the class of all sets with positive reach. Subsequently we collect some basic facts about sets of positive reach. Proofs can be found in [12], [28], [20]. For surveys see [15], and in connection with convex bodies also [19]. Our general reference for results and terminology of convex geometry is [24].

If $0 < \epsilon < \operatorname{reach}(X)$, then we define $p_X : X^\epsilon \to X$ to be the nearest point map (the metric projection onto X), and $\sigma_X : X^\epsilon \setminus X \to S^{d-1}$, the spherical image map with values in the unit sphere S^{d-1}, is defined by

$$\sigma_X(y) := \|y - p_X(y)\|^{-1}(y - p_X(y)).$$

In the following we write $\operatorname{bd} A$ for the topological boundary of a set $A \subset \mathbb{R}^d$. Let $F_X : X^\epsilon \to \mathbb{R}^d \times S^{d-1}$ be given by $F_X := (p_X, \sigma_X)$. Moreover, we set $\mathcal{N}(X) := F_X(\operatorname{bd} X^\epsilon)$. Then $F_X|_{\operatorname{bd} X^\epsilon}$ is a bi-Lipschitz homeomorphism with inverse

$$t_\epsilon : \mathcal{N}(X) \to \operatorname{bd} X^\epsilon, \quad (x, u) \mapsto x + \epsilon u.$$

Hence, the so-called generalized unit normal bundle $\mathcal{N}(X)$ is locally $(d-1)$-rectifiable, i.e., $\mathcal{N}(X)$ is countably $(d-1)$-rectifiable in the sense of Federer [13] and $\mathcal{H}^{d-1}(\mathcal{N}(X) \cap M) < \infty$ for any compact set $M \subset \mathbb{R}^d \times S^{d-1}$. See also [13] for other notions of rectifiability as well as for basic concepts of measure theory. Subsequently the σ-algebra of Borel sets of a topological space $(T, \mathcal{T})$ will be denoted by $\mathfrak{B}(T)$.

In the following, the set X is always assumed to have positive reach. The common set of differentiability points of the three mappings p_X, σ_X, F_X is denoted by $\mathcal{D}_X$. Since $\mathcal{H}^{d-1}(\operatorname{bd} X^\epsilon \setminus \mathcal{D}_X) = 0$, cf. [20], generalized curvatures $k_1(x,u), \ldots, k_{d-1}(x,u) \in (-\infty, \infty]$ can be defined for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(X)$. In fact, if $(x,u) \in \mathcal{N}(X)$ is such that $x + \epsilon u \in \mathcal{D}_X$, then

$$k_i(x,u) = \begin{cases} \dfrac{k_i(x+\epsilon u)}{1 - \epsilon k_i(x+\epsilon u)}, & \text{if } k_i(x+\epsilon u) < \epsilon^{-1}, \\ \\ \infty, & \text{if } k_i(x+\epsilon u) = \epsilon^{-1}, \end{cases}$$

for $i \in \{1, \ldots, d-1\}$, independent of the particular choice of $\epsilon \in (0, \operatorname{reach}(X))$. The curvatures $k_1(x+\epsilon u), \ldots, k_{d-1}(x+\epsilon u)$ are defined as the eigenvalues of the symmetric linear map $D\sigma_X(x+\epsilon u)|_{u^\perp}$, cf. [28], [20]. Henceforth we shall set

$$\frac{k_i}{\sqrt{1+k_i^2}} = 1 \quad \text{and} \quad \frac{1}{\sqrt{1+k_i^2}} = 0,$$

if $k_i = \infty$. Expressions of the form

$$\frac{k_{i_1} \cdots k_{i_r}}{\prod_{i=1}^{d-1} \sqrt{1+k_i^2}}$$

have to be calculated as

$$\prod_{l=1}^{r} \frac{k_{i_l}}{\sqrt{1+k_{i_l}^2}} \prod_{t=r+1}^{d-1} \frac{1}{\sqrt{1+k_{j_t}^2}},$$

where $\{i_1, \ldots, i_r\}$ and $\{j_{r+1}, \ldots, j_{d-1}\}$ are complementary subsets of the set $\{1, \ldots, d-1\}$.

The geometric relevance of these generalized curvatures becomes clear, if the generalized curvature measures $\Theta_0(X, \cdot), \ldots, \Theta_{d-1}(X, \cdot)$ are introduced and represented as integrals over the unit normal bundle $\mathcal{N}(X)$, see equation (5) below. These generalized curvature measures are signed measures which are defined (at least) on bounded sets $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$, and which are concentrated on the unit normal bundle $\mathcal{N}(X)$. They can be obtained as the coefficients of the local Steiner formula

$$\lambda^d(M_\rho(X,\eta)) = \frac{1}{d} \sum_{r=0}^{d-1} \rho^{d-r} \binom{d}{r} \Theta_r(X,\eta). \tag{4}$$

Here, we assume $0 < \rho < \operatorname{reach}(X)$, λ^d denotes the d-dimensional Lebesgue measure, and we set

$$M_\rho(X,\eta) := \{x \in \mathbb{R}^d \mid 0 < d_X(x) \le \rho,\, F_X(x) \in \eta\}.$$

The following useful representation of the generalized curvature measures, which involves the curvatures on the unit normal bundle, is due to M. Zähle [28]. In our terminology her result can be written in the form

$$\binom{d-1}{r}\Theta_{d-1-r}(X,\eta) = \int_{\mathcal{N}(X)\cap\eta} \sum_{1\le i_1<\ldots<i_r\le d-1} \frac{k_{i_1}(x,u)\cdots k_{i_r}(x,u)}{\prod_{i=1}^{d-1}\sqrt{1+k_i(x,u)^2}}\, \mathrm{d}\mathcal{H}^{d-1}(x,u), \tag{5}$$

if η is a bounded Borel set of $\mathbb{R}^d \times S^{d-1}$ and $r \in \{0,\ldots,d-1\}$. By specializing η we obtain Federer's curvature measures

$$C_r(X,\beta) := \Theta_r(X,\beta\times S^{d-1}), \quad r \in \{0.\ldots,d-1\},$$

if $\beta \in \mathfrak{B}(\mathbb{R}^d)$ is bounded, and the surface area measures

$$S_r(X,\omega) := \Theta_r(X,\mathbb{R}^d\times\omega), \quad r \in \{0,\ldots,d-1\},$$

if $\omega \in \mathfrak{B}(S^{d-1})$ and $X \ne \mathbb{R}^d$ is a nonempty closed convex subset of $\mathbb{R}^d$. The last definition is possible, since for a convex set X of positive reach the equations (4) and (5) still hold for unbounded sets η. In the case where $X = K \in \mathcal{K}^d$, a further specialization of the generalized curvature measures reproduces Minkowski's classical quermassintegrals, which can be defined by

$$W_{d-r}(K) := \frac{1}{d}\Theta_r(K,\mathbb{R}^d\times S^{d-1}), \quad r \in \{0,\ldots,d-1\}.$$

In addition, one usually defines $W_0(K)$ as the volume, i.e. the d-dimensional Lebesgue measure $\lambda^d(K)$, of $K \in \mathcal{K}^d$.

Now, for a set X of positive reach the closed convex normal cones

$$N(X,x) := \{\lambda u \in \mathbb{R}^d \mid \lambda \ge 0,\, (x,u) \in \mathcal{N}(X)\}, \quad x \in X,$$

and the closed sets

$$F(X,u) := \{x \in X \mid (x,u) \in \mathcal{N}(X)\}, \quad u \in S^{d-1},$$

which are the convex support sets of X, if X is a convex body (cf. [24]), lead to classifications for the singularities of a set X with positive reach, respectively for the singularities of a convex body $K \in \mathcal{K}^d$. The set of r-singular boundary points of X is defined by

$$\Sigma^r(X) := \{x \in X \mid \dim(N(X,x) \cap S^{d-1}) \geq d-1-r\},$$

if $r \in \{0,\dots,d-1\}$, and $\Sigma^{-1}(X) := \emptyset$. Hence, the classical ridge points of order r of X are exactly the points in $\Sigma^r(X) \setminus \Sigma^{r-1}(X)$. And similarly, for a convex body $K \in \mathcal{K}^d$, we set

$$\Sigma_r(K) := \{u \in S^{d-1} \mid \dim F(K,u) \geq d-1-r\},$$

if $r \in \{0,\dots,d-1\}$, and $\Sigma_{-1}(K) := \emptyset$. This defines the set of r-singular unit normal vectors of K. The last definition could also be extended to the case of a set X with positive reach, but it seems to be more natural to restrict the investigation of singular normal vectors to the setting of convex geometry.

3 Singular boundary points

The proof of the central Theorem 3.2, which will be given in the present section, essentially relies on an application of Federer's coarea formula. In Lemma 3.1 we make sure that the assumptions are fulfilled that are necessary for the applicability of the coarea formula. Approximate tangent spaces, approximate differentials and Jacobians, which are mentioned in the proof of Theorem 3.2, are introduced in Federer's book [13, 3.2.16–3.2.22].

Lemma 3.1. *For a set $X \subset \mathbb{R}^d$ of positive reach and $r \in \{0,\dots,d-1\}$, the set $\Sigma^r(X)$ of r-singular boundary points of X is a countably r-rectifiable Borel set.*

Proof. For $\tau > 0$ define

$$\Sigma^{r,\tau}(X) := \{x \in X \mid \;\exists\; u_1,\dots,u_{d-r} \in N(X,x) \cap S^{d-1} : \det\left(\langle u_i, u_j\rangle_{i,j=1}^{d-r}\right) \geq \tau\}.$$

Let $x_i \in \Sigma^{r,\tau}(X)$, $i \in \mathbb{N}$, and assume $x_i \to x$ for $i \to \infty$. Since X is closed, we have $x \in X$. Choose $\epsilon \in (0, \operatorname{reach}(X))$. Note that if $u_i \in N(X,x_i) \cap S^{d-1}$, $i \in \mathbb{N}$, and $u_i \to u$ for $i \to \infty$, then $u \in N(X,x) \cap S^{d-1}$. In fact, from $p_X(x_i + \epsilon u_i) = x_i$ for each $i \in \mathbb{N}$ we obtain $p_X(x + \epsilon u) = x$. Here, Theorem 4.8 (8) and (12) from [12] are used. This shows $u \in N(X,x) \cap S^{d-1}$ according to another application of Theorem 4.8 (12) in [12]. Thus a compactness argument yields that $\Sigma^{r,\tau}(X)$ is a closed subset of $\mathbb{R}^d$. Since

$$\Sigma^r(X) = \bigcup_{n \in \mathbb{N}} \Sigma^{r,n^{-1}}(X),$$

we obtain that $\Sigma^r(X)$ is Borel measurable.

The proof of the rectifiability statement is already contained in Remark 4.15 (3) of [12]. □

Remark. A different proof for a similar statement has been proposed by Kohlmann [20, Theorem 2.12] and independently by Anzellotti & Ossanna [5, Theorem 2.1]. Their approach, however, leads to a slightly weaker statement in the present context.

Theorem 3.2. *Let $X \subset \mathbb{R}^d$ be a set of positive reach, let $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ be bounded, and $r \in \{0, \dots, d-1\}$. Then*

$$\int_{\Sigma^r(X)} \mathcal{H}^{d-1-r}(N(X,x) \cap \eta_x) \, \mathrm{d}\mathcal{H}^r(x) = \binom{d-1}{r} \Theta_r(X, (\Sigma^r(X) \times S^{d-1}) \cap \eta). \tag{6}$$

Proof. According to Lemma 3.1 the set $\Sigma^r(X)$ can be written as a disjoint union of countably many r-rectifiable Borel sets. Hence, in order to prove the theorem we can assume that $\Sigma^r(X)$ itself is an r-rectifiable Borel set. Define

$$\mathcal{N}(\Sigma^r(X)) := \{(x,u) \in \mathcal{N}(X) \mid x \in \Sigma^r(X)\}.$$

Then, in particular, $\mathcal{N}(\Sigma^r(X))$ is an $(\mathcal{H}^{d-1}, d-1)$-rectifiable Borel set. Since $\mathcal{N}(\Sigma^r(X)) \subset \mathcal{N}(X)$ and because $\mathcal{N}(X)$ is locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable, Theorem 3.2.19 in [13] implies for the corresponding approximate tangent spaces that

$$\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(\Sigma^r(X)), v) = \mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(X), v),$$

for $\mathcal{H}^{d-1}$ almost all $v \in \mathcal{N}(\Sigma^r(X))$. In addition, for $\mathcal{H}^{d-1}$ almost all $v \in \mathcal{N}(\Sigma^r(X))$, at least $(d-1-r)$ of the generalized curvatures $k_1(v), \dots, k_{d-1}(v)$ are equal to ∞. This can be proved as follows (cf. [20, Corollary 2.11]): let $v = (x,u) \in \mathcal{N}(\Sigma^r(X))$ be such that $\sigma_X|_{\mathrm{bd}\, X^\epsilon}$ is differentiable at $x + \epsilon u$, where $0 < \epsilon < \mathrm{reach}(X)$ is arbitrarily chosen. Since $\dim N(X,x) \geq d - r$ and because $N(X,x)$ is convex, there are at least $(d-1-r)$ linearly independent unit vectors $v_1, \dots, v_{d-1-r} \perp u$ such that $D\sigma_X(x+\epsilon u)(v_i) = \epsilon^{-1} v_i$ for $i \in \{1, \dots, d-1-r\}$. Thus the dimension of the eigenspace of $D\sigma_X(x+\epsilon u)|_{u^\perp}$ associated with the eigenvalue ϵ^{-1} is at least $(d-1-r)$.

Here and in the following we can assume without loss of generality that $-\infty < k_1(v) \leq \ldots \leq k_{d-1}(v) \leq \infty$, if these curvatures are defined for $v \in \mathcal{N}(X)$. This means that $k_1(v), \dots, k_r(v) \in (-\infty, \infty]$ and $k_{r+1}(v) = \ldots = k_{d-1}(v) = \infty$, for $\mathcal{H}^{d-1}$ almost all $v \in \mathcal{N}(\Sigma^r(X))$.

First of all we consider the case $r > 0$. The case $r = 0$ is formally included in the subsequent arguments, if products over empty index sets are interpreted as 1. However, for the sake of clarity the case $r = 0$ is also considered separately at the end of the proof.

Now we define the projection map

$$\pi_1 : \mathcal{N}(\Sigma^r(X)) \to \Sigma^r(X), \quad (x,u) \mapsto x.$$

The approximate Jacobian of π_1 can be calculated according to

$$\mathrm{ap}J_r\pi_1(x,u) = \prod_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma^r(X))$.

In fact, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma^r(X))$, the $(d-1)$ vectors in $\mathbb{R}^d \times \mathbb{R}^d$ which are given by

$$w_i(x,u) := \left(\frac{1}{\sqrt{1+k_i(x,u)^2}} u_i, \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} u_i \right), \quad i \in \{1,\ldots,d-1\},$$

represent an orthonormal basis of the $(d-1)$-dimensional approximate tangent space $\mathrm{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(\Sigma^r(X)),(x,u))$, where $u_1,\ldots,u_{d-1}$ is a suitable orthonormal basis of $u^\perp$, cf. [28], [20], [19]. Hence, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma^r(X))$,

$$\mathrm{ap}D\pi_1(x,u)(w_i(x,u)) = \frac{1}{\sqrt{1+k_i(x,u)^2}} u_i,$$

$i \in \{1,\ldots,d-1\}$. This implies

$$w_{r+1}(x,u),\ldots,w_{d-1}(x,u) \in \mathrm{kernel}(\mathrm{ap}D\pi_1(x,u)),$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma^r(X))$. Again for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma^r(X))$ we distinguish two cases. If $k_r(x,u) < \infty$, then

$$(\mathrm{kernel}(\mathrm{ap}D\pi_1(x,u)))^\perp = \mathrm{lin}\{w_1(x,u),\ldots,w_r(x,u)\},$$

and thus

$$\begin{aligned} \mathrm{ap}J_r\pi_1(x,u) &= \left\| \bigwedge_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}} u_i \right\| \\ &= \prod_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}}. \end{aligned}$$

If, however, $k_r(x,u) = \infty$, then we get

$$\mathrm{ap}J_r\pi_1(x,u) = 0 = \prod_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}}.$$

In this situation, Federer's coarea formula [13, Theorem 3.2.22] can be applied. We obtain

$$\int_{\mathcal{N}(\Sigma^r(X))\cap\eta} \prod_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,u)$$
$$= \int_{\Sigma^r(X)} \mathcal{H}^{d-1-r}(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^r(x). \tag{7}$$

On the other hand, we know from equation (5) that

$$\int_{\mathcal{N}(\Sigma^r(X))\cap\eta} \sum_{1\le i_1<\ldots<i_{d-1-r}\le d-1} \frac{k_{i_1}\cdots k_{i_{d-1-r}}}{\prod_{i=1}^{d-1}\sqrt{1+k_i^2}} \,\mathrm{d}\mathcal{H}^{d-1}$$
$$= \binom{d-1}{r}\Theta_r(X,(\Sigma^r(X)\times S^{d-1})\cap\eta) \tag{8}$$

holds true. In the sum under the integral at most one summand is positive for $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(\Sigma^r(X))$. In fact, the summand corresponding to $i_1=r+1,\ldots,i_{d-1-r}=d-1$ is equal to

$$\frac{k_{r+1}(x,u)\cdots k_{d-1}(x,u)}{\prod_{i=1}^{d-1}\sqrt{1+k_i(x,u)^2}} = \prod_{i=1}^{r}\frac{1}{\sqrt{1+k_i(x,u)^2}},$$

and this can be positive. In the remaining cases we have $i_1\le r$, and thus the number of factors in the denominator which are equal to ∞ is greater than the corresponding number in the numerator. In order to see this, distinguish the two cases $k_{i_1}(x,u)=\infty$ and $k_{i_1}(x,u)<\infty$. This yields that

$$\frac{k_{i_1}(x,u)\cdots k_{i_{d-1-r}}(x,u)}{\prod_{i=1}^{d-1}\sqrt{1+k_i(x,u)^2}} = 0,$$

if $i_1\le r$. Thus equation (8) can be simplified to

$$\int_{\mathcal{N}(\Sigma^r(X))\cap\eta} \prod_{i=1}^{r} \frac{1}{\sqrt{1+k_i(x,u)^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,u)$$
$$= \binom{d-1}{r}\Theta_r(X,(\Sigma^r(X)\times S^{d-1})\cap\eta). \tag{9}$$

The statement of the Theorem now follows from equations (7) and (9), if $r>0$.

Finally, let $r=0$. Then $\Sigma^0(X)=\{x_\iota \mid \iota\in I\}$, where I is at most countable. In addition, for $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(\Sigma^0(X))$,

$$k_1(x,u)=\ldots=k_{d-1}(x,u)=\infty.$$

Hence,

$$\begin{aligned}
&\Theta_0(X,(\Sigma^0(X)\times S^{d-1})\cap\eta)\\
&= \int_{\mathcal{N}(\Sigma^0(X))\cap\eta}\prod_{i=1}^{d-1}\frac{k_i}{\sqrt{1+k_i^2}}\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \mathcal{H}^{d-1}(\mathcal{N}(\Sigma^0(X))\cap\eta)\\
&= \sum_{\iota\in I}\mathcal{H}^{d-1}((\{x_\iota\}\times N(X,x_\iota))\cap\eta)\\
&= \sum_{\iota\in I}\mathcal{H}^{d-1}(N(X,x_\iota)\cap\eta_{x_\iota})\\
&= \int_{\Sigma^0(X)}\mathcal{H}^{d-1}(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^0(x).
\end{aligned}$$

This settles the case $r=0$. □

Remark. The preceding proof also shows that Theorem 3.2 holds true for any set $\eta\in\mathfrak{B}(\mathbb{R}^d\times S^{d-1})$, since

$$\prod_{i=1}^{r}\frac{1}{\sqrt{1+k_i(x,u)^2}}\geq 0$$

for $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(X)$. But then, of course, both sides of equation (6) in Theorem 3.2 can be infinite.

From equation (7) in the proof of Theorem 3.2 we immediately obtain the following corollary, which is similar to a result on semi-convex functions, see Theorem 4.1 (4.2) in [2].

Corollary 3.3. *Let $X\subset\mathbb{R}^d$ be a set of positive reach, $r\in\{0,\dots,d-1\}$, and let $\eta\in\mathfrak{B}(\mathbb{R}^d\times S^{d-1})$. Then*

$$\int_{\Sigma^r(X)}\mathcal{H}^{d-1-r}(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^r(x)\leq\mathcal{H}^{d-1}(\mathcal{N}(\Sigma^r(X))\cap\eta).$$

Theorem 3.2 provides an explicit description of the restriction of the curvature measure $\Theta_j(X,\cdot)$ to the set $\Sigma^r(X)\times S^{d-1}$, if $r=j\in\{0,\dots,d-1\}$. In Corollary 3.4 and Theorem 3.8 below we investigate the cases where $r\neq j$.

Corollary 3.4. *Let $X\subset\mathbb{R}^d$ be a set of positive reach, $r\in\{0,\dots,d-2\}$, $j\in\{r+1,\dots,d-1\}$, and let $\eta\in\mathfrak{B}(\mathbb{R}^d\times S^{d-1})$ be bounded. Then*

$$\Theta_j(X,(\Sigma^r(X)\times S^{d-1})\cap\eta)=0.$$

Proof. Again we use the representation (5) for the generalized curvature measures in order to get

$$\int_{\mathcal{N}(\Sigma^r(X))\cap\eta} \sum_{1\le i_1<\ldots<i_{d-1-j}\le d-1} \frac{k_{i_1}\cdots k_{i_{d-1-j}}}{\prod_{i=1}^{d-1}\sqrt{1+k_i^2}}\,\mathrm{d}\mathcal{H}^{d-1}$$

$$= \binom{d-1}{j}\Theta_j(X,(\Sigma^r(X)\times S^{d-1})\cap\eta). \tag{10}$$

For $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(\Sigma^r(X))$ at least $(d-1-r)$ of the generalized curvatures $k_1(x,u),\ldots,k_{d-1}(x,u)$ are equal to ∞. Since we assumed that $d-1-j<d-1-r$, each summand under the integral in equation (10) vanishes. □

Our next corollary is a local Steiner formula for special parallel sets.

Corollary 3.5. *Let $X\subset\mathbb{R}^d$ be a set of positive reach, $r\in\{0,\ldots,d-1\}$, and let $\eta\in\mathfrak{B}(\mathbb{R}^d\times S^{d-1})$ be bounded. Then*

$$\begin{aligned}\lambda^d(M_\rho(X,\hat\eta_r)) &= \frac{1}{d}\sum_{j=0}^{r-1}\rho^{d-j}\binom{d}{j}\Theta_j(X,\hat\eta_r)+\\ &\quad+\frac{1}{d-r}\rho^{d-r}\int_{\Sigma^r(X)}\mathcal{H}^{d-1-r}(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^r(x),\end{aligned}$$

if $\hat\eta_r:=(\Sigma^r(X)\times S^{d-1})\cap\eta$ and $\rho\in(0,\mathrm{reach}(X))$.

Proof. The statement immediately follows from equation (4), Theorem 3.2, and Corollary 3.4. □

Remarks.

1. If $X=K\in\mathcal{K}^d$, then

$$\begin{aligned}\frac{1}{d}\rho^d\Theta_0(K,\hat\eta_r) &= \frac{1}{d}\rho^d\int_{\mathcal{N}(K)\cap\hat\eta_r}\prod_{i=1}^{d-1}\frac{k_i}{\sqrt{1+k_i^2}}\,\mathrm{d}\mathcal{H}^{d-1}\\ &= \frac{1}{d}\rho^d\int_{S^{d-1}}\int_{\pi_2^{-1}(\{u\})}\mathbf{1}_{\hat\eta_r}\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{H}^{d-1}(u)\\ &= \frac{1}{d}\rho^d\mathcal{H}^{d-1}(\{u\in S^{d-1}\mid(F(K,u)\times\{u\})\cap\hat\eta_r\neq\emptyset\})\\ &= \lambda^d(\{tu\in\mathbb{R}^d\mid(F(K,u)\times\{u\})\cap\hat\eta_r\neq\emptyset,\ t\in[0,\rho]\}).\end{aligned}$$

Here we have applied Federer's coarea formula to the projection map $\pi_2:\mathcal{N}(K)\to S^{d-1}$, $(x,u)\mapsto u$. Note that the approximate Jacobian of π_2 is

$$\mathrm{ap}J_{d-1}\pi_2(x,u)=\prod_{i=1}^{d-1}\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$, see the proof of Theorem 4.3. In addition, Theorem 2.2.9 of [24] has been used.

Thus, for $d=3$, $r=1$, and $\eta = \beta \times S^2$ with a Borel set $\beta \subset \Sigma^1(K) \setminus \Sigma^0(K)$, where K is a convex body in $\mathbb{R}^3$, we obtain Theorem 4.2 of [9] as a special case of our Corollary 3.5.

2. If $r = d-1$, then Corollary 3.5 is just the local Steiner formula (4), since $\Sigma^{d-1}(X) = \operatorname{bd} X$ and hence $M_\rho(X, \hat{\eta}_{d-1}) = M_\rho(X, \eta)$ as well as $\Theta_j(X, \hat{\eta}_{d-1}) = \Theta_j(X, \eta)$.

   For $r=0$, the statement of Corollary 3.5 has to be interpreted as

$$\begin{aligned} \lambda^d(M_\rho(X,\hat{\eta}_0)) &= \frac{1}{d}\rho^d \Theta_0(X,\hat{\eta}_0) \\ &= \frac{1}{d}\rho^d \int_{\Sigma^0(X)} \mathcal{H}^{d-1}(N(X,x) \cap \eta_x)\, \mathrm{d}\mathcal{H}^0(x). \end{aligned}$$

Finally, the next corollary summarizes two remarks from Section 1, which both are immediate consequences of Theorem 3.2.

Corollary 3.6. *Let $K \in \mathcal{K}^d$ and $r \in \{0,\ldots,d-1\}$. Then*

$$\int_{\Sigma^r(K)} \mathcal{H}^{d-1-r}(N(K,x) \cap S^{d-1})\, \mathrm{d}\mathcal{H}^r(x) \le d\binom{d-1}{r} W_{d-r}(K),$$

and, for $\tau > 0$,

$$\mathcal{H}^r(\Sigma^r(K,\tau)) \le \frac{d}{\tau}\binom{d-1}{r} W_{d-r}(K).$$

In Corollary 3.4 we considered the restriction of the generalized curvature measure $\Theta_j(X,\cdot)$ to the set $\Sigma^r(X) \times S^{d-1}$ under the condition $j \ge r+1$. Now we ask, what can be said for $j \le r-1$. An answer is given in Theorem 3.8 below. But first recall that, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \Sigma^r(X) \times S^{d-1}$,

$$k_{r+1}(x,u) = \ldots = k_{d-1}(x,u) = \infty.$$

Let X be a given set of positive reach, $0 < \epsilon < \operatorname{reach}(X)$, and $r \in \{0,\ldots,d-1\}$. In order to be able to state and prove Theorem 3.8 we shall need the sets $\mathcal{N}^r(X)$ and $\mathcal{N}^r(\Sigma^r(X))$ defined by

$$\mathcal{N}^r(X) := \{(x,u) \in \mathcal{N}(X) \mid x + \epsilon u \in \mathcal{D}_X \text{ and } k_1(x,u),\ldots,k_r(x,u) < \infty\}$$

and

$$\mathcal{N}^r(\Sigma^r(X)) := \mathcal{N}^r(X) \cap (\Sigma^r(X) \times S^{d-1}).$$

Note that the definition of $\mathcal{N}^r(X)$ is independent of the special choice of $\epsilon > 0$.

Lemma 3.7. *Let $X \subset \mathbb{R}^d$ be a set of positive reach, and let $r \in \{0, \dots, d-1\}$. Then the set $\mathcal{N}^r(\Sigma^r(X))$ is Borel measurable.*

Proof. Choose $0 < \epsilon < \operatorname{reach}(X)$. Then $\mathcal{N}^r(X)$ is equal to the set

$$\{(x,u) \in \mathcal{N}(X) \mid x + \epsilon u \in \mathcal{D}_X \text{ and } \operatorname{rank}(\operatorname{ap}D\pi_1(x,u)) \geq r\}.$$

But this set is equal to the Borel set

$$\{(x,u) \in \mathcal{N}(X) \mid (x,u) \in (t_\epsilon)^{-1}(\mathcal{D}_X) \text{ and } \\ \dim Dp_X(t_\epsilon(x,u))(\operatorname{Tan}(\operatorname{bd} X^\epsilon, t_\epsilon(x,u))) \geq r\},$$

cf. [19, §2]. These remarks together with Lemma 3.1 imply the statement of the lemma. □

Theorem 3.8. *Let $X \subset \mathbb{R}^d$ be a set of positive reach, $r \in \{1, \dots, d-1\}$, $j \in \{0, \dots, r-1\}$, and let $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ be bounded. Then*

$$\binom{d-1}{j} \Theta_j(X, \mathcal{N}^r(\Sigma^r(X)) \cap \eta) \\ = \int_{\Sigma^r(X)} \int_{N(X,x)\cap\eta_x} \sum_{1 \leq i_1 < \ldots < i_{r-j} \leq r} \times \\ \times k_{i_1}(x,u) \cdots k_{i_{r-j}}(x,u) \, \mathrm{d}\mathcal{H}^{d-1-r}(u) \, \mathrm{d}\mathcal{H}^r(x).$$

Proof. The generalized curvature measure

$$\binom{d-1}{j} \Theta_j(X, \mathcal{N}^r(\Sigma^r(X)) \cap \eta)$$

can be written as

$$\int_{\mathcal{N}^r(\Sigma^r(X))\cap\eta} \sum_{1 \leq i_1 < \ldots < i_{d-1-j} \leq d-1} \frac{k_{i_1} \cdots k_{i_{d-1-j}}}{\prod_{i=1}^{d-1} \sqrt{1 + k_i^2}} \, \mathrm{d}\mathcal{H}^{d-1}. \tag{11}$$

But all those summands under the integral in (11) vanish which correspond to indices $(i_1, \dots, i_{d-1-j})$ that do not fulfill the condition

$$i_{r+1-j} = r+1, \dots, i_{d-1-j} = d-1.$$

Note that $r + 1 - j \geq 2$. Thus, (11) can be simplified to

$$\int_{\mathcal{N}^r(\Sigma^r(X))\cap\eta} \sum_{1 \leq i_1 < \ldots < i_{r-j} \leq r} \frac{k_{i_1} \cdots k_{i_{r-j}}}{\prod_{i=1}^{r} \sqrt{1 + k_i^2}} \, \mathrm{d}\mathcal{H}^{d-1}.$$

Since we integrate over $\mathcal{N}^r(\Sigma^r(X))$, this is the same as

$$\int_{\mathcal{N}^r(\Sigma^r(X))\cap\eta} \mathrm{ap}J_r\pi_1 \sum_{1\le i_1<\ldots<i_{r-j}\le r} k_{i_1}\cdots k_{i_{r-j}}\,\mathrm{d}\mathcal{H}^{d-1}.$$

If we set $N^r(X,x) := (\mathcal{N}^r(X))_x$, an application of Federer's coarea formula yields

$$\begin{aligned}&\binom{d-1}{j}\Theta_j(X,\mathcal{N}^r(\Sigma^r(X))\cap\eta)\\ &\quad= \int_{\Sigma^r(X)}\int_{N^r(X,x)\cap\eta_x}\sum_{1\le i_1<\ldots<i_{r-j}\le r}\times\\ &\qquad\times k_{i_1}(x,u)\cdots k_{i_{r-j}}(x,u)\,\mathrm{d}\mathcal{H}^{d-1-r}(u)\,\mathrm{d}\mathcal{H}^r(x).\end{aligned}$$

The proof is completed, if we observe that

$$\begin{aligned}0 &= \int_{\mathcal{N}(\Sigma^r(X))\setminus\mathcal{N}^r(\Sigma^r(X))} \mathrm{ap}J_r\pi_1(x,u)\,\mathrm{d}\mathcal{H}^{d-1}(x,u)\\ &= \int_{\Sigma^r(X)}\mathcal{H}^{d-1-r}(N(X,x)\cap S^{d-1}\setminus N^r(X,x))\,\mathrm{d}\mathcal{H}^r(x).\end{aligned}$$

□

Remark. A statement similar to that of Theorem 3.8 has been found by Anzellotti & Ossanna [5, Theorem 3.3 (i)] in the case $d=3$, $r=1$, and $j=0$.

Now we turn our attention to the disintegration of the measure $\Theta_{d-2}(X,\cdot)$ for a set X with positive reach. First we have to check a measurability statement.

Lemma 3.9. *Let $X\subset\mathbb{R}^d$ be a set of positive reach, and let $r\in\{0,\ldots,d-1\}$. Then, for an arbitrary Borel set $\eta\in\mathfrak{B}(\mathbb{R}^d\times S^{d-1})$, the mapping*

$$h(\cdot,\eta):\Sigma^r(X)\setminus\Sigma^{r-1}(X)\to\mathbb{R},\quad x\mapsto\mathcal{H}^{d-1-r}(N(X,x)\cap\eta_x),$$

is Borel measurable.

Proof. We first consider the case of a closed set η. Let $\mathcal{F}$ denote the space of closed subsets of $\mathbb{R}^d$ endowed with the usual topology, see e.g. [21] or [25]. M. Zähle [27, Theorem 2.1.3] has shown that the map $\mathcal{F}\to\mathbb{R}$, $F\mapsto\mathcal{H}^s(F)$, is Borel measurable for any $s\ge 0$. Hence, it is sufficient to prove the measurability of the map $X\to\mathcal{F}$, $x\mapsto N(X,x)\cap\eta_x$. Since the intersection map $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$, $(F,G)\mapsto F\cap G$, is Borel measurable [21, 1-2, Corollary 1], it remains to prove that $x\mapsto N(X,x)$ and $x\mapsto\eta_x$ are Borel measurable. For that purpose we show that these maps are upper semi-continuous. We shall only carry out the proof for the first map, the proof for the second map is straightforward then.

Let $x_i, x_0 \in X$, $i \in \mathbb{N}$, and $x_i \to x_0$ for $i \to \infty$. Let

$$u \in \limsup_{i\to\infty} N(X, x_i).$$

For $n \in \mathbb{N}$ set $B(u, n^{-1}) := \{x \in \mathbb{R}^d \mid \|u - x\| \leq n^{-1}\}$. Let n be fixed for the moment. According to [21, Proposition 1-2-3] there is a subsequence $(i_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that, for all $j \in \mathbb{N}$,

$$N(X, x_{i_j}) \cap B(u, n^{-1}) \neq \emptyset.$$

Thus we can choose $u_{i_j} \in N(X, x_{i_j}) \cap B(u, n^{-1})$ for all $j \in \mathbb{N}$. We may assume, without loss of generality, that $u_{i_j} \to \bar{u}$ for $j \to \infty$. Let $\epsilon_0 \in (0, \mathrm{reach}(X))$ and $\epsilon := (\|u\|+1)^{-1}\epsilon_0$. Observe that $x_{i_j} + \epsilon u_{i_j}$, $x_0 + \epsilon\bar{u} \in X^{\epsilon_0}$. Since $x_{i_j} \to x_0$ and thus $p_X(x_{i_j} + \epsilon u_{i_j}) \to p_X(x_0 + \epsilon\bar{u})$ for $j \to \infty$, and because of $x_{i_j} = p_X(x_{i_j} + \epsilon u_{i_j})$ for all $j \in \mathbb{N}$, we get – as in the proof of Lemma 3.1 – that $\bar{u} \in N(X, x_0) \cap B(u, n^{-1}) \neq \emptyset$. This implies $u \in N(X, x_0)$, since $n \in \mathbb{N}$ was arbitrary. According to [21, Proposition 1-2-4] this proves the upper semi-continuity, and hence the lemma for closed sets.

Now consider the class $\mathcal{A}$ of all sets $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$ such that $h(\cdot, \eta)$ is Borel measurable. $\mathcal{A}$ contains the closed subsets of $\mathbb{R}^d \times S^{d-1}$. If $(\eta_i)_{i\in\mathbb{N}}$ is a sequence of sets in $\mathcal{A}$ with $\eta_i \cap \eta_j = \emptyset$ for $i \neq j$, then $\cup_{i\in\mathbb{N}}\eta_i \in \mathcal{A}$. In fact, simply observe that η_x is always a Borel set for $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$, and

$$\mathcal{H}^{d-1-r}\left(N(X, x) \cap (\cup_{i\in\mathbb{N}}\eta_i)_x\right) = \sum_{i\in\mathbb{N}} \mathcal{H}^{d-1-r}\left(N(X, x) \cap (\eta_i)_x\right).$$

Here and for the remaining part of the proof we always assume that x is chosen from the set $\Sigma^r(X) \setminus \Sigma^{r-1}(X)$.

If $\eta \in \mathcal{A}$, then also $(\mathbb{R}^d \times S^{d-1}) \setminus \eta \in \mathcal{A}$. To see this, note that

$$N(X, x) \cap ((\mathbb{R}^d \times S^{d-1}) \setminus \eta)_x = N(X, x) \cap S^{d-1} \setminus (N(X, x) \cap \eta_x),$$

and hence

$$\begin{aligned}&\mathcal{H}^{d-1-r}(N(X, x) \cap ((\mathbb{R}^d \times S^{d-1}) \setminus \eta)_x)\\ &\quad = \mathcal{H}^{d-1-r}(N(X, x) \cap S^{d-1}) - \mathcal{H}^{d-1-r}(N(X, x) \cap \eta_x).\end{aligned}$$

Observe that $\dim N(X, x) = d - r$, and thus $\mathcal{H}^{d-1-r}(N(X, x) \cap S^{d-1}) < \infty$.

Hence, we have verified that $\mathcal{A}$ is a Dynkin class which contains the closed sets. This proves $\mathcal{A} = \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$, see [8, Theorem 1.6.1]. □

Remark. In the course of the proof for Lemma 3.9 we have shown the following additional fact: if $X \subset \mathbb{R}^d$ is a set of positive reach, if $s \geq 0$, and if $\eta \subset \mathbb{R}^d \times S^{d-1}$ is a closed set, then the map

$$X \to \mathbb{R}, \quad x \mapsto \mathcal{H}^s(N(X, x) \cap \eta_x),$$

is Borel measurable.

For the proof and the statement of Theorem 3.10 we write $\operatorname{reg} X$ for the set of all $x \in X$ such that $\dim N(X,x) = 1$, if $X \subset \mathbb{R}^d$ has positive reach. Hence, for $x \in \operatorname{reg} X$, we have

$$\operatorname{card}(N(X,x) \cap S^{d-1}) \in \{1,2\}.$$

Note, in particular, that if K is a convex body with nonempty interior, then

$$\operatorname{card}(N(K,x) \cap S^{d-1}) = 1$$

for all $x \in \operatorname{reg} K$. Therefore the following theorem especially holds for convex bodies with nonempty interior.

Theorem 3.10. *Let $X \subset \mathbb{R}^d$ be a set of positive reach. Assume that $N(X,x)$ contains exactly one unit vector for each $x \in \operatorname{reg} X$. Then, for any bounded set $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$,*

$$\Theta_{d-2}(X,\eta) = \int_{\operatorname{bd} X} \frac{\mathcal{H}^{\dim N(X,x)-1}(N(X,x) \cap \eta_x)}{\mathcal{H}^{\dim N(X,x)-1}(N(X,x) \cap S^{d-1})} \, \mathrm{d}C_{d-2}(X,x).$$

Proof. For a nonnegative Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}$ (with compact support) we obtain from a special case of Theorem 3.2 the subsequent equation

$$\begin{aligned} &(d-1) \int_{\Sigma^{d-2}(X)} f(x) \, \mathrm{d}C_{d-2}(X,x) \\ &\quad = \int_{\Sigma^{d-2}(X)} f(x) \, \mathcal{H}^1(N(X,x) \cap S^{d-1}) \, \mathrm{d}\mathcal{H}^{d-2}(x). \end{aligned} \tag{12}$$

We will apply equation (12) with the function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(x) := \mathbf{1}_{\Sigma^{d-2}(X) \setminus \Sigma^{d-3}(X)}(x) \frac{\mathcal{H}^1(N(X,x) \cap \eta_x)}{\mathcal{H}^1(N(X,x) \cap S^{d-1})}.$$

According to Lemma 3.1 and Lemma 3.9 this function is Borel measurable. We shall also use the fact that

$$C_{d-2}(X, \Sigma^{d-3}(X)) = 0 = \mathcal{H}^{d-2}(\Sigma^{d-3}(X)),$$

which follows from Corollary 3.4 and Lemma 3.1. Let $\sigma_X(x)$ be defined by $\{\sigma_X(x)\} = N(X,x) \cap S^{d-1}$ for $x \in \operatorname{reg} X$. Because of the assumption of the theorem, this is a uniquely determined unit vector. But then the mapping $\sigma_X : \operatorname{reg} X \to S^{d-1}$ is even continuous. The theorem will now be proved by first decomposing η, respectively $\Theta_{d-2}(X,\eta)$. Then Theorem 3.2 and equation (12) will be used.

$$
\begin{aligned}
&\Theta_{d-2}(X,\eta)\\
=\ &\Theta_{d-2}(X,(\Sigma^{d-2}(X)\times S^{d-1})\cap\eta)+\Theta_{d-2}(X,(\operatorname{reg}X\times S^{d-1})\cap\eta)\\
=\ &\frac{1}{d-1}\int_{\Sigma^{d-2}(X)}\mathcal{H}^1(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^{d-2}(x)+\\
&\qquad+\int_{\operatorname{reg}X\times S^{d-1}}\mathbf{1}_\eta(x,\sigma_X(x))\,\mathrm{d}\Theta_{d-2}(X,x,u)\\
=\ &\frac{1}{d-1}\int_{\Sigma^{d-2}(X)\setminus\Sigma^{d-3}(X)}\mathcal{H}^1(N(X,x)\cap\eta_x)\,\mathrm{d}\mathcal{H}^{d-2}(x)+\\
&\qquad+\int_{\operatorname{reg}X\times S^{d-1}}\mathbf{1}_\eta(x,\sigma_X(x))\,\mathrm{d}\Theta_{d-2}(X,x,u)\\
=\ &\frac{1}{d-1}\int_{\Sigma^{d-2}(X)\setminus\Sigma^{d-3}(X)}\frac{\mathcal{H}^1(N(X,x)\cap\eta_x)}{\mathcal{H}^1(N(X,x)\cap S^{d-1})}\times\\
&\qquad\qquad\qquad\times\mathcal{H}^1(N(X,x)\cap S^{d-1})\,\mathrm{d}\mathcal{H}^{d-2}(x)+\\
&\qquad+\int_{\operatorname{reg}X}\mathbf{1}_\eta(x,\sigma_X(x))\,\mathrm{d}C_{d-2}(X,x)\\
=\ &\int_{\Sigma^{d-2}(X)\setminus\Sigma^{d-3}(X)}\frac{\mathcal{H}^1(N(X,x)\cap\eta_x)}{\mathcal{H}^1(N(X,x)\cap S^{d-1})}\,\mathrm{d}C_{d-2}(X,x)+\\
&\qquad+\int_{\operatorname{reg}X}\frac{\mathcal{H}^0(N(X,x)\cap\eta_x)}{\mathcal{H}^0(N(X,x)\cap S^{d-1})}\,\mathrm{d}C_{d-2}(X,x)\\
=\ &\int_{\operatorname{bd}X}\frac{\mathcal{H}^{\dim N(X,x)-1}(N(X,x)\cap\eta_x)}{\mathcal{H}^{\dim N(X,x)-1}(N(X,x)\cap S^{d-1})}\,\mathrm{d}C_{d-2}(X,x).
\end{aligned}
$$

This proves the theorem. □

Remark. Theorem 3.2 implies that the fibre measures λ_x in the disintegration $(C_{d-2}(X,\cdot),\lambda_x)$ of $\Theta_{d-2}(X,\cdot)$ can be chosen according to

$$\lambda_x(\omega)=\frac{\mathcal{H}^{\dim N(X,x)-1}(N(X,x)\cap\omega)}{\mathcal{H}^{\dim N(X,x)-1}(N(X,x)\cap S^{d-1})},\quad \omega\in\mathfrak{B}(S^{d-1}),$$

if $x\in\operatorname{bd}X$, and for $x\notin\operatorname{bd}X$ we can for example choose an arbitrary but fixed probability measure on $\mathfrak{B}(S^{d-1})$ for λ_x, provided the assumptions of Theorem

3.2 are fulfilled.

The following special case deserves to be mentioned separately.

Corollary 3.11. *Let $K \in \mathcal{K}^d$ be a convex body with nonempty interior, and let $\omega \in \mathfrak{B}(S^{d-1})$. Then*

$$S_{d-2}(K,\omega) = \int_{\operatorname{bd} K} \frac{\mathcal{H}^{\dim N(K,x)-1}(N(K,x)\cap\omega)}{\mathcal{H}^{\dim N(K,x)-1}(N(K,x)\cap S^{d-1})}\,\mathrm{d}C_{d-2}(K,x).$$

4 Singular normal vectors

A convex body in particular is a set of positive reach. Hence, everything which has been proved in Section 3 for sets of positive reach especially holds for convex bodies. In the setting of convex geometry there is a strict correspondence between the singular normal vectors of a convex body $K \in \mathcal{K}^d$ which contains the origin in its interior ($o \in \operatorname{int} K$) and the singular boundary points of the polar body K^*, where

$$K^* := \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq 1 \quad \forall y \in K\}.$$

Thus it is not surprising that theorems can be established similar to those of Section 3 which reflect this duality.

Lemma 4.1. *For a convex body $K \in \mathcal{K}^d$ and $r \in \{0,\dots,d-1\}$, the set $\Sigma_r(K)$ of r-singular normal vectors of K is a countably r-rectifiable Borel set.*

Proof. First, let us assume that $\operatorname{int} K \neq \emptyset$. Then we can assume that $o \in \operatorname{int} K$, because all notions involved are invariant with respect to translations. It is easy to see, cf., e.g., [18, Lemma 2.1], that $u \in S^{d-1}$ is an r-singular normal vector of K, if and only if $h_K(u)^{-1}u$ is an r-singular boundary point of the polar body K^*. By h_K we denote the support function of the convex body K. In other words $\Sigma_r(K) = \eta(\Sigma^r(K^*))$, where $\eta : \operatorname{bd} K^* \to S^{d-1}$, $x \mapsto \|x\|^{-1}x$. This proves the lemma for a convex body with interior points, since η is a bi-Lipschitz homeomorphism.

The general situation is easily reduced to the preceding one by applying the result of the first part of the proof in the affine hull of a given convex body $K \in \mathcal{K}^d$. □

For the sake of completeness we mention the following proposition, although it will not be used in the sequel.

Proposition 4.2. *Let $K \in \mathcal{K}^d$ and $r \in \{0,\dots,d-1\}$. Then*

$$\Sigma_r(K) \subset M_0 \cup \bigcup_{i\in\mathbb{N}} M_i,$$

where $\mathcal{H}^r(M_0) = 0$ and, for each $i \in \mathbb{N}$, M_i is an r-dimensional embedded submanifold of S^{d-1} which is of class C^2.

Proof. The proof follows from an application of Alberti's Theorem 1 [1] to the restriction of the support function h_K to a finite number of suitably chosen hyperplanes which do not contain the origin. In addition, Theorem 1.7.4 from Schneider's book [24] is used. Alternatively, a proof can be given which uses the C^2-rectifiability of the set of singular boundary points of the polar body and the arguments of the proof for Lemma 4.1 □

In Section 2 the generalized curvature measures have been defined for sets with positive reach as measures on bounded sets of $\mathfrak{B}(\mathbb{R}^d \times S^{d-1})$. As far as convex bodies are concerned the restriction to bounded sets is no longer necessary. This remark should be kept in mind in the following.

Theorem 4.3. *Let $K \in \mathcal{K}^d$ be a convex body, let $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$, and $r \in \{0, \ldots, d-1\}$. Then*

$$\int_{\Sigma_r(K)} \mathcal{H}^{d-1-r}(F(K,u) \cap \eta^u) \, \mathrm{d}\mathcal{H}^r(u)$$
$$= \binom{d-1}{r} \Theta_{d-1-r}(K, (\mathbb{R}^d \times \Sigma_r(K)) \cap \eta),$$

where $\eta^u := \{x \in \mathbb{R}^d \mid (x,u) \in \eta\}$.

Proof. As in the proof of Theorem 3.2 the set $\Sigma_r(K)$ can be assumed to be an r-rectifiable Borel set. Define

$$\mathcal{N}(\Sigma_r(K)) := \{(x,u) \in \mathcal{N}(K) \mid u \in \Sigma_r(K)\}.$$

Then $\mathcal{N}(\Sigma_r(K))$ and $\mathcal{N}(K)$ are $(\mathcal{H}^{d-1}, d-1)$-rectifiable Borel sets, $\mathcal{N}(\Sigma_r(K)) \subset \mathcal{N}(K)$, and the corresponding approximate tangent spaces coincide for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$. In addition, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$, at least $(d-1-r)$ of the generalized curvatures vanish, i.e., $k_1(x,u) = \ldots = k_{d-1-r}(x,u) = 0$ and $0 \le k_{d-r}(x,u) \le \ldots \le k_{d-1}(x,u) \le \infty$. To see this, let $(x,u) \in \mathcal{N}(\Sigma_r(K))$ be such that $\sigma_K|_{\mathrm{bd}\, K^\epsilon}$ is differentiable at $x + \epsilon u$, where $0 < \epsilon < \infty$ is arbitrarily chosen. Since $F(K,u)$ is convex and $\dim F(K,u) \ge d-1-r$, there are at least $(d-1-r)$ linearly independent unit vectors $v_1, \ldots, v_{d-1-r} \perp u$ such that $D\sigma_K(x+\epsilon u)(v_i) = 0$ for $i \in \{1, \ldots, d-1-r\}$. Thus the dimension of the eigenspace of $D\sigma_K(x+\epsilon u)|_{u^\perp}$ associated with the eigenvalue 0 is at least $(d-1-r)$. Also observe that, due to the convexity of K, all generalized curvatures are nonnegative.

Let us assume $r > 0$. For the projection map

$$\pi_2 : \mathcal{N}(\Sigma_r(K)) \to \Sigma_r(K), \quad (x,u) \mapsto u,$$

we prove that

$$\mathrm{ap}J_r\pi_2(x,u) = \prod_{i=d-r}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}.$$

In fact, if we adopt the notation from the proof of Theorem 3.2, we get, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$,

$$w_1(x,u), \ldots, w_{d-1-r}(x,u) \in \mathrm{kernel}(\mathrm{ap}D\pi_2(x,u)),$$

since

$$\mathrm{ap}D\pi_2(x,u)(w_i(x,u)) = \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} u_i,$$

for $i \in \{1,\ldots,d-1\}$ and $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$. For $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$ we distinguish two cases. If $k_{d-r}(x,u) > 0$, then

$$(\mathrm{kernel}(\mathrm{ap}D\pi_2(x,u)))^{\perp} = \mathrm{lin}\{w_{d-r}(x,u), \ldots, w_{d-1}(x,u)\},$$

and thus

$$\begin{aligned} \mathrm{ap}J_r\pi_2(x,u) &= \left\| \bigwedge_{i=d-r}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} u_i \right\| \\ &= \prod_{i=d-r}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}. \end{aligned}$$

If, however, $k_{d-r}(x,u) = 0$, then

$$\mathrm{ap}J_r\pi_2(x,u) = 0 = \prod_{i=d-r}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}.$$

Now, Federer's coarea formula yields

$$\begin{aligned} &\int_{\mathcal{N}(\Sigma_r(K))\cap\eta} \prod_{i=d-r}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} \, \mathrm{d}\mathcal{H}^{d-1}(x,u) \\ &\quad = \int_{\Sigma_r(K)} \mathcal{H}^{d-1-r}(F(K,u)\cap\eta^u) \, \mathrm{d}\mathcal{H}^r(u). \end{aligned} \tag{13}$$

We also have

$$\begin{aligned} &\int_{\mathcal{N}(\Sigma_r(K))\cap\eta} \sum_{1\le i_1<\ldots<i_r\le d-1} \frac{k_{i_1}\cdots k_{i_r}}{\prod_{i=1}^{d-1}\sqrt{1+k_i^2}} \, \mathrm{d}\mathcal{H}^{d-1} \\ &\quad = \binom{d-1}{r} \Theta_{d-1-r}(K, (\mathbb{R}^d \times \Sigma_r(K))\cap\eta). \end{aligned} \tag{14}$$

The sum under the integral corresponding to $i_1 = d-r, \dots, i_r = d-1$ is equal to

$$\frac{k_{d-r}(x,u)\cdots k_{d-1}(x,u)}{\prod_{i=1}^{d-1}\sqrt{1+k_i(x,u)^2}} = \prod_{i=d-r}^{d-1}\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}.$$

In the remaining cases we have $i_1 \le d-r-1$, and hence $k_{i_1}(x,u) = 0$. Thus we see that equation (14) can be simplified to

$$\int_{\mathcal{N}(\Sigma_r(K))\cap\eta}\prod_{i=d-r}^{d-1}\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}\,\mathrm{d}\mathcal{H}^{d-1}(x,u)$$
$$= \binom{d-1}{r}\Theta_{d-1-r}(K,(\mathbb{R}^d\times\Sigma_r(K))\cap\eta). \qquad (15)$$

Equations (13) and (15) taken together complete the proof of the theorem in the case $r > 0$.

Now, let $r = 0$. Then $\Sigma_0(K) = \{u_\iota \mid \iota \in I\}$, where I is at most countable. Further on, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_0(K))$,

$$k_1(x,u) = \ldots = k_{d-1}(x,u) = 0.$$

Hence,

$$\begin{aligned}
&\Theta_{d-1}(K,(\mathbb{R}^d\times\Sigma_0(K))\cap\eta)\\
&\quad= \int_{\mathcal{N}(\Sigma_0(K))\cap\eta}\prod_{i=1}^{d-1}\frac{1}{\sqrt{1+k_i^2}}\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad= \mathcal{H}^{d-1}(\mathcal{N}(\Sigma_0(K))\cap\eta)\\
&\quad= \sum_{\iota\in I}\mathcal{H}^{d-1}((F(K,u_\iota)\times\{u_\iota\})\cap\eta)\\
&\quad= \sum_{\iota\in I}\mathcal{H}^{d-1}(F(K,u_\iota)\cap\eta^{u_\iota})\\
&\quad= \int_{\Sigma_0(K)}\mathcal{H}^{d-1}(F(K,u)\cap\eta^u)\,\mathrm{d}\mathcal{H}^0(u),
\end{aligned}$$

and this finishes the proof. □

An immediate conclusion can be drawn from equation (13) in the proof of Theorem 4.3.

Corollary 4.4. *Let $K \in \mathcal{K}^d$, $r \in \{0,\dots,d-1\}$, and $\eta \in \mathfrak{B}(\mathbb{R}^d\times S^{d-1})$. Then*

$$\int_{\Sigma_r(K)}\mathcal{H}^{d-1-r}(F(K,u)\cap\eta^u)\,\mathrm{d}\mathcal{H}^r(u) \le \mathcal{H}^{d-1}(\mathcal{N}(\Sigma_r(K))\cap\eta).$$

Next we shall investigate the restriction of $\Theta_j(K,\cdot)$ to the set $\mathbb{R}^d \times \Sigma_r(K)$, if $j \neq r$. The corresponding results are contained in Corollary 4.5 and in Theorem 4.9.

Corollary 4.5. *Let $K \in \mathcal{K}^d$, $r \in \{0,\dots,d-2\}$, and $j \in \{0,\dots,d-2-r\}$. Then we have $\Theta_j(K,\mathbb{R}^d \times \Sigma_r(K)) = 0$.*

Proof. The assumption guarantees that $d-1-j \geq r+1$. For $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(\Sigma_r(K))$ at least $(d-1-r)$ of the generalized curvatures $k_1(x,u),\dots,k_{d-1}(x,u)$ are equal to 0. Hence, each summand in the representation of $\Theta_j(K,\mathbb{R}^d \times \Sigma_r(K))$ as an integral over the unit normal bundle $\mathcal{N}(K)$ contains at least one factor which is equal to 0. □

There is also a special form of the local Steiner formula in the context of singular normal vectors.

Corollary 4.6. *Let $K \in \mathcal{K}^d$, $r \in \{0,\dots,d-1\}$, and $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$. Then*

$$\begin{aligned}\lambda^d(M_\rho(K,\check{\eta}_r)) &= \frac{1}{r+1}\rho^{r+1}\int_{\Sigma_r(K)} \mathcal{H}^{d-1-r}(F(K,u)\cap\eta^u)\,\mathrm{d}\mathcal{H}^r(u) + \\ &\quad + \frac{1}{d}\sum_{j=d-r}^{d-2}\rho^{d-j}\binom{d}{j}\Theta_j(K,\check{\eta}_r) + \\ &\quad + \rho\mathcal{H}^{d-1}(\{x \in \mathrm{bd}\,K \mid (\{x\}\times N(K,x))\cap\check{\eta}_r \neq \emptyset\}),\end{aligned}$$

if $\check{\eta}_r := (\mathbb{R}^d \times \Sigma_r(K))\cap\eta$.

Proof. First, the local Steiner formula (4) and Corollary 4.5 are used. In addition, an application of the coarea formula to the map $\pi_1 : \mathcal{N}(K) \to \mathrm{bd}\,K$, $(x,u) \mapsto x$, and Theorem 2.2.4 from [24] yield that

$$\begin{aligned}\frac{1}{d}\rho d\Theta_{d-1}(K,\check{\eta}_r) &= \rho\int_{\mathcal{N}(K)\cap\check{\eta}_r}\prod_{i=1}^{d-1}\frac{1}{\sqrt{1+k_i^2}}\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \rho\int_{\mathrm{bd}\,K}\int_{\pi_1^{-1}(\{x\})}\mathbf{1}_{\check{\eta}_r}\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \rho\mathcal{H}^{d-1}(\{x \in \mathrm{bd}\,K \mid (\{x\}\times N(K,x))\cap\check{\eta}_r \neq \emptyset\}).\end{aligned}$$

Observe that the approximate Jacobian of π_1 is given by

$$\mathrm{ap}J_{d-1}\pi_1(x,u) = \prod_{i=1}^{d-1}\frac{1}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$, see the proof of Theorem 3.2. □

Remark. If $r = d-1$, then Corollary 4.6 coincides with the local Steiner formula, since then $\Sigma_{d-1}(K) = S^{d-1}$ and hence $\check{\eta}_{d-1} = \eta$.

For $r = 0$, the statement of Corollary 4.6 has to be interpreted as

$$\begin{aligned} \lambda^d(M_\rho(K, \check{\eta}_0)) &= \rho \int_{\Sigma_0(K)} \mathcal{H}^{d-1}(F(K,u) \cap \eta^u)\, \mathrm{d}\mathcal{H}^0(u) \\ &= \rho \mathcal{H}^{d-1}(\{x \in \mathrm{bd}\, K \mid (\{x\} \times N(K,x)) \cap \check{\eta}_0 \neq \emptyset\}). \end{aligned}$$

By specializing the result of Theorem 4.3 we now obtain as a corollary estimates from above for the r-dimensional Hausdorff measure of certain subsets $\Sigma_r(K,\tau)$ of the set of r-singular normal vectors. As in Section 1, the definition of these subsets takes into account the strength of the singularities. To be precise, let $\Sigma_r(K,\tau)$ be the subset of $\Sigma_r(K)$ which is defined by

$$\Sigma_r(K,\tau) := \{u \in S^{d-1} \mid \mathcal{H}^{d-1-r}(F(K,u)) \geq \tau\},$$

for $\tau > 0$. Note that $F(K,u)$ is equal to the reverse spherical image of K at u, cf. [24].

Corollary 4.7. *Let $K \in \mathcal{K}^d$ and $r \in \{0, \dots, d-1\}$. Then*

$$\int_{\Sigma_r(K)} \mathcal{H}^{d-1-r}(F(K,u))\, \mathrm{d}\mathcal{H}^r(u) \leq d\binom{d-1}{r} W_{r+1}(K),$$

and, for $\tau > 0$,

$$\mathcal{H}^r(\Sigma_r(K,\tau)) \leq \frac{d}{\tau}\binom{d-1}{r} W_{r+1}(K).$$

Before we can state Theorem 4.9 two definitions are required. For $K \in \mathcal{K}^d$ let

$$\mathcal{N}_r(K) := \{(x,u) \in \mathcal{N}(K) \mid x + u \in \mathcal{D}_K \text{ and } k_{d-r}(x,u), \dots, k_{d-1}(x,u) > 0\}$$

and

$$\mathcal{N}_r(\Sigma_r(K)) := \mathcal{N}_r(K) \cap (\mathbb{R}^d \times \Sigma_r(K)).$$

Hence, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}_r(\Sigma_r(K))$, we have

$$k_1(x,u) = \ldots = k_{d-1-r}(x,u) = 0 \quad \text{and} \quad k_{d-r}(x,u), \dots, k_{d-1}(x,u) > 0.$$

Since the proof for Lemma 4.8 is very much the same as the one for Lemma 3.7, we omit the argument.

Lemma 4.8. *Let $K \in \mathcal{K}^d$, and let $r \in \{0, \dots, d-1\}$. Then $\mathcal{N}_r(\Sigma_r(K))$ is a Borel measurable set.*

Theorem 4.9. *Let $K \in \mathcal{K}^d$, $r \in \{1, \ldots, d-1\}$, $j \in \{d-r, \ldots, d-1\}$, and let $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$. Then*

$$\begin{aligned}&\binom{d-1}{j}\Theta_j(K, \mathcal{N}_r(\Sigma_r(K)) \cap \eta)\\ &= \int_{\Sigma_r(K)} \int_{F(K,u)\cap\eta^u} \sum_{d-r\le i_1<\ldots<i_{r+j+1-d}\le d-1} \times\\ &\qquad \times r_{i_1}(x,u)\cdots r_{i_{r+j+1-d}}(x,u)\,\mathrm{d}\mathcal{H}^{d-1-r}(x)\,\mathrm{d}\mathcal{H}^r(u),\end{aligned}$$

where $r_i(x,u) := k_i(x,u)^{-1} \in [0,\infty)$ for $i \in \{d-r, \ldots, d-1\}$ and $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$.

Proof. The proof of Theorem 4.9 is very similar to that of Theorem 3.8. Thus we only mention that the products $r_{i_1}(x,u)\cdots r_{i_{r+j+1-d}}(x,u)$, which appear in the statement of the theorem, are almost everywhere well-defined, since

$$\begin{aligned}0 &= \int_{\mathcal{N}(\Sigma_r(K))\setminus\mathcal{N}_r(\Sigma_r(K))} \mathrm{ap}J_r\pi_2(x,u)\,\mathrm{d}\mathcal{H}^{d-1}(x,u)\\ &= \int_{\Sigma_r(K)} \mathcal{H}^{d-1-r}(F(K,u)\setminus F_r(K,u))\,\mathrm{d}\mathcal{H}^r(u).\end{aligned}$$

Here we have set $F_r(K,u) := (\mathcal{N}_r(K))^u$. □

In the introduction we described the problem of finding an explicit representation for the fibre measures λ_x in the disintegration $(C_r(K,\cdot),\lambda_x)$ of the generalized curvature measure $\Theta_r(K,\cdot)$, where $K \in \mathcal{K}^d$ is a convex body with nonempty interior and $r \in \{0,\ldots,d-1\}$. Thus, we considered the disintegration of $\Theta_r(K,\cdot)$ which is induced by the projection π_1 onto the first component of $\mathbb{R}^d \times S^{d-1}$. A solution for this problem was given in Section 3 for the case $r = d-2$. The case $r = d-1$ can easily be treated by Theorem 3.2.

Similarly, we now pose the corresponding question for the disintegration of $\Theta_r(K,\cdot)$, for a convex body $K \in \mathcal{K}^d$ and $r \in \{0,\ldots,d-1\}$, which is induced by the projection $\pi_2 : \mathbb{R}^d \times S^{d-1} \to S^{d-1}$, $(x,u) \mapsto u$, onto the second component of $\mathbb{R}^d \times S^{d-1}$. Since $S_r(K,\cdot) = \Theta_r(K,\cdot)\circ\pi_2^{-1}$, we again obtain from abstract measure theory that there are probability measures λ^u on $\mathfrak{B}(\mathbb{R}^d)$, for each $u \in S^{d-1}$, such that

$$\Theta_r(K,\eta) = \int_{S^{d-1}} \int_{\mathbb{R}^d} \mathbf{1}_\eta(x,u)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}S_r(K,u),$$

for $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$. Note that the measures λ^u in the disintegration $(S_r(K,\cdot),\lambda^u)$ of $\Theta_r(K,\cdot)$ may depend on the index $r \in \{0,\ldots,d-1\}$ and that they are uniquely determined up to a set of $S_r(K,\cdot)$ measure zero. The remaining part of this article is devoted to determining an explicit expression for the measures λ^u in the disintegration $(S_1(K,\cdot),\lambda^u)$ of $\Theta_1(K,\cdot)$, i.e., to a solution of the just stated problem for $r = 1$. The case $r = 0$ immediately follows from Theorem 4.3.

First of all a measurability statement is required which is provided by the next Lemma.

Lemma 4.10. *Let $K \in \mathcal{K}^d$ and $r \in \{0, \dots, d-1\}$. Then, for an arbitrary Borel set $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$, the mapping*

$$g(\cdot, \eta) : \Sigma_r(K) \setminus \Sigma_{r-1}(K) \to \mathbb{R}, \quad u \mapsto \mathcal{H}^{d-1-r}(F(K,u) \cap \eta^u),$$

is Borel measurable.

Proof. Again the proof is similar to that of the corresponding Lemma 3.9 in Section 3. Here, more generally one shows that the map

$$S^{d-1} \to \mathbb{R}, \quad u \mapsto \mathcal{H}^s(F(K,u) \cap \eta^u),$$

$s \geq 0$, is Borel measurable for closed sets $\eta \subset \mathbb{R}^d \times S^{d-1}$ by first verifying the upper semi-continuity of the mapping

$$S^{d-1} \to \mathcal{F}, \quad u \mapsto F(K,u).$$

Then the usual Dynkin class argument is applied. In particular note that $\mathcal{H}^{d-1-r}(F(K,u)) < \infty$ for $u \in \Sigma_r(K) \setminus \Sigma_{r-1}(K)$. □

Theorem 4.11. *Let $K \in \mathcal{K}^d$ and $\eta \in \mathfrak{B}(\mathbb{R}^d \times S^{d-1})$. Then*

$$\Theta_1(K, \eta) = \int_{S^{d-1}} \frac{\mathcal{H}^{\dim F(K,u)}(F(K,u) \cap \eta^u)}{\mathcal{H}^{\dim F(K,u)}(F(K,u))} \, \mathrm{d}S_1(K,u).$$

Proof. If $f : S^{d-1} \to \mathbb{R}$ is a nonnegative Borel measurable function, a special case of Theorem 4.3 yields

$$\begin{aligned}(d-1) &\int_{\Sigma_{d-2}(K)} f(u) \, \mathrm{d}S_1(K,u) \\ &= \int_{\Sigma_{d-2}(K)} f(u) \, \mathcal{H}^1(F(K,u)) \, \mathrm{d}\mathcal{H}^{d-2}(u). \qquad (16)\end{aligned}$$

Equation (16) will be applied with the function $f : S^{d-1} \to \mathbb{R}$ given by

$$f(u) := \mathbf{1}_{\Sigma_{d-2}(K) \setminus \Sigma_{d-3}(K)}(u) \frac{\mathcal{H}^1(F(K,u) \cap \eta^u)}{\mathcal{H}^1(F(K,u))}.$$

The measurability of f is implied by Lemma 4.1 and Lemma 4.10. In addition, we shall use that

$$S_1(K, \Sigma_{d-3}(K)) = 0 = \mathcal{H}^{d-2}(\Sigma_{d-3}(K)),$$

see Corollary 4.5 and Lemma 4.1. Write $\operatorname{regn} K$ for the set of all $u \in S^{d-1}$ such that $\dim F(K,u) = 0$, and let $\tau_K(u)$ be defined by $\{\tau_K(u)\} = F(K,u)$ for $u \in \operatorname{regn} K$. For the proof of the theorem we now decompose η respectively $\Theta_1(K,\eta)$, and subsequently we use Theorem 4.3 and equation (16).

$$\begin{aligned}
& \Theta_1(K,\eta) \\
= \; & \Theta_1(K,(\mathbb{R}^d \times \Sigma_{d-2}(K)) \cap \eta) + \Theta_1(K,(\mathbb{R}^d \times \operatorname{regn} K) \cap \eta) \\
= \; & \frac{1}{d-1} \int_{\Sigma_{d-2}(K)} \mathcal{H}^1(F(K,u) \cap \eta^u) \, \mathrm{d}\mathcal{H}^{d-2}(u) + \\
& \quad + \int_{\mathbb{R}^d \times \operatorname{regn} K} \mathbf{1}_\eta(\tau_K(u),u) \, \mathrm{d}\Theta_1(K,x,u) \\
= \; & \frac{1}{d-1} \int_{\Sigma_{d-2}(K) \setminus \Sigma_{d-3}(K)} \mathcal{H}^1(F(K,u) \cap \eta^u) \, \mathrm{d}\mathcal{H}^{d-2}(u) + \\
& \quad + \int_{\mathbb{R}^d \times \operatorname{regn} K} \mathbf{1}_\eta(\tau_K(u),u) \, \mathrm{d}\Theta_1(K,x,u) \\
= \; & \int_{\Sigma_{d-2}(K) \setminus \Sigma_{d-3}(K)} \frac{\mathcal{H}^1(F(K,u) \cap \eta^u)}{\mathcal{H}^1(F(K,u))} \, \mathrm{d}S_1(K,u) \\
& \quad + \int_{\operatorname{regn} K} \frac{\mathcal{H}^0(F(K,u) \cap \eta^u)}{\mathcal{H}^0(F(K,u))} \, \mathrm{d}S_1(K,u) \\
= \; & \int_{S^{d-1}} \frac{\mathcal{H}^{\dim F(K,u)}(F(K,u) \cap \eta^u)}{\mathcal{H}^{\dim F(K,u)}(F(K,u))} \, \mathrm{d}S_1(K,u).
\end{aligned}$$

□

Remark. From Theorem 4.11 we obtain that the fibre measures λ^u in the disintegration $(S_1(K,\cdot),\lambda^u)$ of $\Theta_1(K,\cdot)$ can be chosen according to

$$\lambda^u(\beta) = \frac{\mathcal{H}^{\dim F(K,u)}(F(K,u) \cap \beta)}{\mathcal{H}^{\dim F(K,u)}(F(K,u))}, \quad \beta \in \mathfrak{B}(\mathbb{R}^d),$$

if $u \in S^{d-1}$.

Specializing Theorem 4.11 we can now express $C_1(K,\cdot)$ as an integral with respect to $S_1(K,\cdot)$.

Corollary 4.12. *Let $K \in \mathcal{K}^d$ and $\beta \in \mathfrak{B}(\mathbb{R}^d)$. Then*

$$C_1(K,\beta) = \int_{S^{d-1}} \frac{\mathcal{H}^{\dim F(K,u)}(F(K,u)\cap\beta)}{\mathcal{H}^{\dim F(K,u)}(F(K,u))}\,\mathrm{d}S_1(K,u).$$

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