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Absolute Continuity for Curvature Measures of Convex Sets I

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Abstract. Let us denote by $C_r(K,\cdot)$, $r \in \{0,\dots,d-1\}$, the curvature measures of a convex body K in Euclidean space $\mathbb{R}^d$ with $d \geq 2$. According to Lebesgue's decomposition theorem the curvature measure of order r of K, $C_r(K,\cdot)$, can be written as the sum of an absolutely continuous measure, $C_r^a(K,\cdot)$, and a singular measure, $C_r^s(K,\cdot)$, with respect to $(d-1)$-dimensional Hausdorff measure. For example, if K is a polytope, then $C_r^a(K,\cdot)=0$ and if K is sufficiently smooth, then $C_r^s(K,\cdot)=0$. For a general convex body K, a description of $C_r^a(K,\cdot)$ in terms of geometric quantities is known. In the present paper, we provide a corresponding explicit representation for the singular part $C_r^s(K,\cdot)$. Further, denote by $\Sigma^r(K)$ the set of r-singular boundary points of the convex body K. It is known that $\Sigma^r(K)$ has σ-finite, but possibly infinite, r-dimensional Hausdorff measure. Provided that the singular part $C_r^s(K,\cdot)$ of the curvature measure of order r vanishes, for a given convex body K, we prove that the r-dimensional Hausdorff measure of $\Sigma^r(K)$ also vanishes. Examples show that in a certain sense this result is sharp. Analogous results are established for surface area measures and singular normal vectors.

1. Introduction

With a general convex body K (nonempty compact convex set) in $\mathbb{R}^d$, $d \geq 2$, two series of measures can be associated, the curvature measures $C_r(K,\cdot)$ and the surface area measures $S_r(K,\cdot)$, where $r \in \{0,\dots,d-1\}$. The former are defined on the σ-algebra $\mathfrak{B}(\mathbb{R}^d)$ of Borel sets of Euclidean space $\mathbb{R}^d$, and the latter are measures on the Borel sets $\mathfrak{B}(S^{d-1})$ of the unit sphere, cf. Schneider [20]. Our aim is to explore the relationship between properties of these measures and the geometry of the corresponding convex bodies.

First, let us consider two extreme cases. If the convex body K is a polytope, then

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the curvature measure $C_r(K,\cdot)$ can be represented by

$$C_r(K,\beta) = \frac{d\kappa_{d-r}}{\binom{d}{r}} \sum_{F\in\mathcal{F}_r(K)} \gamma(F,K)\mathcal{H}^r(F\cap\beta),$$

if $\beta \in \mathfrak{B}(\mathbb{R}^d)$ and $r \in \{0,\dots,d-1\}$. The quantity $\gamma(F,K)$ is the external angle of K at the face F, $\mathcal{H}^r$ denotes r-dimensional Hausdorff measure, $\mathcal{F}_r(K)$ is the set of r-dimensional faces of K, and κ_i is the volume of the i-dimensional unit ball. See Schneider [20] for precise definitions. In particular, this representation shows that the support of the measure $C_r(K,\cdot)$, $\operatorname{supp} C_r(K,\cdot)$, can be covered by finitely many r-dimensional planes, if K is a polytope. What about the reverse statement? Is a convex body K with $\dim K \geq r+1$ a polytope, provided that $r \in \{0,\dots,d-1\}$ and $\operatorname{supp} C_r(K,\cdot)$ is contained in the union of finitely many r-dimensional planes? The answer is in the affirmative and a proof will be given in Section 4. The corresponding result for surface area measures is due to Goodey and Schneider [8], see also Theorem 4.6.4 in Schneider [20].

Next, assume that the convex body K is of class C^2, that is, the boundary of K is a $(d-1)$-dimensional submanifold of class C^2. Write $\operatorname{bd} K$ for the topological boundary of K. It is well known that then

$$C_r(K,\beta) = \int_{\operatorname{bd} K\cap\beta} H_{d-1-r}(K,x)\,\mathrm{d}\mathcal{H}^{d-1}(x), \quad \beta \in \mathfrak{B}(\mathbb{R}^d),$$

where $H_{d-1-r}(K,\cdot)$ is the normalized elementary symmetric function of order $(d-1-r)$ of the principle curvatures of the boundary of K. More generally, if, for a given convex body K, there is a Borel measurable function $f : \operatorname{bd} K \to [0,\infty)$ such that

$$C_r(K,\beta) = \int_{\operatorname{bd} K\cap\beta} f(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)$$

holds for all $\beta \in \mathfrak{B}(\mathbb{R}^d)$, then $C_r(K,\cdot)$ is said to be absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure. In this situation we write $C_r(K,\cdot) \ll \mathcal{H}^{d-1}$. Since the restriction of the $(d-1)$-dimensional Hausdorff measure to the boundary of K, $\mathcal{H}^{d-1} \llcorner \operatorname{bd} K$, is a σ-finite measure, the Radon-Nikodym theorem, Theorem 4.2.2 in [2, p. 132], yields that $C_r(K,\cdot) \ll \mathcal{H}^{d-1}$ holds true if and only if $\mathcal{H}^{d-1}(\beta) = 0$ implies $C_r(K,\beta) = 0$ for all $\beta \in \mathfrak{B}(\mathbb{R}^d)$.

Now the following questions arise naturally. What can be said about a convex body K for which $C_r(K,\cdot)$ is absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure? Furthermore, are there geometric properties of a convex body K that are equivalent to the assumption that the r-th curvature measure $C_r(K,\cdot)$, for some $r \in \{0,\dots,d-2\}$, is absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure*? Answers to these questions can be viewed as contributions to the general problem of retrieving geometric information about a convex body K from partial information about a functional $f(K)$ which is associated with the convex body

*This problem was pointed out by Rolf Schneider at a workshop held at Grado/Italy.

K. The present purpose is to initiate a thorough investigation of the cases where $f(K) = C_r(K, \cdot)$ or $f(K) = S_r(K, \cdot)$, for some $r \in \{0, \dots, d-1\}$, and $f(K)$ is assumed to be absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure.

A first observation is the following. Let K be an arbitrary convex body and $r \in \{0, \dots, d-1\}$. Since $C_r(K, \cdot)$ is concentrated on $\operatorname{bd} K$ and $\mathcal{H}^{d-1} \llcorner \operatorname{bd} K$ is σ-finite, the Lebesgue decomposition theorem yields that the measure $C_r(K, \cdot)$ can be decomposed into an absolutely continuous part, $C_r^a(K, \cdot)$, and a singular part, $C_r^s(K, \cdot)$, with respect to the restriction of $(d-1)$-dimensional Hausdorff measure to the boundary of K. Whereas the description of $C_r^a(K, \cdot)$ in terms of geometric quantities is essentially known, a corresponding explicit representation for the singular part $C_r^s(K, \cdot)$ is provided in Section 3. This representation is obtained by integrating suitably defined curvatures over restricted subsets of the unit normal bundle of the convex body K. It is this representation which prepares the ground for further investigations.

For a polytope K, for instance, we get $C_r(K, \cdot) = C_r^s(K, \cdot)$. In fact, this holds true for an arbitrary convex body K if and only if $H_{d-1-r}(K, x) = 0$ for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$. According to results of Zamfirescu and Aleksandrov, see Theorem 2.6.2 and the subsequent remarks in Schneider [20], this condition is satisfied for most convex bodies. The relation $C_r(K, \cdot) = C_r^a(K, \cdot)$ holds true, provided that K is of class C^2. This, however, is not a necessary condition. It will turn out that a convex body with an absolutely continuous curvature measure of order r can still have plenty of singular boundary points. On the other hand, absolute continuity implies, vaguely speaking, a severe restriction for the size (in a measure theoretic sense) of the set of singular boundary points.

To be more precise, let us denote by $N(K, x)$ the normal cone of the convex body K at the boundary point x. Furthermore, let $\Sigma^r(K)$ be the set of r-singular boundary points of K, that is,

$$\Sigma^r(K) := \{x \in \operatorname{bd} K : \dim N(K, x) \geq d - r\}, \tag{1.1}$$

for $r \in \{0, \dots, d-1\}$. Recall that $\Sigma^r(K)$ has σ-finite, but possibly infinite, r-dimensional Hausdorff measure. The restriction of $C_r(K, \cdot)$ to a set $\beta \in \mathfrak{B}(\mathbb{R}^d)$ is denoted by $C_r(K, \cdot) \llcorner \beta$. In Section 4 it will be proved that

$$C_r(K, \cdot) \llcorner \beta \ll \mathcal{H}^{d-1} \llcorner \beta \Longrightarrow \mathcal{H}^r(\Sigma^r(K) \cap \beta) = 0,$$

if $\beta \in \mathfrak{B}(\mathbb{R}^d)$ and $r \in \{0, \dots, d-2\}$. Examples in Section 5 show that in a certain sense this result cannot be improved.

Analogous results are obtained for surface area measures and singular normal vectors. Although this similarity of results and methods suggests an underlying duality, it will be necessary to provide independent proofs. The role of duality will be further studied in subsequent papers. As far as proofs are concerned, we use tools and concepts from convex geometry and some geometric measure theory. In particular, Federer's coarea formula is applied repeatedly, and also M. Zähle's [22] representation of (generalized) curvature measures by means of the unit normal bundle and generalized curvature functions plays a prominent role. Finally, it should be emphasized that our main results concerning curvature measures remain true, if nonempty closed convex sets $K \subset \mathbb{R}^d$, $K \neq \mathbb{R}^d$, are considered instead of convex bodies.

2. Preliminary remarks

Our general setting is given by the Euclidean space $\mathbb{R}^d$, $d \geq 2$, with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|$. As usual, let $B := \{x \in \mathbb{R}^d : |x| \leq 1\}$ be the unit ball and $S^{d-1} := \operatorname{bd} B$ the unit sphere. We investigate the interplay between geometry and measure theory. For terminology and results of measure theory we essentially follow the presentation in the books of Federer [7] and Evans and Gariepy [4], our main reference for convexity is Schneider's book [20]. The connection between convex geometry and measure theory has already been investigated in [22], [10], [11], [12]. To these references we refer for all notions which are not explicitly defined here.

In the following, we write $\mathcal{K}^d$ for the set of all convex bodies. For a convex body $K \in \mathcal{K}^d$, we shall use the support function $h_K = h(K, \cdot)$, the (extended) spherical image map σ_K, which is defined on $\mathbb{R}^d \setminus K$ and on the set $\operatorname{reg} K$ of regular boundary points, and the reverse spherical image map ξ_K, which is defined on the set $\operatorname{regn} K$ of regular normal vectors, cf. [20] and [10]. The support plane $H(K, u)$, for $u \in S^{d-1}$, and the corresponding halfspaces $H^-(K, u)$ and $H^+(K, u)$ are defined as in [20]. Furthermore, the normal cone $N(K, x)$, $x \in \operatorname{bd} K$, the support set $F(K, u)$, $u \in S^{d-1}$, and the generalized unit normal bundle $\mathcal{N}(K)$ are defined as in [22], [10], [12].

It is important for our method that, for a convex body $K \in \mathcal{K}^d$, generalized curvatures $k_i(x, u) \in [0, \infty]$, $i \in \{1, \ldots, d-1\}$, on the unit normal bundle $\mathcal{N}(K)$ can be defined for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$. See Zähle [22] and Kohlmann [15] or [10], [12] for the details. Henceforth, we will always assume that the ordering of these curvatures is such that

$$0 \leq k_1(x, u) \leq \ldots \leq k_{d-1}(x, u) \leq \infty. \tag{2.1}$$

For a normal boundary point $x \in \operatorname{bd} K$, due to Aleksandrov's theorem [1], there exist generalized principal curvatures $k_1(K, x), \ldots, k_{d-1}(K, x) \in [0, \infty)$. The reference to the convex body K is omitted, if there is no danger of ambiguity. These curvatures are the eigenvalues of the generalized second order differential of a suitable convex function which locally represents the boundary of K at x. Similarly, for a unit vector $u \in S^{d-1}$ such that h_K is second order differentiable at u, one can introduce the generalized principal radii of curvature $r_1(K, u), \ldots, r_{d-1}(K, u)$ as the eigenvalues of the second order differential $d^2 h_K(u)|_{u^\perp}$ of h_K at u. Details can be found, for example, in §2 of [11].

Normalized elementary symmetric functions of these curvatures and radii of curvature are defined, for suitable $x \in \operatorname{bd} K$ and $u \in S^{d-1}$, by

$$H_j(K, x) := \binom{d-1}{j}^{-1} \sum_{1 \leq i_1 < \ldots < i_j \leq d-1} k_{i_1}(K, x) \cdots k_{i_j}(K, x), \tag{2.2}$$

and

$$P_j(K, u) := \binom{d-1}{j}^{-1} \sum_{1 \leq i_1 < \ldots < i_j \leq d-1} r_{i_1}(K, u) \cdots r_{i_j}(K, u), \tag{2.3}$$

if $j \in \{1, \ldots, d-1\}$, and for $j = 0$ the right-hand sides are defined to be equal to one. For the expression on the right-hand side of equation (2.3) we can also write

$D_j h(K,u)$, which is a short notation for the special mixed discriminant

$$D(d^2 h_K(u)|_{u^\perp}[j], d^2 h_B(u)|_{u^\perp}[d-1-j]),$$

cf. Schneider [20, §2.5] or Leichtweiß [17, §3]. Note that in [17] a different normalization is used for $D_j h(K,u)$. Up to the factor $\binom{d-1}{j}$, this is also equal to the sum of the principal minors of order j of the Hessian matrix of h_K at u.

We are mainly interested in the curvature measures $C_r(K,\cdot)$ and the surface area measures $S_r(K,\cdot)$ of a convex body $K \in \mathcal{K}^d$, where $r \in \{0,\dots,d-1\}$. Their common generalizations, the generalized curvature measures $\Theta_r(K,\cdot)$, will only appear in a hidden form in the course of the proofs of Section 4. A thorough introduction to all these measures is provided in Schneider's book [20]. Some of the concepts and arguments from [10] and [12], concerning curvature and surface area measures, are also essential in the present context. It should be emphasized that

$$C_{d-1}(K,\beta) = \mathcal{H}^{d-1}(\operatorname{bd} K \cap \beta), \quad \beta \in \mathfrak{B}(\mathbb{R}^d), \tag{2.4}$$

if $\dim K = d$,

$$C_{d-1}(K,\beta) = 2\mathcal{H}^{d-1}(\operatorname{bd} K \cap \beta), \quad \beta \in \mathfrak{B}(\mathbb{R}^d), \tag{2.5}$$

if $\dim K = d-1$, and $C_{d-1}(K,\cdot) = 0$, if $\dim K \le d-2$. Moreover, the relation

$$S_0(K,\omega) = \mathcal{H}^{d-1}(\omega), \quad \omega \in \mathfrak{B}(S^{d-1}), \tag{2.6}$$

holds for all convex bodies. In particular, all curvature and surface area measures are defined on Borel sets. By a standard procedure of measure theory, however, these measures can be extended to (outer) Radon measures in the sense of Evans and Gariepy [4]. Thus we can apply the definitions for absolute continuity "$\ll$" or the restriction "$\llcorner$" of measures from [4, p. 40 and p. 2].

As already mentioned, the Lebesgue decomposition theorem, see [2, Theorem 4.3.1], yields that, for a convex body K, the curvature measure $C_r(K,\cdot)$ can be decomposed with respect to the restriction of $(d-1)$-dimensional Hausdorff measure to the boundary of K into an absolutely continuous part $C_r^a(K,\cdot)$ and a singular part $C_r^s(K,\cdot)$. The theorem also states the uniqueness of this decomposition, but the general measure theoretic result does not provide further geometric information. In the present situation, however, it is known that

$$C_r^a(K,\gamma) = \int_{\operatorname{bd} K \cap \gamma} H_{d-1-r}(K,x)\, \mathrm{d}\mathcal{H}^{d-1}(x), \quad \gamma \in \mathfrak{B}(\mathbb{R}^d), \tag{2.7}$$

for $r \in \{0,\dots,d-1\}$.

Analogous statements hold true for surface area measures, for which a similar notation is used. In particular, the absolutely continuous part $S_r^a(K,\cdot)$ of the surface area measure $S_r(K,\cdot)$ with respect to $(d-1)$-dimensional Hausdorff measure on the unit sphere is given by

$$S_r^a(K,\alpha) = \int_\alpha P_r(K,u)\, \mathrm{d}\mathcal{H}^{d-1}(u), \quad \alpha \in \mathfrak{B}(S^{d-1}), \tag{2.8}$$

for $r \in \{0, \ldots, d-1\}$.

These representations can be deduced, for example, from Hilfssatz 3.6 in Schneider [19] and from the proof of Hilfssatz 2 in Leichtweiß [16] (see also Aleksandrov [1]), if, in addition, results from the theory of the differentiation of measures, cf. [7], [4], [9], are used. A different proof, which provides additional insight, is contained in Section 3. Finally, although Federer's book [7] is the basic reference for results from geometric measure theory, the surveys which are contained in [10, §2] and [12, §2] should be sufficient for the present purpose.

3. Lebesgue decompositions

The starting point for our derivation of the Lebesgue decomposition of the curvature measure $C_r(K, \cdot)$ with respect to $(d-1)$-dimensional Hausdorff measure is the following lemma. It shows, for a general convex body K, how the curvatures on the unit normal bundle of K and the generalized principal curvatures on the boundary of the convex body K are connected.

Lemma 3.1. *Let $K \in \mathcal{K}^d$, and let x be a normal boundary point of K. Then,*

$$k_i(x, \sigma_K(x)) = k_i(K, x) \in [0, \infty), \quad i \in \{1, \ldots, d-1\},$$

if the ordering is chosen properly. In particular, this holds for $\mathcal{H}^{d-1}$ almost all $x \in \operatorname{bd} K$.

Proof. Let $x \in \operatorname{bd} K$ be a normal boundary point. Fix some $\epsilon > 0$ and set $u := \sigma_K(x)$. Then Corollary 4.2 and the preceding remarks from Noll [18] imply that, for all $\epsilon > 0$, $x + \epsilon u$ is a normal boundary point of $K^\epsilon := K + \epsilon B$, and

$$k_i(K^\epsilon, x + \epsilon u) = \frac{k_i(K, x)}{1 + \epsilon k_i(K, x)}, \quad i \in \{1, \ldots, d-1\}, \tag{3.1}$$

if the ordering is chosen properly. Hence, in particular, the spherical image map $\sigma_{K^\epsilon}|_{\operatorname{bd} K^\epsilon} : \operatorname{bd} K^\epsilon \to S^{d-1}$ is differentiable at $x + \epsilon u$, and the eigenvalues of the corresponding second fundamental form

$$\mathrm{II}_{x+\epsilon u} : \begin{cases} \operatorname{Tan}^{d-1}(\operatorname{bd} K^\epsilon, x + \epsilon u) \times \operatorname{Tan}^{d-1}(\operatorname{bd} K^\epsilon, x + \epsilon u) & \to \quad \mathbb{R} \\ (v, w) \mapsto \langle D\sigma_{K^\epsilon}(x + \epsilon u)(v), w \rangle \end{cases}$$

are given by $k_1(K^\epsilon, x + \epsilon u), \ldots, k_{d-1}(K^\epsilon, x + \epsilon u)$.

On the other hand, the definition of the generalized curvatures $k_i(x, u)$, $i \in \{1, \ldots, d-1\}$, and the fact that σ_{K^ϵ} is differentiable at $x + \epsilon u$ yield

$$k_i(K^\epsilon, x + \epsilon u) = \frac{k_i(x, u)}{1 + \epsilon k_i(x, u)}, \quad i \in \{1, \ldots, d-1\}, \tag{3.2}$$

if again the ordering is chosen properly. Note that $\sigma_{K^\epsilon} = \sigma_K$ on $\mathrm{bd}\, K^\epsilon$. A comparison of the equations (3.1) and (3.2) completes the argument, since $\mathcal{H}^{d-1}$ almost all boundary points of K are normal. □

In order to obtain an explicit expression for the singular part of the curvature measure $C_r(K, \cdot)$ of the convex body $K \in \mathcal{K}^d$ with respect to $(d-1)$-dimensional Hausdorff measure, we define the sets

$$\mathcal{N}^a(K) := \{(x, u) \in \mathcal{N}(K) : k_{d-1}(x, u) < \infty\}$$

and

$$\mathcal{N}^s(K) := \{(x, u) \in \mathcal{N}(K) : k_{d-1}(x, u) = \infty\}.$$

The measurability of these sets with respect to $(d-1)$-dimensional Hausdorff measure follows, for example, from the proof of Lemma 3.7 in [12].

Moreover, for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$, let

$$\mathbb{H}_{d-1-r}(K, (x, u)) := \binom{d-1}{r}^{-1} \sum_{1 \le i_1 < \ldots < i_{d-1-r} \le d-1} \frac{k_{i_1}(x, u) \cdots k_{i_{d-1-r}}(x, u)}{\prod_{i=1}^{d-1} \sqrt{1 + k_i(x, u)^2}},$$

if $r \in \{0, \ldots, d-2\}$, and

$$\mathbb{H}_0(K, (x, u)) := \prod_{i=1}^{d-1} \frac{1}{\sqrt{1 + k_i(x, u)^2}}.$$

For the interpretation of the preceding definitions in the case that some of the curvatures are infinite, we refer to [12, §2].

Theorem 3.2. *For an arbitrary convex body $K \in \mathcal{K}^d$, $r \in \{0, \ldots, d-1\}$, and for each $\beta \in \mathfrak{B}(\mathbb{R}^d)$,*

$$C_r^a(K, \beta) = \int_{\mathrm{bd}\, K \cap \beta} H_{d-1-r}(K, x)\, \mathrm{d}\mathcal{H}^{d-1}(x)$$

and

$$C_r^s(K, \beta) = \int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x) \mathbb{H}_{d-1-r}(K, (x, u))\, \mathrm{d}\mathcal{H}^{d-1}(x, u).$$

Proof. Let $\gamma \in \mathfrak{B}(\mathbb{R}^d)$ be arbitrarily chosen, and denote by $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(x, y) \mapsto x$, the projection onto the first component. We also write π_1 for the restriction of this map to $\mathcal{N}(K)$. Using a special case of equation (9) from [10], we get

$$C_r(K, \gamma) = \int_{\mathcal{N}(K)} \mathbf{1}_\gamma(x) \mathbb{H}_{d-1-r}(K, (x, u))\, \mathrm{d}\mathcal{H}^{d-1}(x, u)$$

$$
\begin{aligned}
&= \int\limits_{\mathcal{N}^a(K)} \mathbf{1}_\gamma(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&\quad + \int\limits_{\mathcal{N}^s(K)} \mathbf{1}_\gamma(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&= \binom{d-1}{r}^{-1} \int\limits_{\mathcal{N}^a(K)} \mathbf{1}_\gamma(x) \sum_{1\le i_1<\ldots<i_{d-1-r}\le d-1} k_{i_1}(x,u)\cdots k_{i_{d-1-r}}(x,u) \\
&\quad \times \operatorname{ap} J_{d-1}\pi_1(x,u)\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&\quad + \int\limits_{\mathcal{N}^s(K)} \mathbf{1}_\gamma(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&= \int\limits_{\operatorname{bd} K\cap\gamma} H_{d-1-r}(K,x)\,\mathrm{d}\mathcal{H}^{d-1}(x) \\
&\quad + \int\limits_{\mathcal{N}^s(K)} \mathbf{1}_\gamma(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u). \qquad (3.3)
\end{aligned}
$$

Here we have used [7, Theorem 3.2.22], equation (6) from [10], i.e.

$$\operatorname{ap} J_{d-1}\pi_1(x,u) = \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+k_i(x,u)^2}},$$

which holds for $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(K)$, and the relation

$$
\begin{aligned}
&H_{d-1-r}(K,x) = \\
&\binom{d-1}{r}^{-1} \sum_{1\le i_1<\ldots<i_{d-1-r}\le d-1} k_{i_1}(x,\sigma_K(x))\cdots k_{i_{d-1-r}}(x,\sigma_K(x)),
\end{aligned}
$$

which holds for $\mathcal{H}^{d-1}$ almost all $x\in\operatorname{bd} K$ according to definition (2.2) and Lemma 3.1. From the decomposition (3.3), equation (2.7), and from the uniqueness of the Lebesgue decomposition we now obtain that, for all $\gamma\in\mathfrak{B}(\mathbb{R}^d)$,

$$C_r^s(K,\gamma) = \int\limits_{\mathcal{N}^s(K)} \mathbf{1}_\gamma(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u). \qquad (3.4)$$

This completes the proof of the theorem. □

Remark 3.3. In the proof of Theorem 3.2 we used the existence and the uniqueness of the Lebesgue decomposition of $C_r(K,\cdot)$ as well as additional information about the absolutely continuous part. The Lebesgue decomposition of $C_r(K,\cdot)$, however, can also be deduced from the proof of Theorem 3.2 itself. In fact, let $\mathcal{M}(K)$ be the Borel set of normal boundary points of K. Then the measure

$$\gamma \mapsto \int_{\mathrm{bd}\, K\cap\gamma} H_{d-1-r}(K,x)\,\mathrm{d}\mathcal{H}^{d-1}(x), \quad \gamma \in \mathfrak{B}(\mathbb{R}^d),$$

is concentrated on $\mathcal{M}(K)$, since $\mathcal{H}^{d-1}(\mathrm{bd}\, K \setminus \mathcal{M}(K)) = 0$. On the other hand,

$$C_r^s(K,\mathcal{M}(K)) = \int_{\mathcal{N}^s(K)} \mathbf{1}_{\mathcal{M}(K)}(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) = 0,$$

since, for $x \in \mathcal{M}(K)$, Lemma 3.1 yields that $k_{d-1}(x,u) < \infty$, if $u \in S^{d-1}$ is the uniquely determined unit vector such that $(x,u) \in \mathcal{N}(K)$, and thus $(x,u) \notin \mathcal{N}^s(K)$.

The proof of an analogous theorem, which involves surface area measures and singular normal vectors, requires a lemma on the connection between radii of curvature and curvatures on the unit normal bundle.

Lemma 3.4. *Let $K \in \mathcal{K}^d$ be a convex body. Then, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$,*

$$k_i(\xi_K(u),u)^{-1} = r_i(K,u) \in [0,\infty), \quad i \in \{1,\dots,d-1\},$$

if the ordering is chosen properly.

Proof. Let S^* be the set of all $u \in S^{d-1}$ such that the support function $h(K,\cdot)$ is not second order differentiable at u. Then Aleksandrov's theorem on the second order differentiability of a convex function states that $\mathcal{H}^{d-1}(S^*) = 0$. Denote by $\mathcal{D}_K$ the set of points in $\mathbb{R}^d \setminus K$ where σ_K is differentiable and, in addition, let $\mathcal{N}^*$ be the set of all $(x,u) \in \mathcal{N}(K)$ such that there is some $\epsilon > 0$ with $x + \epsilon u \notin \mathcal{D}_K$. Then also $\mathcal{H}^{d-1}(\mathcal{N}^*) = 0$, see, for example, [10, §2]. Denote by $\pi_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(y,z) \mapsto z$, the projection onto the second component. Hence, we have $\mathcal{H}^{d-1}(\pi_2(\mathcal{N}^*)) = 0$. Now, choose $u \in S^{d-1} \setminus (S^* \cup \pi_2(\mathcal{N}^*))$. Then there is a unique $x \in \mathbb{R}^d$ such that $(x,u) \in \mathcal{N}(K)$, and obviously $(x,u) \notin \mathcal{N}^*$. Thus, for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, the support function h_K is second order differentiable at u and $\xi_K(u) + \epsilon u \in \mathcal{D}_K$ for all $\epsilon > 0$.

It is sufficient to prove the lemma for such a unit vector u. But for such a vector u, the spherical image map σ_K is differentiable at $\xi_K(u) + \epsilon u$, and hence $\xi_K(u) + \epsilon u$ is a normal boundary point of K^ϵ. Since, in addition, $h(K^\epsilon,\cdot)$ is second order differentiable at u, the second remark after Lemma 2.5 in [11] implies

$$k_i(K^\epsilon, \xi_K(u) + \epsilon u)^{-1} = r_i(K^\epsilon,u) \in (0,\infty), \quad i \in \{1,\dots,d-1\}, \tag{3.5}$$

for all $\epsilon > 0$, if the ordering is chosen properly. Set $x := \xi_K(u)$, and note that

$$r_i(K^\epsilon,u) = r_i(K,u) + \epsilon. \tag{3.6}$$

From $x + \epsilon u \in \mathcal{D}_K$ and from equation (3.5) we infer that

$$k_i(K^\epsilon, x + \epsilon u) = \frac{k_i(x,u)}{1 + \epsilon k_i(x,u)} > 0, \quad i \in \{1,\dots,d-1\}, \tag{3.7}$$

if again the ordering is chosen properly. It is not excluded that $k_i(x,u)=\infty$ for some $i\in\{1,\dots,d-1\}$. Indeed, in this case the right-hand side of the last equation is equal to ϵ^{-1}.

Finally, the equations (3.5), (3.6) and (3.7) imply that, for an arbitrary $\epsilon>0$,

$$r_i(K,u)+\epsilon=\left(\frac{k_i(x,u)}{1+\epsilon k_i(x,u)}\right)^{-1}=\frac{1}{k_i(x,u)}+\epsilon,\quad i\in\{1,\dots,d-1\},$$

from which the lemma follows. □

Observe that for the proof of the preceding lemma it was not sufficient to assume that h_K is second order differentiable at $u\in S^{d-1}$.

Now we define the subsets

$$\mathcal{N}_a(K):=\{(x,u)\in\mathcal{N}(K):k_1(x,u)>0\}$$

and

$$\mathcal{N}_s(K):=\{(x,u)\in\mathcal{N}(K):k_1(x,u)=0\}$$

of the unit normal bundle of the convex body $K\in\mathcal{K}^d$. For the measurability of these sets we again refer to [12].

Theorem 3.5. *For an arbitrary convex body $K\in\mathcal{K}^d$, $r\in\{0,\dots,d-1\}$, and for each $\omega\in\mathfrak{B}(S^{d-1})$,*

$$S_r^a(K,\omega)=\int_\omega P_r(K,u)\,\mathrm{d}\mathcal{H}^{d-1}(u)$$

and

$$S_r^s(K,\omega)=\int_{\mathcal{N}_s(K)}\mathbf{1}_\omega(u)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u).$$

Proof. We continue to write π_2 for the restriction of the projection map π_2 to $\mathcal{N}(K)$. Note that, for $\mathcal{H}^{d-1}$ almost all $(x,u)\in\mathcal{N}(K)$,

$$\mathrm{ap}\,J_{d-1}\pi_2(x,u)=\prod_{i=1}^{d-1}\frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}},$$

see the proof of Theorem 4.3 in [12]. Let $\alpha\in\mathfrak{B}(S^{d-1})$. Then we get, if again a special case of equation (9) from [10] is used,

$$\begin{aligned}S_r(K,\alpha)&\\&=\int_{\mathcal{N}(K)}\mathbf{1}_\alpha(u)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u)\\&=\int_{\mathcal{N}_a(K)}\mathbf{1}_\alpha(u)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u)\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{\mathcal{N}_s(K)} \mathbf{1}_\alpha(u) \mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&= \binom{d-1}{r}^{-1} \int\limits_{\mathcal{N}_a(K)} \mathbf{1}_\alpha(u) \sum_{1\le j_1<\ldots<j_r\le d-1} \Big(k_{j_1}(x,u)\cdots k_{j_r}(x,u)\Big)^{-1} \\
&\quad \times \operatorname{ap} J_{d-1}\pi_2(x,u)\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&\quad + \int\limits_{\mathcal{N}_s(K)} \mathbf{1}_\alpha(u) \mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) \\
&= \int\limits_\alpha P_r(K,u)\,\mathrm{d}\mathcal{H}^{d-1}(u) \\
&\quad + \int\limits_{\mathcal{N}_s(K)} \mathbf{1}_\alpha(u) \mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u). \qquad (3.8)
\end{aligned}
$$

In the last step we have used [7, Theorem 3.2.22] and the relation

$$
P_r(K,u) = \binom{d-1}{r}^{-1} \sum_{1\le j_1<\ldots<j_r\le d-1} \left(k_{j_1}(\xi_K(u),u)\cdots k_{j_r}(\xi_K(u),u)\right)^{-1},
$$

which holds for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ according to Lemma 3.4. Observe that $k_1(\xi_K(u),u) > 0$ for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. Also note that (here we omit the argument)

$$
\frac{k_{i_1}\cdots k_{i_{d-1-r}}}{\prod_{j=1}^{d-1}\sqrt{1+k_j^2}} = \frac{1}{k_{j_1}\cdots k_{j_r}} \prod_{j=1}^{d-1} \frac{k_j}{\sqrt{1+k_j^2}},
$$

if $k_1,\ldots,k_{d-1} > 0$ and $\{j_1,\ldots,j_r\}$ is the set complementary to $\{i_1,\ldots,i_{d-1-r}\}$ with respect to $\{1,\ldots,d-1\}$. This holds true even if some of the curvatures are equal to ∞, confer the preliminary remarks in [12, §2].

From the decomposition (3.8), equation (2.8), and from the uniqueness of the Lebesgue decomposition we can now deduce that, for all $\alpha \in \mathfrak{B}(S^{d-1})$,

$$
S_r^s(K,\alpha) = \int\limits_{\mathcal{N}_s(K)} \mathbf{1}_\alpha(u) \mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u). \qquad (3.9)
$$

This finishes the proof of the theorem. □

Remark 3.6. It is also possible, with almost no additional effort, to deduce the Lebesgue decomposition of $S_r(K,\cdot)$ with respect to $\mathcal{H}^{d-1} \llcorner S^{d-1}$ from Lemma 3.4 and the proof of Theorem 3.5. The reasoning is similar to the one described in Remark 3.3. Instead of the set of normal boundary points, now one has to consider a Borel set $\mathcal{S}(K)$ consisting of unit vectors for which the conclusion of Lemma 3.4 holds true, and such that $\mathcal{H}^{d-1}(S^{d-1} \setminus \mathcal{S}(K)) = 0$ is fulfilled. Such a set exists, since the statement of Lemma 3.4 holds true for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$ and because $\mathcal{H}^{d-1}$ is a Borel regular measure with $\mathcal{H}^{d-1}(S^{d-1}) < \infty$.

4. Absolute continuity and singularities

The class of all convex bodies $K \in \mathcal{K}^d$ for which $C_r(K, \cdot)$ is purely singular, i.e., for which $C_r^a(K, \cdot) = 0$ holds true, obviously is too large to admit a fruitful geometric description. The subclass of d-dimensional polytopes, however, can be characterized by a simple property of the support of the measure $C_r(K, \cdot)$, see Theorem 4.1 below. Moreover, if $C_r(K, \cdot)$ fulfills this property, then it is immediately implied that $C_r^a(K, \cdot) = 0$.

Subsequently, we denote by $\mathcal{K}_o^d$ the set of all convex bodies with nonempty interiors, $\mathbf{A}(d, j)$, $j \in \{0, \dots, d\}$, is the set of all j-dimensional (affine) planes in $\mathbb{R}^d$, and μ_j denotes a suitably normalized Haar measure on $\mathbf{A}(d, j)$, cf. [20].

Theorem 4.1. *Let $K \in \mathcal{K}^d$, $r \in \{0, \dots, d-1\}$, and let $\dim K \geq r + 1$. Further, assume that the support of the curvature measure $C_r(K, \cdot)$ can be covered by finitely many r-dimensional planes. Then K is a polytope.*

Proof. For the proof we can assume that $\dim K = d$, that is, $K \in \mathcal{K}_o^d$. To see this, recall that the measures

$$\Phi_r(K, \cdot) := \frac{\binom{d}{r}}{d\kappa_{d-r}} C_r(K, \cdot)$$

do not depend on the dimension of the surrounding space.

First, let $r = d - 1$. Then we get

$$\operatorname{bd} K \subset \bigcup_{i=1}^{k} H_i,$$

for suitable hyperplanes $H_i \subset \mathbb{R}^d$. With respect to $(d-1)$-dimensional Hausdorff measure almost every boundary point of K lies in a hyperplane H_i, for some $i \in \{1, \dots, k\}$, which is a support plane of K. Therefore, by a simple continuity argument, we can assume that all planes H_i are support planes, that is, for all $i \in \{1, \dots, k\}$, $H_i = H(K, u_i)$ for some $u_i \in S^{d-1}$. Hence,

$$K \subset \bigcap_{i=1}^{k} H^-(K, u_i) =: P.$$

Assume that there is some $x \in P \setminus K$. Since $\dim K = d$, there is some $x_0 \in \operatorname{int} K$, and hence also $x_0 \in \operatorname{int} P$. This yields $(x, x_0] \subset \operatorname{int} P$. But then there is some $y \in \operatorname{bd} K \cap (x, x_0]$, and thus $y \in \operatorname{int} P$. In particular,

$$y \in \operatorname{bd} K \setminus \bigcup_{i=1}^{k} H_i,$$

a contradiction. This proves $K = P$.

Now, we consider the case $r = 0$. The assumption then says that

$$\operatorname{supp} C_0(K,\cdot) = M := \{x_1,\dots,x_k\} \subset \operatorname{bd} K.$$

Thus, we obtain for the convex hull of M, $P := \operatorname{conv} M \subset K$. We want to show that $P = K$. Assume that there is some $u \in S^{d-1}$ such that $h(P,u) < h(K,u)$. Then we can choose a positive number $\epsilon > 0$ such that $\beta := (H(K,u)^+ - \epsilon u) \cap \operatorname{bd} K$ and $\operatorname{supp} C_0(K,\cdot)$ are disjoint sets. But then $\sigma(K,\beta)$ contains a relatively open subset of S^{d-1}, and thus

$$C_0(K,\beta) = \mathcal{H}^{d-1}(\sigma(K,\beta)) > 0,$$

a contradiction. This proves $h(P,u) \geq h(K,u)$ for all $u \in S^{d-1}$, and thus $P = K$.

Finally, let $1 \leq r \leq d-2$. Assume that $\operatorname{supp} C_r(K,\cdot)$ is covered by the r-dimensional planes $A_1,\dots,A_k \in \mathbf{A}(d,r)$. Set

$$\beta := \mathbb{R}^d \setminus (A_1 \cup \ldots \cup A_k).$$

Then $C_r(K,\beta) = 0$, and hence Theorem 4.5.5 from Schneider [20] implies

$$\int_{\mathbf{A}(d,\,d-r)} C_0(K \cap E, \beta \cap E)\,\mathrm{d}\mu_{d-r}(E) = 0.$$

In other words,

$$C_0(K \cap E, \beta \cap E) = 0,$$

for μ_{d-r} almost all $E \in \mathbf{A}(d,d-r)$.

Let $i \in \{1,\dots,k\}$ and assume that E meets A_i in at least two points. This implies that the dimension of the intersection of the associated linear subspaces $\operatorname{lin} E$ and $\operatorname{lin} A_i$ is at least one. Since $\dim(\operatorname{lin} E) = d-r$ and $\dim(\operatorname{lin} A_i) = r$, this yields that $\operatorname{lin} E$ and $\operatorname{lin} A_i$ are in special position, cf. Schneider [20, §4.5]. From this we infer that the set of all $E \in \mathbf{A}(d,d-r)$ such that E meets A_i in at least two points has μ_{d-r} measure zero. To see this, Lemma 4.5.1 from [20] can be used. Thus, for μ_{d-r} almost all $E \in \mathbf{A}(d,d-r)$, $\operatorname{supp} C_0(K \cap E,\cdot)$ can be covered by a set which contains at most k points.

According to the second part of the proof, this implies that, for μ_{d-r} almost all $E \in \mathbf{A}(d,d-r)$, $K \cap E$ is a polytope with at most k vertices. By approximation the same statement holds true for any section $K \cap E$ of K by a $(d-r)$-dimensional plane E. Observe that $d-r \geq 2$. Now the result follows from Theorem 4.7 in Klee [14]. □

From this characterization of polytopes involving curvature measures we obtain two simple corollaries which are similar in spirit to recent results of Schneider [21]. The first corollary can be viewed as an analogue for curvature measures of Theorem 3.2 in [21]. But it also fits well into the present framework. In fact, the assumption of Corollary 4.2 in particular means that the curvature measure of order r of the convex body K is absolutely continuous (with bounded density) with respect to a special singular measure which is derived from the r-th curvature measure of a polytope.

Corollary 4.2. *Let $0 \le r \le d-1$, and let $P \in \mathcal{K}^d$ be a polytope. If $K \in \mathcal{K}^d$ is a convex body of dimension $\geq r+1$ satisfying*

$$C_r(K,\cdot) \le C_r(P,\cdot),$$

then K is equal to P.

Proof. The assumption implies

$$\operatorname{supp} C_r(K,\cdot) \subset \operatorname{supp} C_r(P,\cdot) = \bigcup_{G \in \mathcal{F}_r(P)} G.$$

Therefore, we obtain from Theorem 4.1 that K is a polytope and

$$\bigcup_{F \in \mathcal{F}_r(K)} F \subset \bigcup_{G \in \mathcal{F}_r(P)} G. \tag{4.1}$$

If $r = d-1$, this yields $\operatorname{bd} K \subset \operatorname{bd} P$, and hence $K = P$, since $\dim K = d$.

Now, let $0 \le r \le d-2$. From (4.1) we get $K \subset P$, since each extreme point of K is contained in some $F \in \mathcal{F}_r(K)$. Obviously, we can assume that $\dim P = d$ (otherwise we work in the affine plane spanned by P).

Assume that $K \neq P$. Then there is some $u \in S^{d-1}$ such that $\dim F(K,u) \geq r+1$ and $h(K,u) < h(P,u)$. Choose $F \in \mathcal{F}_r(K)$ with $F \subset F(K,u)$. Further, let $x_0 \in \operatorname{relint} F$ be arbitrarily chosen. According to (4.1) there is some $G \in \mathcal{F}_r(P)$ such that $x_0 \in G$. But then we even get $F \subset G$. Moreover, $K \subset P$ implies that $N(P,G) \subset N(K,F)$, cf. [20, p. 72] for the notation. The inclusion is strict. In fact, we have $u \in N(K,F)$, but $u \notin N(P,G)$, since $h(P,u) > h(K,u)$. This yields $\gamma(F,K) > \gamma(G,P)$, and thus $C_r(K,F) > C_r(P,F)$, a contradiction. □

The proof of our second corollary is essentially the same as the proof for the corresponding Theorem 3.8 in Schneider [21]. One merely has to use the weak continuity of the curvature measures and the preceding Corollary 4.2. For the terminology concerning statements about 'most' convex bodies, see [20, §2.6].

Corollary 4.3. *Let $0 \le r \le d-1$. For most convex bodies $L \in \mathcal{K}^d$, the inequality*

$$C_r(K,\cdot) \le C_r(L,\cdot)$$

for a convex body $K \in \mathcal{K}^d$ of dimension $\geq r+1$ implies that K is equal to L.

Remark 4.4. By an extension of the method of proof for Theorem 4.1, the following can be shown: Let $K \in \mathcal{K}_o^d$ and $r \in \{0,\dots,d-2\}$, let $A_1,\dots,A_k \in \mathbf{A}(d,r+1)$ be such that $A_i \cap \operatorname{int} K \neq \emptyset$ for all $i \in \{1,\dots,k\}$, and assume that

$$\operatorname{supp} C_r(K,\cdot) \subset \bigcup_{i=1}^{k} A_i.$$

Then K is a polytope.

Remark 4.5. Let $r \in \{0, \dots, d-2\}$. Let $K, L \in \mathcal{K}_o^d$ be convex bodies with the origin o as an interior point, and let the radial projection $f : \operatorname{bd} L \to \operatorname{bd} K$ be defined by $f(x) = \lambda(x)x \in \operatorname{bd} K$ with $\lambda(x) > 0$ for $x \in \operatorname{bd} L$. Assume that

$$C_r(K, f(\beta)) \leq C_r(L, \beta)$$

holds for all Borel sets $\beta \subset \operatorname{bd} L$. Then the following can be said:

If $r = 0$, then K and L differ only by a dilatation with centre o. This follows from a result of Aleksandrov, see Theorem 7.2.12 in Schneider [20].

If $1 \leq r \leq d-2$ and L is a polytope, then K also is a polytope. This is implied by the preceding remark.

However, one can find suitable simplices K and L which fulfill the preceding assumptions, but do not differ only by a dilatation.

Now, we come to the main results of this section. In the following theorem we consider a convex body K for which $C_r(K, \cdot)$ is absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure. More generally, we will merely assume that this condition is fulfilled for the restriction of $C_r(K, \cdot)$ to an arbitrarily chosen Borel set β. Under this assumption we will show that $\Sigma^r(K) \cap \beta$ is invisible, if one tries to observe this set with the help of an s-dimensional Hausdorff measure and $s \geq r$. The Cantor-type example of Section 5 demonstrates that this is no longer true for $s < r$.

Finally recall from Section 2 that, in the assumption of Theorem 4.6, we can write $C_{d-1}(K, \cdot)$ instead of the restriction of $(d-1)$-dimensional Hausdorff measure to the boundary of the convex body K.

Theorem 4.6. *Let $K \in \mathcal{K}^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $r \in \{0, \dots, d-2\}$. Then, the condition*

$$C_r(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

implies

$$\mathcal{H}^r(\Sigma^r(K) \cap \beta) = 0.$$

Proof. According to the assumption of the theorem, $C_r^s(K, \cdot) \llcorner \beta = 0$. From Theorem 3.2 we then obtain that this is equivalent to the condition

$$\int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x) \mathbb{H}_{d-1-r}(K, (x, u)) \, d\mathcal{H}^{d-1}(x, u) = 0. \tag{4.2}$$

Observe that up to a set of $\mathcal{H}^{d-1}$ measure zero we have, cf. the proof of Theorem 3.2 from [12], that

$$(\Sigma^r(K) \times S^{d-1}) \cap \mathcal{N}(K) \subset \mathcal{N}^s(K), \tag{4.3}$$

since

$$\Sigma^r(K) = \{x \in \operatorname{bd} K : \dim N(K, x) \geq d - r\}$$

and $r \leq d-2$. Therefore, (4.2) and (4.3) together with a special case of equation (9) from [10] imply

$$C_r(K, \Sigma^r(K) \cap \beta) = 0.$$

According to Theorem 3.2 from [12] we deduce

$$\int_{\Sigma^r(K)\cap\beta} \mathcal{H}^{d-1-r}(N(K,x)\cap S^{d-1})\,\mathrm{d}\mathcal{H}^r(x) = 0. \tag{4.4}$$

Since $\mathcal{H}^{d-1-r}(N(K,x)\cap S^{d-1}) > 0$ for all $x \in \Sigma^r(K)\cap\beta$, equation (4.4) already implies $\mathcal{H}^r(\Sigma^r(K)\cap\beta) = 0$. □

Remark 4.7. In the preceding proof, we implicitly showed that, for $K \in \mathcal{K}^d$ and $r \in \{0,\dots,d-1\}$, the measures $C_r(K,\cdot)\llcorner\Sigma^r(K)$ and $\mathcal{H}^r\llcorner\Sigma^r(K)$ have the same sets of measure zero. A similar statement holds for surface area measures and singular normal vectors.

For the proof of an analogous theorem, which involves surface area measures and singular normal vectors, first recall from [12] that

$$\Sigma_r(K) := \{u \in S^{d-1} : \dim F(K,u) \geq d-1-r\}$$

denotes the set of r-singular normal vectors of the convex body K. Also note that $S_0(K,\cdot)$ can be used instead of the restriction of $(d-1)$-dimensional Hausdorff measure to the unit sphere, see equation (2.6) of Section 2.

Theorem 4.8. *Let $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and $r \in \{1,\dots,d-1\}$. Then, the condition*

$$S_r(K,\cdot)\llcorner\omega \ll S_0(K,\cdot)\llcorner\omega$$

implies

$$\mathcal{H}^{d-1-r}(\Sigma_{d-1-r}(K)\cap\omega) = 0.$$

Proof. Theorem 3.5 yields that the assumption of Theorem 4.8 is equivalent to

$$\int_{\mathcal{N}_s(K)} \mathbf{1}_\omega(u)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathrm{d}\mathcal{H}^{d-1}(x,u) = 0. \tag{4.5}$$

Observe that up to a set of $(d-1)$-dimensional Hausdorff measure zero,

$$(\mathbb{R}^d \times \Sigma_{d-1-r}(K)) \cap \mathcal{N}(K) \subset \mathcal{N}_s(K), \tag{4.6}$$

since

$$\Sigma_{d-1-r}(K) = \{u \in S^{d-1} : \dim F(K,u) \geq r\}$$

and $r \geq 1$, cf. the proof of Theorem 4.3 from [12]. Therefore, we obtain from (4.5), (4.6) and again from a special case of equation (9) from [10] that

$$S_r(K, \Sigma_{d-1-r}(K)\cap\omega) = 0.$$

Now, Theorem 4.3 from [12] yields

$$\int_{\Sigma_{d-1-r}(K)\cap\omega} \mathcal{H}^r(F(K,u))\,\mathrm{d}\mathcal{H}^{d-1-r}(u) = 0. \tag{4.7}$$

Since $\mathcal{H}^r(F(K,u)) > 0$ for all $u \in \Sigma_{d-1-r}(K) \cap \omega$, we finally get from equation (4.7) that $\mathcal{H}^{d-1-r}(\Sigma_{d-1-r}(K) \cap \omega) = 0$. □

5. Examples

In the following two examples, we discuss the optimality of the preceding theorems. The first example shows that there exists a closed convex set $K \subset \mathbb{R}^d$, $K \neq \mathbb{R}^d$, for which the set of r-singular boundary points $\Sigma^r(K)$ contains an $(r-1)$-dimensional plane, although $C_r(K,\cdot) \ll C_{d-1}(K,\cdot)$ holds true.

Example 5.1. Consider the function

$$f : \mathbb{R}^{d-1} \to \mathbb{R}, \quad (x_1, \ldots, x_{d-1}) \mapsto \left\{ (x_r^2)^\alpha + \sum_{i=r+1}^{d-1} x_i^2 \right\}^{\frac{1}{2}},$$

where $r \in \{1, \ldots, d-2\}$ and $\alpha > 1$. Theorem 1.5.10 from Schneider [20] can be used to show that the continuous function f is convex. The epigraph K of f defines a closed convex set in $\mathbb{R}^d$. Let $(e_1, \ldots, e_d)$ be the standard basis of $\mathbb{R}^d$, and identify $\mathbb{R}^{d-1}$ with $\mathbb{R}^{d-1} \times \{0\}$. It is easy to see that $\dim N(K,x) = d - r$ for all points $x = (x_1, \ldots, x_{r-1}, 0, \ldots, 0) \in \mathbb{R}^{d-1}$. In fact, this follows from

$$\operatorname{lin}\{e_1, \ldots, e_{r-1}, e_r\} \subset S(K,x),$$

where $S(K,x)$ is the support cone of K at x, [20, §2.2], and from

$$-(e_d \pm e_i) \in N(K,x), \quad i \in \{r+1, \ldots, d-1\}.$$

Furthermore, note that $\operatorname{bd} K \setminus (\mathbb{R}^{r-1} \times \{o\} \times \{0\})$ is a hypersurface of class C^2 (thus consisting of normal boundary points only), and that the relations

$$\dim (N(K,x) \cap S^{d-1}) = d - r - 1$$

as well as

$$N(K,x) \subset \operatorname{lin}\{e_{r+1}, \ldots, e_d\}$$

hold true for all $x \in \mathbb{R}^{r-1} \times \{o\}$. Hence, the set of all $(x,u) \in \mathcal{N}(K)$, such that $k_{d-1}(x,u) < \infty$ is violated, has $\mathcal{H}^{d-1}$ measure zero. But then Theorem 3.2 yields that $C_r(K,\cdot) \ll C_{d-1}(K,\cdot)$. One can even check by a direct calculation that $C_r(K,\cdot)$ does not have a bounded density with respect to $C_{d-1}(K,\cdot)$. In addition, from

$$\{(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : x_r = \ldots = x_{d-1} = 0\} \subset \Sigma^r(K)$$

we get that

$$\mathcal{H}^{r-1}(\Sigma^r(K)) = \begin{cases} \infty, & \text{if } r \in \{2, \ldots, d-2\}, \\ 1, & \text{if } r = 1. \end{cases}$$

One can also construct convex bodies with $\mathcal{H}^0(\Sigma^1(K)) = \infty$ and such that $C_1(K,\cdot) \ll C_{d-1}(K,\cdot)$. Cap bodies of a ball can, for instance, be used.

By modifying this example, convex bodies with similar properties can be found. In particular, a convex body K need not be smooth, even if $C_j(K,\cdot) \ll C_{d-1}(K,\cdot)$ holds true for all $j \in \{0,\ldots,d-2\}$. Examples in the context of surface area measures are then obtained by using polarity. This follows from the results of a subsequent paper [13].

Our next example demonstrates that, in general, Theorem 3.2, and hence also Theorem 3.5, cannot be improved. To see this, let $\epsilon \in (0,1)$ be fixed. Subsequently, we define a convex body $K \in \mathcal{K}_0^3$ and a Borel set $\beta \subset \operatorname{bd} K$ such that $C_1(K,\cdot) \llcorner \beta \ll C_2(K,\cdot) \llcorner \beta$, $\mathcal{H}^2(\beta) > 0$ and $\dim_{\mathrm{H}}(\Sigma^1(K) \cap \beta) = \epsilon$. By $\dim_{\mathrm{H}}(M)$ we denote the Hausdorff dimension of a subset M of $\mathbb{R}^d$. Since ϵ can be arbitrarily close to 1, this justifies the initial statement.

Example 5.2. Let $C \subset [0,\pi]$ be a closed Cantor-like set such that $\dim_{\mathrm{H}}(C) = \epsilon$. For a construction of such a set we refer to [3]. Let (e_1, e_2, e_3) be the standard basis of $\mathbb{R}^3$. Then define the bi-Lipschitz map

$$\gamma : [0,\pi] \to S^2, \quad t \mapsto \cos(t)e_1 + \sin(t)e_2,$$

and, for $t \in [0,\pi]$, the half spaces

$$H^-(t) := \{x \in \mathbb{R}^3 : \langle x - \gamma(t), \gamma(t) + e_3 \rangle \le 0\}.$$

Then we set

$$K := B \cap \bigcap_{t \in C} H^-(t) \in \mathcal{K}_0^3$$

and

$$\beta := \{x \in S^2 : \langle x, e_3 \rangle \le 0\}.$$

If $x \in \beta \setminus \gamma(C)$, then x is a normal boundary point of K, since $\gamma(C)$ is a compact set. If, however, $x \in \gamma(C)$, then $\dim N(K,x) = 2$, since certainly $\dim N(K,x) \ge 2$ and $\operatorname{lin}\{\gamma'(\gamma^{-1}(x))\} \subset S(K,x)$. Thus we get $\beta \cap \Sigma^1(K) = \gamma(C)$ and $\dim_{\mathrm{H}}(\gamma(C)) = \epsilon$. Further,

$$\begin{aligned} &\{(x,u) \in \mathcal{N}(K) : x \in \gamma(C)\} \\ &\subset \left\{(\gamma(t), \cos(\lambda)\gamma(t) + \sin(\lambda)e_3) : t \in C,\ \lambda \in \left[0, \frac{\pi}{2}\right]\right\} =: N, \end{aligned}$$

and obviously $\mathcal{H}^2(N) = 0$. But then, as in the preceding example, Theorem 3.2 yields that $C_1(K,\cdot) \llcorner \beta \ll C_2(K,\cdot) \llcorner \beta$.

The preceding construction can be generalized to yield, for any $r \in \{1,\ldots,d-2\}$ and any $\epsilon \in (0,1)$, a convex body $K \in \mathcal{K}_0^d$ and a Borel set $\beta \subset \operatorname{bd} K$ such that $C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$, $\mathcal{H}^{d-1}(\beta) > 0$ and $\dim_{\mathrm{H}}(\Sigma^r(K) \cap \beta) = r - 1 + \epsilon$. In fact, let $C \subset [0,\pi]$ be defined as above, set

$$C_r := C \times [0,\pi] \times \ldots \times [0,\pi] \subset \mathbb{R}^r,$$

and define $\gamma_r : [0, \pi]^r \to \mathbb{R}^d$ recursively by

$$\gamma_r(t_1, \ldots, t_r) = \cos(t_r)\,(\gamma_{r-1}(t_1, \ldots, t_{r-1})) + \sin(t_r) e_{r+1}, \quad r \geq 2,$$

and $\gamma_1(t_1) := \gamma(t_1)$, if $(e_1, \ldots, e_d)$ denotes the standard basis of $\mathbb{R}^d$. Further, for $j \in \{r+2, \ldots, d\}$ we set

$$H_j^-(t_1, \ldots, t_r) := \{x \in \mathbb{R}^d : \langle x - \gamma_r(t_1, \ldots, t_r), \gamma_r(t_1, \ldots, t_r) + e_j \rangle \leq 0\}.$$

Then the convex body

$$K := B \cap \bigcap_{j=r+2}^{d} \bigcap_{(t_1, \ldots, t_r) \in C_r} H_j^-(t_1, \ldots, t_r)$$

and the set

$$\beta := \bigcap_{j=r+2}^{d} \{x \in S^{d-1} : \langle x, e_j \rangle \leq 0\}$$

fulfill all requirements. Note that $\dim_{\mathrm{H}}(C_r) = r - 1 + \epsilon$. This is a nontrivial statement which follows, for instance, from Theorem 2.10.45 in Federer [6] or from Corollary 7.4 in Falconer [5].

These considerations naturally lead to the problem of determining the value of the constant

$$\sigma^r := \sup\{\dim_{\mathrm{H}}(\Sigma^r(K)) : K \in \mathcal{K}^d,\ C_r(K, \cdot) \ll C_{d-1}(K, \cdot)\},$$

where $r \in \{1, \ldots, d-2\}$. An obvious estimate is $\sigma^r \in [r-1, r]$, but the precise value seems to be unknown. We conjecture that $\sigma^r = r$.

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