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# Absolute continuity for curvature measures of convex sets II

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## 1. Introduction

A central and challenging problem in geometry is to find the basic relationships between (suitably defined) curvatures of a geometric object and the local geometric shape of the object which is considered. In one direction, one asks for geometric properties of a set which can be retrieved, provided some specific information is available about the curvatures which are associated with the set. But it is also important to obtain inferences in the reverse direction. Here one wishes to find characteristic properties of the curvatures which can be deduced from knowledge of the local geometric shape of the sets involved.

In convex geometry, where one strives to avoid a priori smoothness assumptions different from those already implied by convexity itself, curvature measures of arbitrary closed convex sets replace the pointwise defined curvature functions of smooth convex surfaces which are used in classical differential geometry. In spite of the lack of differentiability assumptions, (at least in principle) the curvature measures encapsulate all relevant information about the sets with which they are associated. In order to investigate these measures, the methods and tools of convex and integral geometry, certain generalized curvature functions and Federer's coarea formula play a decisive rôle.

Our general framework is determined by the geometry of convex sets in Euclidean space $\mathbb{R}^d$ $(d \geq 2)$. In this setting, *local Steiner formulae* are used to introduce the *curvature measures* $C_r(K, \cdot)$ of a (non-empty) closed convex set $K \subset \mathbb{R}^d$, for $r \in \{0, \ldots , d-1\}$, as Radon measures on the $\sigma$-algebra of Borel subsets of $\mathbb{R}^d$. These measures, as well as their spherical counterparts, the (intermediate) *surface area measures* $S_r(K, \cdot)$, have been the subject of numerous investigations over the last 30 years. This can be seen, e.g., from the books of Schneider [41] and Schneider & Weil [44], which are recommended for an introduction to this subject, as well as from the surveys by Schneider [42] and Schneider & Wieacker [46]. A considerable number of these investigations can be understood as contributions to the following fundamental question, which has also been pointed out in [43].

Which geometric consequences can be inferred for a closed convex set $K$, provided some specific measure theoretic information on the curvature measure $C_r(K, \cdot)$, for some $r \in \{0, \ldots , d-1\}$, is available? For example, what can be said about the set of *singular boundary points* of a closed convex set $K$ if the singular part of some curvature measure of $K$ vanishes?

Of course, the curvature measures of special classes of convex bodies (non-empty compact convex sets) such as bodies with smooth boundaries (of differentiability class $C^2$) or polytopes are fairly well understood. For arbitrary closed convex sets, a systematic investigation was initiated in [24], which aims at establishing a precise connection between the *local geometric shape*, in particular the *boundary structure*, of a given convex set $K$ and the *absolute continuity* of some curvature measure $C_r(K, \cdot)$, $r \in \{0, \ldots , d-2\}$, of $K$ with respect to the *boundary measure* $C_{d-1}(K, \cdot)$ of $K$ (see Sect. 2 for some definitions). There, based on the previous work [23], the interplay between the absolute continuity of some curvature measure of a convex set and the measure theoretic size of the set of singular boundary points of this set has been elucidated. It is the purpose of the present paper to continue this line of research.

One of the basic roots of the present research can be traced back to a result of Aleksandrov. Let $K \subseteq \mathbb{R}^3$ be a full-dimensional convex body, and suppose that the *specific curvature* of $K$ is bounded, that is, there is a constant $\lambda \in \mathbb{R}$ such that $C_0(K, \cdot) \leq \lambda\, C_2(K, \cdot)$. Then $K$ is *smooth* (has a unique *support plane* through each boundary point); see [2] or [3, p. 445]. Obviously, the assumption of bounded specific curvature precisely means that the Gaussian curvature measure $C_0(K, \cdot)$ is absolutely continuous with respect to the boundary measure $C_2(K, \cdot)$ and the density function is bounded by a constant. Aleksandrov's result has been discussed in the books by Busemann [10, pp. 32–34] and Pogorelov [33, pp. 57–60] or in Schneider's survey [38]. These authors also raised the question whether suitable generalizations of this result could be established in higher dimensions. But only recently, an extension of Aleksandrov's result to higher dimensions and all curvature measures has been found by Burago & Kalinin [8]. As a consequence of their result, it follows that the assumption

$$C_r(K,\cdot) \le \lambda\, C_{d-1}(K,\cdot)\ , \tag{1}$$

for a closed convex set $K \subset \mathbb{R}^d$ with non-empty interior, a constant $\lambda \in \mathbb{R}$ and $r \in \{0,\dots,d-2\}$, implies that the dimension of the normal cone of $K$ at an arbitrary boundary point $x$ is $d-1-r$ at the most. In the important case of the mean curvature measure, that is for $r = d-2$, Bangert [6] and the present author [25] have independently (and by different approaches) obtained a much stronger characterization, saying that condition (1) holds if and only if a suitable ball rolls freely inside $K$. Thus it becomes apparent that the absolute continuity (with bounded density) of some curvature measure of a convex body $K$ with respect to the boundary measure of $K$ allows one to deduce a certain degree of regularity for the boundary surface of $K$.

The much more restrictive assumption

$$C_r(K,\cdot) = \lambda\, C_{d-1}(K,\cdot)\ , \tag{2}$$

for a convex body $K \subseteq \mathbb{R}^d$ with non-empty interior, a constant $\lambda \in \mathbb{R}$ and $r \in \{0,\dots,d-2\}$, yields that $K$ must be a ball. This result, which was first proved by Schneider [39], represents a substantial generalization of the classical Liebmann-Süss theorem to the non-smooth setting of convex geometry. A different proof and extensions to spaces of constant curvature or to certain combinations of curvature measures have been given by Kohlmann [29], [28]. For closed convex sets with non-empty interiors, Kohlmann (see [26], [27]) has also studied (weak) stability and splitting results under pinching conditions of the form

$$\alpha\, C_{d-1}(K,\cdot) \le C_r(K,\cdot) \le \beta\, C_{d-1}(K,\cdot)\ , \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$ are properly chosen constants. Furthermore, Bangert [6] has obtained an optimal splitting result in the case $r = d-2$. In some special situations, diameter bounds have been obtained; see, e.g., the contributions by Diskant [13], Lang [30], and Bangert [6]. Conditions of the form (3) can be used to state stability results, which have been explored by various authors; see Diskant [12], Schneider [40], Arnold [4], Kohlmann [26], [27], and the literature cited there. Actually, in some of these papers arguments are implicitly used which involve the absolute continuity of some curvature measure. It is the purpose of the present paper to investigate the relationship between the rather weak measure theoretic assumption of the absolute continuity of some curvature measure and the geometry of the associated convex set. In particular, we are concerned with *regularity results*. Thus we also provide the basis for subsequent work [25], in which the case of absolute continuity with bounded densities and some applications to *stability results* are treated.

For some of the results mentioned in the preceding paragraphs corresponding theorems are known for surface area measures. The degree of similarity between statements of results and methods of proof for curvature and surface area measures depends on the particular case which is considered. For example, recent approaches to characterizations of balls or stability results for curvature measures differ from the proofs of corresponding results for surface area measures. Moreover, surface area measures are distinguished by their connection to mixed volumes. Results for surface area measures which are in the spirit of the above mentioned theorems of Aleksandrov and Burago & Kalinin will be contained in [25] for the first time. There the interplay and analogy between surface area and curvature measures is, in fact, exploited as a technique of proof. A careful analysis of the nature of this analogy suggests an underlying *duality*, which will also be described more precisely in [25]. As a prerequisite for this subsequent work and since the results are interesting in their own right, we shall establish results concerning the absolute continuity of surface area measures which are dual (in a vague sense) to those for curvature measures.

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## 2. Notation and statement of results

The starting point for the present investigation is Theorem 2.1 below. In order to state it and to describe our main results, we fix some notation. Let $\mathcal{C}^d$ be the set of all non-empty closed convex sets $K \subset \mathbb{R}^d$. Let $\mathcal{H}^s$, $s \geq 0$, denote the $s$-dimensional Hausdorff measure in a Euclidean space. Which space is meant, will be clear from the context. The unit sphere of $\mathbb{R}^d$ with respect to the Euclidean norm $|\cdot|$ is denoted by $S^{d-1}$. If $K \in \mathcal{C}^d$ and $x \in \text{bd}\, K$ (the boundary of $K$), then the *normal cone* of $K$ at $x$ is denoted by $N(K, x)$; see [41] for notions of convex geometry which are not explicitly defined here. For our approach, the (generalized) *unit normal bundle* $\mathcal{N}(K)$ of a convex set $K \in \mathcal{C}^d$ plays an important rôle. It is defined as the set of all pairs $(x, u) \in \text{bd}\, K \times S^{d-1}$ such that $u \in N(K, x)$. Walter (see [49] or [50]) showed that this set represents a (strong) $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{R}^d$. For $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$, one can introduce *generalized curvatures* $k_i(x, u)$, $i \in \{1, \ldots, d-1\}$. These generalized curvatures can be obtained as limits of curvatures which are defined on the boundaries of the outer parallel sets of $K$. They are non-negative, since $K$ is convex. But they are merely defined almost everywhere on $\mathcal{N}(K)$, since the boundaries of the outer parallel sets of $K$ are submanifolds which are of class $C^{1,1}$, but need not be of class $C^2$. More explicitly, for any $\epsilon > 0$ let $K_\epsilon$ be the set of all $z \in \mathbb{R}^d$ whose distance from $K$ is at most $\epsilon$. For $y \in \text{bd } K_\epsilon$ let $\sigma_K(y)$ denote the exterior unit normal vector of $K_\epsilon$ at $y$. Then, for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$, the spherical image map $\sigma_K|\text{bd } K_\epsilon$ is differentiable at $x + \epsilon u$ for all $\epsilon > 0$ (see [49]), and therefore curvatures $k_1(x + \epsilon u), \ldots , k_{d-1}(x + \epsilon u)$ are defined as the eigenvalues of the symmetric linear map $D\sigma_K(x + \epsilon u)$ restricted to the orthogonal complement of $u$. Hence, for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$ and any $\epsilon > 0$, we can define

$$k_i(x, u) := \lim_{t \downarrow 0} \frac{k_i(x + \epsilon u)}{1 + (t - \epsilon)k_i(x + \epsilon u)} ,$$

$i \in \{1, \ldots , d-1\}$, independent of the particular choice of $\epsilon > 0$ (see [53]). We shall always assume that the ordering of these curvatures is such that

$$0 \leq k_1(x, u) \leq \ldots \leq k_{d-1}(x, u) \leq \infty . \tag{4}$$

In addition, we set $k_0(x, u) := 0$ and $k_d(x, u) := \infty$ for all $(x, u) \in \mathcal{N}(K)$. More details of this construction, in the more general context of sets with positive reach, can be found in M. Zähle [53] and in [23], [24].

The curvature measures of a general convex set $K$ cannot be expressed in terms of curvature functions which are defined (almost everywhere) on the boundary of $K$. However, the generalized curvature functions can be used to describe curvature measures in an appropriate way. This is the reason why, for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$, we define certain weighted elementary symmetric functions of generalized curvatures on $\mathcal{N}(K)$ by

$$\mathbb{H}_j(K, (x, u)) := \binom{d-1}{j}^{-1} \sum_{1 \leq i_1 < \ldots < i_j \leq d-1} \frac{k_{i_1}(x, u) \cdots k_{i_j}(x, u)}{\prod_{i=1}^{d-1} \sqrt{1 + k_i(x, u)^2}}$$

if $j \in \{1, \ldots , d-1\}$, and

$$\mathbb{H}_0(K, (x, u)) := \prod_{i=1}^{d-1} \frac{1}{\sqrt{1 + k_i(x, u)^2}} .$$

In the following, we refer to Chapter 1 of [14] for the basic notation and results concerning measure theory. However, there is one minor difference. For us a Radon measure in $\mathbb{R}^d$ will be defined on the Borel subsets of $\mathbb{R}^d$, whereas in [14] Radon measures are understood to be outer measures defined on all subsets of $\mathbb{R}^d$. The simple connection between these two points of view is as follows. A Radon measure $\mu$ in the sense of [14] yields a Radon measure in our sense simply by restricting $\mu$ to the $\sigma$-algebra of Borel sets.

On the other hand, a Radon measure $\mu$ on the Borel sets of $\mathbb{R}^d$ can be extended as a Radon measure $\bar{\mu}$ to all subsets of $\mathbb{R}^d$ by setting

$$\bar{\mu}(A) := \inf \left\{ \mu(B) : A \subseteq B \,, B \in \mathfrak{B}(\mathbb{R}^d) \right\} \ .$$

Here and subsequently, we denote by $\mathfrak{B}(X)$ the $\sigma$-algebra of Borel sets of an arbitrary topological space $X$. The preceding discussion shows that we can simply refer to Radon measures (on $\mathbb{R}^d$) without further explanations. Similar remarks apply to Radon measures on $S^{d-1}$.

Now let $\mu$ and $\nu$ be two Radon measures on $\mathbb{R}^d$. If $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathfrak{B}(\mathbb{R}^d)$, then we say that $\mu$ is *absolutely continuous* with respect to $\nu$, and we write $\mu \ll \nu$. By the Radon-Nikodym theorem, $\mu \ll \nu$ if and only if there is a non-negative Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mu(A) = \int_A f(x) \, \nu(dx)$$

for all $A \in \mathfrak{B}(\mathbb{R}^d)$. In particular, the *density function* $f$ is locally integrable with respect to $\nu$. Furthermore, we say that $\mu$ is *singular* with respect to $\nu$ if there is a Borel set $B \subseteq \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus B) = 0 = \nu(B)$, and in this case we write $\mu \perp \nu$. Certainly, this is a symmetric relation. A version of the Lebesgue decomposition theorem says that for arbitrary Radon measures $\mu$ and $\nu$ there are two Radon measures $\mu^a$ and $\mu^s$ such that $\mu = \mu^a + \mu^s$, $\mu^a \ll \nu$ and $\mu^s \perp \nu$. Moreover, the absolutely continuous part $\mu^a$ and the singular part $\mu^s$ (of $\mu$ with respect to $\nu$) are uniquely determined by these conditions. We shall also consider the restriction $(\mu \llcorner A)(\cdot) := \mu(A \cap \cdot)$ of a Radon measure $\mu$ to a set $A \in \mathfrak{B}(\mathbb{R}^d)$, which is again a Radon measure. Similar definitions and statements apply to measures on the Borel subsets of the unit sphere, where the surface area measures of convex bodies are defined.

These notions and results will now be applied to the curvature measures of a convex set $K \in \mathcal{C}^d$. As these measures are locally finite and concentrated on bd $K$, the curvature measure $C_r(K, \cdot)$, for any $r \in \{0, \dots , d-1\}$, can be written as the sum of two measures, that is,

$$C_r(K, \cdot) = C_r^a(K, \cdot) + C_r^s(K, \cdot) \ ,$$

where $C_r^a(K, \cdot)$ is absolutely continuous and $C_r^s(K, \cdot)$ is singular with respect to the boundary measure $C_{d-1}(K, \cdot)$. Recall that if $K \in \mathcal{C}^d$, then

$$C_{d-1}(K, \cdot) = \mathcal{H}^{d-1} \llcorner \operatorname{bd} K$$

if $K$ has non-empty interior or dim $K \leq d-2$. If dim $K = d-1$, then

$$C_{d-1}(K, \cdot) = 2(\mathcal{H}^{d-1} \llcorner \operatorname{bd} K) \ .$$

Subsequently, we often say that the $r$-th curvature measure of a convex set is absolutely continuous, by which we wish to express that this measure is absolutely continuous with respect to the boundary measure of the set.

The following result, which was proved in [24, Theorem 3.2], gives an explicit description of the singular part $C_r^s(K,\cdot)$ in terms of the generalized curvature functions on the unit normal bundle of $K$.

**Theorem 2.1.** *For a convex set $K \in \mathcal{C}^d$, $r \in \{0,\dots,d-1\}$, and $\beta \in \mathfrak{B}(\mathbb{R}^d)$,*

$$C_r^s(K,\beta) = \int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x)\mathbb{H}_{d-1-r}(K,(x,u))\,\mathcal{H}^{d-1}(d(x,u)) \tag{5}$$

*if $\mathcal{N}^s(K)$ is the set of all $(x,u) \in \mathcal{N}(K)$ such that $k_{d-1}(x,u) = \infty$.*

In Sect. 3, we shall show how Theorem 2.1 can be used to prove a useful condition which is necessary and sufficient for the absolute continuity of the $r$-th curvature measure of a convex set. It is appropriate to state such a characterization (Theorem 2.2) as a local result for curvature measures which are restricted to an arbitrary Borel subset of $\mathbb{R}^d$. Indeed, the absolute continuity of these measures merely depends on the local shape of the associated convex set. The following theorem will also play a key rôle in [25].

**Theorem 2.2.** *Let $K \in \mathcal{C}^d$, $r \in \{0,\dots,d-1\}$, and $\beta \in \mathfrak{B}(\mathbb{R}^d)$. Then*

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta \tag{6}$$

*if and only if*

$$k_{d-1}(x,u) < \infty \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty\ , \tag{7}$$

*for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $x \in \beta$.*

It should be emphasized that condition (7) can be checked by simply counting the number of curvatures which satisfy $k_i(x,u) = 0$ or $k_i(x,u) = \infty$, respectively. Also note that in the present situation condition (6) can be paraphrased by saying that the Radon measure $C_r(K,\cdot)$ on $\mathbb{R}^d$ is $(d-1)$-rectifiable. This terminology is used, e.g., in [34, p. 603], [32, p. 228], or [16], where the $(d-1)$-rectifiability of a general Radon measure $\mu$ is characterized in terms of properties of the $(d-1)$-dimensional densities of $\mu$. However, these investigations do not seem to be directly related to the present work.

As defined in the introduction, a convex body is a non-empty compact convex subset of $\mathbb{R}^d$. Let $\mathcal{K}^d$ denote the set of all convex bodies. In the special but important case of the curvature measure $C_0(K,\cdot)$ of a convex body $K$, we obtain (again from Theorem 2.1) a characterization of absolute continuity which involves a spherical supporting property of $K$. This property will be described by using the set $\mathrm{expn}^*K$ of *directions of nearest boundary points* of $K$. Formally, this is the set of all unit vectors $u \in S^{d-1}$ for which there exist points $x \in \mathrm{int}\, K$ and $y \in \mathrm{bd}\, K$ such that $|y-x| = \mathrm{dist}(x, \mathrm{bd}\, K)$ and $y - x = |y-x|u$. In other words, $u \in \mathrm{expn}^*K$ if and only if a non-degenerate ball which is contained in $K$ contains a boundary point of $K$ with exterior unit normal vector $u$. In the following, we shall say that $K \in \mathcal{K}^d$ is *supported from inside by a* $d$*-dimensional ball in direction* $u$ if and only if $u \in \mathrm{expn}^*K$. Since we are dealing with a local result, we shall also need the *spherical image* $\sigma(K, \beta)$ of a convex body $K$ at a set $\beta \subseteq \mathbb{R}^d$. Moreover, $D_{d-1}h(K,u)$ denotes the product of the principal radii of curvature of $K$ at $u$. This product is defined for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. It can be calculated as the determinant of the Hessian of the *support function* $h(K,\cdot)$ of $K \in \mathcal{K}^d$ restricted to the orthogonal complement of $u$. For explicit definitions we refer to [41].

**Theorem 2.3.** *For a convex body $K \in \mathcal{K}^d$ and $\beta \in \mathfrak{B}(\mathbb{R}^d)$, the following three conditions are equivalent:*

(a) $C_0(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$;
(b) $D_{d-1}h(K,u) > 0$ *for* $\mathcal{H}^{d-1}$ *almost all* $u \in \sigma(K,\beta)$;
(c) $\mathcal{H}^{d-1}(\sigma(K,\beta) \setminus \mathrm{expn}^*K) = 0$.

*In addition, for* $\gamma \in \mathfrak{B}(\mathbb{R}^d)$,

$$C_0^s(K,\gamma) = \mathcal{H}^{d-1}\left(\{u \in \sigma(K,\gamma) : D_{d-1}h(K,u) = 0\}\right)$$

*and*

$$C_0^a(K,\gamma) = \mathcal{H}^{d-1}\left(\{u \in \sigma(K,\gamma) : D_{d-1}h(K,u) > 0\}\right) \ .$$

Statement (b) of Theorem 2.3 is an analytic and statement (c) a geometric way of characterizing the absolute continuity of the Gaussian curvature measure. In fact, the geometric condition (c) can be viewed as a substantially weakened form of a condition requiring a suitable ball to roll freely inside $K$.

Using a *Crofton intersection formula* and various integral-geometric transformations, we extend Theorem 2.3 to curvature measures of any order. The corresponding result, Theorem 2.4, will be proved in Sect. 5. It can be interpreted as a two-step procedure for verifying the absolute continuity of curvature measures of convex bodies with non-empty interiors. For the curvature measure of order $d-r$ of a convex body $K$ and $r \in \{2, \dots, d-1\}$, the procedure essentially works as follows. First, one has to choose an $r$-dimensional affine subspace $E$ intersecting the interior of $K$. Second, one has to select a unit vector $u$ from the spherical image of the intersection

$K \cap E$ and check whether $u \in \operatorname{expn}^*(K \cap E)$. The precise formulation involves the suitably normalized Haar measure $\mu_r$ on the homogeneous space $\mathbf{A}(d, r)$ of $r$-dimensional affine subspaces in $\mathbb{R}^d$. Furthermore, here and in the following a prime which is attached to a quantity indicates that this quantity has to be calculated with respect to an appropriate affine or linear subspace. We denote the set of convex bodies with non-empty interiors by $\mathcal{K}_o^d$, and $U(E)$ is the unique linear subspace which is parallel to a given affine subspace $E$.

**Theorem 2.4.** *Let $K \in \mathcal{K}_0^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $r \in \{2, \dots, d-1\}$. Then*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

*if and only if, for $\mu_r$ almost all $E \in \mathbf{A}(d, r)$ such that $E \cap \operatorname{int} K \neq \emptyset$, the intersection $K \cap E$ is supported from inside by an $r$-dimensional ball in $\mathcal{H}^{r-1}$ almost all directions of the set $\sigma'(K \cap E, \beta \cap E) \subseteq U(E)$.*

The main tool for establishing such an extension in Sect. 5 is the special case $s = r$ of the following theorem, which is of interest in its own right. It refers to the set $\mathcal{C}_o^d$ of closed convex sets in $\mathcal{C}^d$ with non-empty interiors.

**Theorem 2.5.** *Let $K \in \mathcal{C}_o^d$, let $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and assume that $r \in \{2, \dots, d-1\}$ and $s \in \{r, \dots, d-1\}$. Then*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

*if and only if*

$$C'_{s-r}(K \cap E, \cdot) \llcorner (\beta \cap E) \ll C'_{s-1}(K \cap E, \cdot) \llcorner (\beta \cap E) \ ,$$

*for $\mu_s$ almost all $E \in \mathbf{A}(d, s)$ such that $E \cap \operatorname{int} K \neq \emptyset$.*

Recall that the prime which is attached to the curvature measure $C'_{s-r}(K \cap E, \cdot)$ means that this measure has to be calculated with respect to the affine hull of $K \cap E$. Thus, for $s = r$, Theorem 2.5 especially says that in the mean curvature case ($r = 2$) absolute continuity can be verified by investigating planar sections of $K$.

With regard to Theorem 2.4 it is natural to ask for a one-step procedure which allows one to decide whether a particular curvature measure of a convex body is absolutely continuous with respect to the boundary measure or not. A result which leads to such a procedure is contained in the ensuing Theorem 2.6. It is based on the following definitions.

Let us fix a convex body $K \in \mathcal{K}^d$ and some $r \in \{0, \dots, d-1\}$. For a unit vector $v \in S^{d-1}$ let $H(K, v)$ denote the support plane of $K$ with exterior normal vector $v$. An *affine subspace* $E \in \mathbf{A}(d, r)$ is said to *touch* $K$ if $E \cap K \neq \emptyset$ and $E \subseteq H(K, v)$ for some $v \in S^{d-1}$. Furthermore, we write $\mathrm{A}(K, d, r)$ for the $((d-r)(r+1)-1)$-rectifiable set of $r$-dimensional affine subspaces of $\mathbb{R}^d$ which touch $K$. On $\mathrm{A}(K,d,r)$ several authors [51], [18], [54], [35], [44] have introduced naturally defined measures. For convex bodies, however, all these measures are essentially equivalent. These contact measures have been used for calculating collision probabilities [37], [52], and they are related to absolute or total curvature measures [36], [5], [45]. Let us denote such a measure by $\mu_r(K,\cdot)$. Some relevant details will be described in Sects. 4 and 5.

Next we define the *spherical image of order* $r$ of $K \in \mathcal{K}_o^d$ at $\beta \in \mathfrak{B}(\mathbb{R}^d)$ for any $r \in \{0,\dots,d-1\}$ by

$$\sigma_r(K,\beta) := \{E \in \mathrm{A}(K,d,r) : \beta \cap \mathrm{bd}\, K \cap E \neq \emptyset\}\ .$$

The case $r=d-1$ leads to the ordinary spherical image, since $\mathrm{A}(K,d,d-1)$ is the set of supporting hyperplanes of $K$ each of which can be identified with its exterior unit normal vector. Let $\omega_i$ denote the surface area of the $(i-1)$-dimensional unit sphere. Then the measure $\mu_r(K,\cdot)$ will be normalized so that the relation

$$C_{d-1-r}(K,\beta) = \frac{\omega_d}{\omega_{d-r}}\,\mu_r(K,\sigma_r(K,\beta))\ , \tag{8}$$

due to Weil [51], holds for all $\beta \in \mathfrak{B}(\mathbb{R}^d)$. Set $u^- := \{tu : t \le 0\}$ if $u \in \mathbb{R}^d \setminus \{o\}$, let $r \in \{2,\dots,d-1\}$, and define $B(z,t) := \{y \in \mathbb{R}^d : |y-z| \le t\}$ if $z \in \mathbb{R}^d$ and $t \ge 0$. Then we say that $K$ is *supported from inside by an* $r$*-dimensional ball at* $E \in \mathbf{A}(K,d,r-1)$ if there is some $p \in K \cap E$, some $u \in S^{d-1} \cap U(E)^\perp$ with $(E+u^-) \cap \mathrm{int}\, K \neq \emptyset$, and some $\rho > 0$ such that $B(p-\rho u,\rho) \cap (E+u^-) \subseteq K$.

Equation (8) provides an integral-geometric interpretation for curvature measures of convex sets. In the present context, it also suggests a characterization of absolute continuity involving touching planes.

**Theorem 2.6.** *Let* $K \in \mathcal{K}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, *and* $r \in \{2,\dots,d-1\}$. *Then*

$$C_{d-r}(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

*if and only if* $K$ *is supported from inside by an* $r$*-dimensional ball at* $\mu_{r-1}(K,\cdot)$ *almost all* $E \in \sigma_{r-1}(K,\beta)$.

Essentially, Theorem 2.6 is deduced from Theorem 2.4 through a succession of auxiliary results. The proof includes arguments from convexity, geometric measure theory and also some basic results about Haar measures. The key idea is to associate with an $r$-dimensional affine subspace $E$ meeting int $K$ and a unit vector $u \in U(E)$ the $(r-1)$-dimensional support plane of $K \cap E$ relative to $E$ with exterior unit normal vector $u$. This support plane then represents an $(r-1)$-dimensional affine subspace which touches $K$.

It has already become apparent that the boundary of a convex body $K \in \mathcal{K}_o^d$ one of whose curvature measures is absolutely continuous with respect to the boundary measure cannot be too irregular. A precise and in a certain sense optimal result in this spirit is stated as Theorem 4.6 in [24]. Another regularity result, which complements the picture, is provided by the following theorem. As usual, we say that $x \in \mathrm{bd}\, K$ is a *regular boundary point* of $K \in \mathcal{K}_o^d$ if there exists precisely one support plane of $K$ passing through $x$.

**Theorem 2.7.** *Let $K \in \mathcal{K}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, $r \in \{2, \ldots, d-1\}$, and assume that*

$$C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta \,.$$

*Then, for $\mu_{r-1}(K, \cdot)$ almost all $E \in \sigma_{r-1}(K, \beta)$, every boundary point of $K$ which lies in $E$ is regular.*

In convex and integral geometry, the surface area measures are at least as important as the curvature measures, and, perhaps, they are even more related to other parts of convexity. The surface area measures $S_r(K, \cdot)$ are defined for convex bodies $K \in \mathcal{K}^d$ and $r \in \{0, \ldots, d-1\}$ as measures on the $\sigma$-algebra of Borel subsets of the unit sphere. In addition, $S_0(K, \cdot)$ is equal to the restriction of the $(d-1)$-dimensional Hausdorff measure to the $\sigma$-algebra of Borel subsets of the unit sphere. Therefore it is natural to study characterizations and implications of the condition

$$S_r(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega,$$

where $\omega \subseteq S^{d-1}$ is an arbitrary Borel set. Indeed, for surface area measures, we obtain results which are similar to those already described for curvature measures. This will be shown in Sects. 3 and 4. In fact, a comparison of results suggests an underlying duality which will be investigated more thoroughly in a subsequent paper [25].

## 3. Characterization of absolute continuity

We have already stressed the point that Theorem 2.1 from the introduction and, similarly, Theorem 3.5 from [24] (see also the proof of Theorem 3.6 in this section) provide explicit expressions for the singular parts of the curvature and surface area measures of suitable convex sets, respectively. These expressions now lead to a first characterization of the absolute continuity for curvature and surface area measures, in terms of generalized curvature functions, if they are combined with Lemma 3.1 below. From these expressions, we can also deduce more geometric characterizations of absolute continuity in the special cases of the Gaussian curvature measure and the surface area measure of order $d-1$. Note that by referring to absolute continuity we always mean absolute continuity with respect to the boundary measure or the surface area measure of order zero, that is, (in both cases) with respect to the suitably restricted $(d-1)$-dimensional Hausdorff measure.

In this section, we shall first consider the case of curvature measures, and then we discuss corresponding results for surface area measures. Section 4 will exclusively be devoted to a thorough study of surface are measures, since for these measures the arguments seem to be slightly easier. Dual results for curvature measures then constitute the subject of Sect. 5.

In the following, we refer to Schneider's book [41] for notation and for notions from convexity which are not defined here. From [14], [23] and [24] we adopt the terminology concerning measure theory. For example, normalized elementary symmetric functions of principal curvatures $H_i(K,\cdot)$ or radii of curvature $D_i h(K,\cdot)$, for suitable convex sets $K$ and $i \in \{0,\ldots,d-1\}$, are defined as in [24]. The conventions for calculations involving '$\infty$' are the same as in [23, §2]. Further, in the case $r=0$ the left-hand side of Eq. (9) below is defined as

$$\prod_{j=1}^{n} \sqrt{1+a_j^2}^{\,-1} .$$

Of course, this is motivated by the expression by which $\mathbb{H}_0(K,(x,u))$ has been defined.

**Lemma 3.1.** *Let $r,n \in \mathbb{N}$, $0 \le r \le n$, $n \ge 1$, and $a_1,\ldots,a_n \in [0,\infty]$. Assume that $a_1 \le \ldots \le a_n$. In addition, we define $a_0 := 0$ and $a_{n+1} := \infty$. Then*

$$\sum_{1\le i_1<\ldots<i_r\le n} \frac{a_{i_1}\cdots a_{i_r}}{\prod_{j=1}^{n}\sqrt{1+a_j^2}} = 0 \tag{9}$$

*if and only if either $a_{n-r+1}=0$ or $a_{n-r}=\infty$.*

*Proof.* First of all, for arbitrary $n \in \mathbb{N}$ the special cases $r=0$ and $r=n$ are easily verified.

The general statement is proved by induction with respect to $n \in \mathbb{N}$. Let $A(n)$ be the statement of the lemma. Statement $A(1)$ has already been proved by considering the special cases $r=0$ and $r=n$. Hence, we assume that $A(n-1)$ has been proved for some $n \ge 2$. We show that $A(n)$ is true. The cases $r=0$ and $r=n$ have already been checked. Thus let $1 \le r \le n-1$. Then the condition

$$\sum_{1\le i_1<\ldots<i_r\le n} \frac{a_{i_1}\cdots a_{i_r}}{\prod_{j=1}^{n}\sqrt{1+a_j^2}} = 0 \tag{10}$$

will be considered in each of the two cases $a_n=\infty$ and $0 \le a_n < \infty$.

If $a_n = \infty$, then Eq. (10) is equivalent to

$$\sum_{1 \le i_1 < \ldots < i_{r-1} \le n-1} \frac{a_{i_1} \cdots a_{i_{r-1}}}{\prod_{j=1}^{n-1} \sqrt{1+a_j^2}} = 0 \; , \tag{11}$$

since all summands in Eq. (10) vanish which correspond to indices $1 \le i_1 < \ldots < i_r < n$ and since

$$\frac{a_n}{\sqrt{1+a_n^2}} = 1 \; .$$

Here, $0 \le r-1 \le n-1$, $n-1 \ge 1$, $a_1, \ldots, a_{n-1} \in [0, \infty]$, and $a_1 \le \ldots \le a_{n-1}$. Since $A(n-1)$ is assumed to be true, Eq. (11) is equivalent to $a_{n-1-(r-1)+1} = 0$ or $a_{n-1-(r-1)} = \infty$, that is, $a_{n-r+1} = 0$ or $a_{n-r} = \infty$.

If $0 \le a_n < \infty$, then Eq. (10) implies that

$$\frac{a_{n-r+1} \cdots a_n}{\prod_{j=1}^{n} \sqrt{1+a_j^2}} = 0 \; .$$

Since $a_1 \le \ldots \le a_n < \infty$, necessarily $a_{n-r+1} = 0$. Conversely, if $0 \le a_n < \infty$ and $a_{n-r+1} = 0$, then $0 = a_1 = \ldots = a_{n-r+1} \le \ldots \le a_n < \infty$, and hence Eq. (10) holds.

This shows that Eq. (10) is equivalent to

$$\left[\begin{array}{c} a_n = \infty \text{ and } (a_{n-r+1} = 0 \text{ or } a_{n-r} = \infty) \\ \text{or} \\ 0 \le a_n < \infty \text{ and } a_{n-r+1} = 0 \; . \end{array}\right.$$

But this exactly is the statement of $A(n)$. □

Recall that in order to simplify the presentation, we complemented the definition of the generalized curvatures on the unit normal bundle of a convex set $K$ by setting $k_0(x,u) := 0$ and $k_d(x,u) := \infty$ for $(x,u) \in \mathcal{N}(K)$. This will help us to avoid the need to distinguish different cases.

*Proof of Theorem 2.2.* By Theorem 2.1, the condition

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

is equivalent to

$$\int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x) \mathbb{H}_{d-1-r}(K,(x,u)) \, \mathcal{H}^{d-1}(d(x,u)) = 0 \; .$$

But this is tantamount to saying that for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $x \in \beta$,

$$(x,u) \notin \mathcal{N}^s(K) \quad \text{or} \quad \mathbb{H}_{d-1-r}(K,(x,u)) = 0 \; . \tag{12}$$

Here and subsequently, we tacitly use the essential fact that the generalized curvature functions which are associated with convex sets are non-negative. From Lemma 3.1 and the definition of the set $\mathcal{N}^s(K)$, we obtain that condition (12) is equivalent to

$$k_{d-1}(x,u) < \infty \quad \text{or} \quad k_{d-1-(d-1-r)+1}(x,u) = 0 \quad \text{or} \quad k_{d-1-(d-1-r)}(x,u) = \infty\ ,$$

which was to be proved. □

The following two corollaries are designed to illustrate Theorem 2.2.

**Corollary 3.2.** *Let* $K \in \mathcal{C}^d$*,* $\beta \in \mathfrak{B}(\mathbb{R}^d)$*, and* $i \in \{0, \dots, d-1\}$*. Then*

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

*for all* $r \in \{i, \dots, d-1\}$ *if and only if*

$$k_{d-1}(x,u) < \infty \quad \text{or} \quad k_i(x,u) = \infty\ ,$$

*for* $\mathcal{H}^{d-1}$ *almost all* $(x,u) \in \mathcal{N}(K)$ *such that* $x \in \beta$*.*

**Corollary 3.3.** *Let* $K \in \mathcal{C}^d$*,* $\beta \in \mathfrak{B}(\mathbb{R}^d)$*, and* $i \in \{0, \dots, d-1\}$*. Then*

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

*for all* $r \in \{0, \dots, i\}$ *if and only if*

$$k_{d-1}(x,u) < \infty \quad \text{or} \quad k_{i+1}(x,u) = 0\ ,$$

*for* $\mathcal{H}^{d-1}$ *almost all* $(x,u) \in \mathcal{N}(K)$ *such that* $x \in \beta$*.*

Theorem 2.2 also yields a sufficient condition for the absolute continuity of all curvature measures of a given convex set $K$. In the following corollary, the assumption on bd $K \cap \beta$ implies that the restriction of the spherical image map $\sigma_K$ to the set $\beta$ is locally Lipschitzian. Recall that the *spherical image map* of a convex set $K \in \mathcal{C}^d_o$ is defined for regular boundary points, and for such a boundary point $x$ it is equal to the unique exterior unit normal vector of $K$ at $x$; see [41, §2.2]. If, in addition, we assume that the Lipschitz constant of $\sigma_K|(\text{bd}\, K \cap \beta)$ is smaller than a constant $c$, then we obtain that $k_i(x) \le c$, for $i \in \{1, \dots, d-1\}$ and $\mathcal{H}^{d-1}$ almost all $x \in \text{bd}\, K \cap \beta$, which yields bounds for the densities of the curvature measures of $K$.

**Corollary 3.4.** *Let* $K \in \mathcal{C}^d_o$*,* $\beta \in \mathfrak{B}(\mathbb{R}^d)$*, and assume that* bd $K \cap \beta$ *is locally of class* $C^{1,1}$*. Then* $C_i(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$ *for all* $i \in \{0, \dots, d-2\}$*.*

*Proof.* Use, for example, Lemma 3.1 from [24] and Theorem 2.2. □

In the special case where $K$ is a convex body of revolution, Theorem 2.2 can be used to establish a simple characteristic condition for the absolute continuity of the curvature measures of $K$. We fix some notation. Let $(e_1,\dots,e_d)$ be an orthonormal basis of $\mathbb{R}^d$. Let $f:(a,b)\to[0,\infty)$ be a concave function, and define the curve

$$\gamma:(a,b)\to\mathbb{R}^2\,,\quad t\mapsto te_d+f(t)e_1\;.$$

The convex body which is obtained by rotating $\gamma$ around the $e_d$-axis and taking the closed convex hull is denoted by $K$, and $K':=K\cap\operatorname{lin}\{e_1,e_d\}$. Furthermore, define the function

$$F:(a,b)\times S^{d-2}\to\mathbb{R}^d\,,\quad (t,u)\mapsto te_d+f(t)u\;,$$

where $S^{d-2}:=S^{d-1}\cap\operatorname{lin}\{e_1,\dots,e_{d-1}\}$.

**Theorem 3.5.** *Let $\alpha\in\mathfrak{B}((a,b))$ and $d\geq 3$. Then the following three conditions are equivalent:*

(a) $C_0(K',\cdot)\llcorner\gamma(\alpha)\ll C_1(K',\cdot)\llcorner\gamma(\alpha)$*;*
(b) $C_i(K,\cdot)\llcorner F(\alpha\times S^{d-2})\ll C_{d-1}(K,\cdot)\llcorner F(\alpha\times S^{d-2})$ *for all* $i\in\{0,\dots,d-2\}$*;*
(c) $C_i(K,\cdot)\llcorner F(\alpha\times\omega)\ll C_{d-1}(K,\cdot)\llcorner F(\alpha\times\omega)$ *for some* $i\in\{0,\dots,d-2\}$ *and some* $\omega\in\mathfrak{B}(S^{d-2})$ *with* $\mathcal{H}^{d-2}(\omega)>0$.

*Proof.* An elementary calculation yields that

$$N(t,u):=\sigma_K(F(t,u))=\frac{u-f'(t)e_d}{\sqrt{1+f'(t)^2}}\,,\quad (t,u)\in(a,b)\times S^{d-2}\,,$$

whenever $f$ is differentiable at $t$. If, in addition, $f$ is $C^1$ on $(a,b)$ and second order differentiable at $t$, then it follows that

$$\frac{\partial N}{\partial t}(t,u)=-\frac{f''(t)}{\sqrt{1+f'(t)^2}^3}\,\frac{\partial F}{\partial t}(t,u)$$

and

$$\frac{\partial N}{\partial u_i}(t,u)=\frac{1}{f(t)\sqrt{1+f'(t)^2}}\,\frac{\partial F}{\partial u_i}(t,u)\,,\quad i\in\{1,\dots,d-2\}\,,$$

where $u\in S^{d-2}$ and $(u_1,\dots,u_{d-2},u)$ is an orthonormal basis of the subspace $\operatorname{lin}\{e_1,\dots,e_{d-1}\}$. If these relations are applied to the parallel bodies of $K$ and $K'$, respectively, then one can see that $k(y,v)$ is defined for some $(y,v)\in\mathcal{N}(K')$ with $y\in\gamma(\alpha)$ if and only if $k_1(y,v),\dots,k_{d-1}(y,v)$ are defined for $(y,v)\in\mathcal{N}(K)$. Moreover, if one of these conditions is fulfilled, then $k_1(y',v'),\dots,k_{d-1}(y',v')$ are also defined, whenever $(y',v')$ is obtained from $(y,v)$ by rotation around the $e_d$-axis. Note that we denote by

$k(\cdot)$ the curvature function on the unit normal bundle of $K'$ and that the assumption (4) on the ordering of the generalized curvatures of $K$ is suspended during the present proof. Nevertheless, Theorem 2.2 remains applicable if interpreted properly. Furthermore, if one of the previous conditions is fulfilled, then we also have

$$k(y,v) = k_{d-1}(y,v) = k_{d-1}(y',v') \tag{13}$$

and

$$k_i(y,v) = k_i(y',v') = d(y,v)^{-1} \ , \quad i \in \{1,\ldots,d-2\} \ , \tag{14}$$

where $d(y,v) := |y-P(y,v)| \in (0,\infty)$ and $\{P(y,v)\} := (y+\mathbb{R}\,v)\cap\mathbb{R}\,e_d$.

The subsequent implications follow from repeated application of Theorem 2.2.

First, (a) is fulfilled if and only if $k(y,v) < \infty$ for $\mathcal{H}^1$ almost all $(y,v) \in \mathcal{N}(K')$ such that $y \in \gamma(\alpha)$. But then we obtain from (13), (14) and a Fubini-type argument that $k_i(y,v) < \infty$ holds for all $i \in \{1,\ldots,d-1\}$ and for $\mathcal{H}^{d-1}$ almost all $(y,v) \in \mathcal{N}(K)$ such that $y \in F(\alpha \times S^{d-2})$. Therefore (b) is true.

Obviously, (b) implies (c).

Finally, assume that (c) is fulfilled. Due to (14) this yields that $k_{d-1}(y,v) < \infty$ must hold for $\mathcal{H}^{d-1}$ almost all $(y,v) \in \mathcal{N}(K)$ such that $(y,v) \in F(\alpha\times\omega)$. If the set of all $(y,v) \in \mathcal{N}(K')$ such that $y \in \gamma(\alpha)$ and $k(y,v) = \infty$ has positive $\mathcal{H}^1$ measure, then the set of $(y,v) \in \mathcal{N}(K)$ such that $y \in F(\alpha\times\omega)$ and $k_{d-1}(y,v) = \infty$ has positive $\mathcal{H}^{d-1}$ measure. This can be seen from $\mathcal{H}^{d-2}(\omega) > 0$ and a Fubini-type argument. This contradiction shows that $k(y,v) < \infty$ for $\mathcal{H}^1$ almost all $(y,v) \in \mathcal{N}(K')$ such that $y \in \gamma(\alpha)$, and hence (a) must be true. $\square$

To see an application of Theorem 3.5 concerning a question of boundary regularity, assume that $C_i(K,\cdot) \llcorner F(\alpha\times\omega) \ll C_{d-1}(K,\cdot) \llcorner F(\alpha\times\omega)$ holds for some $i \in \{0,\ldots,d-2\}$, an intervall $\alpha \subseteq (a,b)$, and some $\omega \in \mathfrak{B}(S^{d-2})$ with $\mathcal{H}^{d-2}(\omega) > 0$. Hence we obtain that $C_0(K',\cdot) \llcorner \gamma(\alpha) \ll C_1(K',\cdot) \llcorner \gamma(\alpha)$, and this implies that $\gamma|\alpha$ is of class $C^1$. But then $F|(\alpha \times S^{d-2})$, too, is of class $C^1$. This should be compared with the immediate conclusion which can be obtained from Theorem 4.6 in [24].

Now, we are going to prove Theorem 2.3. Recall from [21] or from the introduction the definition of the set $\mathrm{expn}^* K$ of directions of nearest boundary points of a convex body $K$, which can be rewritten in the form

$$\mathrm{expn}^* K = \left\{ u \in S^{d-1} : B(x,\rho(K-x,u)) \subseteq K \text{ for some } x \in \mathrm{int}\, K \right\} \ ,$$

where $\rho(K-x,\cdot)$ denotes the *radial function* of $K$ with respect to $x$.

*Proof of Theorem 2.3.* The equivalence of (b) and (c) follows from Lemma 2.7 in [21] if the second order differentiability almost everywhere of the support function $h(K,\cdot)$ is used; see [1] and the Notes for Chap. 2.5 in [41].

It remains to prove that (a) $\Leftrightarrow$ (b). But (a) is equivalent to

$$\int_{\mathcal{N}^s(K)} \mathbf{1}_\beta(x)\mathbb{H}_{d-1}(K,(x,u))\,\mathcal{H}^{d-1}(d(x,u)) = 0\ . \tag{15}$$

Recall that a unit vector $u \in S^{d-1}$ is said to be a *regular normal vector* of $K$ if the *support set* $F(K,u)$ of $K$ with exterior normal vector $u$ consists of a single point. This point is denoted by $\tau_K(u)$, and the corresponding map $\tau_K$, which is defined on the set of regular normal vectors, is called the *reverse spherical image map* of $K$. It is known that $\mathcal{H}^{d-1}$ almost all unit vectors are regular normal vectors of a given convex body $K$; see Theorem 2.2.9 in [41].

An application of the coarea formula (Theorem 3.2.22 in [17]) to the projection map $\pi_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\pi_2(x,y) := y$, shows that (15) is equivalent to

$$\begin{aligned}\mathcal{H}^{d-1}(\{u \in S^{d-1} : \tau_K(u) \in \beta \quad \text{and}\\ k_{d-1}(\tau_K(u),u) = \infty\}) = 0\ .\end{aligned} \tag{16}$$

Further, Lemma 3.4 from [24] implies that

$$k_{d-1}(\tau_K(u),u) = \infty \Longleftrightarrow D_{d-1}h(K,u) = 0\ , \tag{17}$$

for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$. This finally yields the equivalence of (a) and (b).

For the proof of the additional statement observe that, for an arbitrary set $\gamma \in \mathfrak{B}(\mathbb{R}^d)$,

$$C_0^s(K,\gamma) = \mathcal{H}^{d-1}(\{u \in \sigma(K,\gamma) : D_{d-1}h(K,u) = 0\})\ .$$

This immediately follows from (16) and (17). Finally, note that

$$C_0^a(K,\gamma) = \mathcal{H}^{d-1}(\sigma(K,\gamma)) - C_0^s(K,\gamma)\ ;$$

compare Eq. (4.2.21) in Schneider [41]. □

*Remark 1.* It should be emphasized that even if $D_{d-1}h(K,u) > 0$ for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, the convex body $K$ is not smooth in general. As a counterexample for dimension $d = 3$ one can choose the polar body $K^*$ of a suitable translate of the convex body $K$ which is defined in Remark 2 below. The fact that the condition $D_2h(K^*,u) > 0$ is fulfilled for $\mathcal{H}^2$ almost all $u \in S^2$ can, for example, be deduced from Theorem 2.2 of [22].

For future investigations of the subject, it will be essential to have characterizations of absolute continuity for both curvature measures and surface area measures. Therefore the remaining part of this section is mainly devoted to briefly establishing results for surface area measures which, in a certain sense, are dual to those already obtained for curvature measures. We shall also provide some explicit examples which can serve to illustrate the abstract results. But these examples also demonstrate that certain conclusions cannot be obtained without additional assumptions.

**Theorem 3.6.** *Let* $K \in \mathcal{K}^d$, $r \in \{0, \dots, d-1\}$, *and* $\omega \in \mathfrak{B}(S^{d-1})$. *Then*

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

*if and only if*

$$k_1(x,u) > 0 \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty \ ,$$

*for* $\mathcal{H}^{d-1}$ *almost all* $(x,u) \in \mathcal{N}(K)$ *such that* $u \in \omega$.

*Proof.* Let $\mathcal{N}_s(K)$ denote the set of all $(x,u) \in \mathcal{N}(K)$ such that $k_1(x,u) = 0$. Then the assumption

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

is equivalent to

$$\int_{\mathcal{N}_s(K)} \mathbf{1}_\omega(u) \mathbb{H}_{d-1-r}(K,(x,u))\, \mathcal{H}^{d-1}(d(x,u)) = 0 \ .$$

This is an immediate consequence of Theorem 3.5 in [24]. In other words, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$,

$$(x,u) \notin \mathcal{N}_s(K) \quad \text{or} \quad \mathbb{H}_{d-1-r}(K,(x,u)) = 0 \ .$$

An application of Lemma 3.1 thus completes the proof. □

As in the case of the Gauss curvature measure $C_0(K,\cdot)$, the absolute continuity of $S_{d-1}(K,\cdot)$ can be characterized by a spherical supporting property. The situation here is 'dual' to the previous one. Condition (c) of Theorem 3.7 below can be interpreted as a substantially weakened form of a condition demanding $K$ to roll freely inside a ball. The statement of this theorem involves the set $\exp^* K$ of *farthest boundary points* of a convex body $K$ (see [21]). This definition implies that $x \in \exp^* K$ holds if and only if the boundary of a ball which contains $K$ passes through $x$. In the following, we say that $K$ is *supported from outside by a* $d$*-dimensional ball at* $x$ if and only if $x \in \exp^* K$. Moreover, recall from [41, §2.2] that $\tau(K,\omega)$ denotes the *reverse spherical image* of $K$ at a set $\omega \subseteq S^{d-1}$. By definition, $\tau(K,\omega)$ is equal to the union of the support sets $F(K,u)$ with $u \in \omega$.

**Theorem 3.7.** *Let $K \in \mathcal{K}^d$ and $\omega \in \mathfrak{B}(S^{d-1})$. Then the following three conditions are equivalent:*

(a) $S_{d-1}(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$;
(b) $H_{d-1}(K,x) > 0$ *for* $\mathcal{H}^{d-1}$ *almost all* $x \in \tau(K,\omega)$;
(c) $\mathcal{H}^{d-1}(\tau(K,\omega) \setminus \exp^* K) = 0$.

*In addition, for* $\alpha \in \mathfrak{B}(S^{d-1})$,

$$S^s_{d-1}(K,\alpha) = \mathcal{H}^{d-1}\left(\{x \in \tau(K,\alpha) : H_{d-1}(K,x) = 0\}\right)$$

*and*

$$S^a_{d-1}(K,\alpha) = \mathcal{H}^{d-1}\left(\{x \in \tau(K,\alpha) : H_{d-1}(K,x) > 0\}\right) \ .$$

*Proof.* The equivalence of (b) and (c) follows from Corollary 3.2 in [21].

It remains to prove that (a) $\Leftrightarrow$ (b). From Theorem 3.5 in [24] it can be seen that (a) is equivalent to

$$\int_{\mathcal{N}_s(K)} \mathbf{1}_\omega(u) \mathbb{H}_0(K,(x,u))\, \mathcal{H}^{d-1}(d(x,u)) = 0 \ .$$

An application of the coarea formula to $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(x,y) \mapsto x$, shows that this precisely means

$$\mathcal{H}^{d-1}(\{x \in \mathrm{bd}\, K : \sigma_K(x) \in \omega \quad \text{and} \quad k_1(x,\sigma_K(x)) = 0\}) = 0$$

if $\sigma_K$ denotes the spherical image map, which is defined for $\mathcal{H}^{d-1}$ almost all boundary points of $K$. In addition, it follows from Lemma 3.1 in [24] that

$$k_1(x,\sigma_K(x)) = 0 \Longleftrightarrow H_{d-1}(K,x) = 0 \ ,$$

for $\mathcal{H}^{d-1}$ almost all $x \in \mathrm{bd}\, K$. This finally implies the equivalence of (a) and (b).

For the proof of the additional statement note that due to Eq. (4.2.24) in [41], the relation $S_{d-1}(K,\alpha) = \mathcal{H}^{d-1}(\tau(K,\alpha))$ holds for an arbitrary Borel set $\alpha \in \mathfrak{B}(S^{d-1})$. $\square$

*Remark 2.* Even if $H_{d-1}(K,x) > 0$ is fulfilled for $\mathcal{H}^{d-1}$ almost all $x \in \mathrm{bd}\, K$, it does not follow that $K$ is strictly convex if $d \geq 3$. The following counterexample is due to Dekster [11]. Denote by $(e_1,e_2,e_3)$ the standard basis of $\mathbb{R}^3$. Let $K$ be the closure of the convex hull of the image set $X((-1,1) \times (-\pi,\pi))$, where

$$X : (-1,1) \times (-\pi,\pi) \to \mathbb{R}^3 \ ,$$
$$(y,t) \mapsto ((1-y^2)\sin t, y, (1-y^2)(1+\cos t)) \ .$$

Then the segment $[-e_2, e_2]$ is contained in the boundary of $K$, although one can show that $H_2(K,x)$ exists even in the sense of classical differential geometry and is positive for all $x \in \operatorname{bd} K \setminus [-e_2, e_2]$. It should be emphasized, however, that there is no positive constant $c$ such that $H_2(K,x) \geq c$ is true for $\mathcal{H}^2$ almost all $x \in \operatorname{bd} K$. Another example for a different purpose is given below.

It should also be observed that strict convexity does not imply that $H_{d-1}(K,x) > 0$ holds for a set of points $x \in \operatorname{bd} K$ which has positive $(d-1)$-dimensional Hausdorff measure. This follows, for instance, from a Baire category argument.

*Remark 3.* Statements analogous (dual) to Corollaries 3.2–3.5 can be proved for surface area measures as well.

In the following we shall describe the construction of a convex body $K \in \mathcal{K}_0^3$ for which $S_2(K,\cdot)$ is absolutely continuous with respect to $S_0(K,\cdot)$, but for which $S_1(K,\cdot)$ is not absolutely continuous. Also note that if $S_1(K,\cdot)$ is absolutely continuous, then $S_2(K,\cdot)$ can still have point masses. An example will be given in [25].

*Example 1.* First of all we define three convex surfaces $F_1, F_2, F_3^\alpha$ by

$$F_1 := \left\{ \left( x, y, |x| + \frac{4}{5}x^2 \left( 1 + \frac{1}{4}y^2 \right) \right) \in \mathbb{R}^3 : x \in [0,1],\ y \in [0,1] \right\} ,$$

$$F_2 := \left\{ \left( x\cos\varphi, x\sin\varphi, x + \frac{4}{5}x^2 \right) \in \mathbb{R}^3 : x \in [0,1],\ \varphi \in [\pi, 2\pi] \right\} ,$$

and, for $\alpha \in (0,1]$,

$$F_3^\alpha := \big\{ (x, 1 + \alpha(1-x^2)t(2-t), (1-t)(|x| + x^2) + 2t) \in \mathbb{R}^3 : \\ x \in [-1,1],\ t \in [0,1] \big\} .$$

Note that $F_3^\alpha \setminus \{(1,0,2),(-1,0,2)\}$ is equal to $F_4^\alpha$, where

$$F_4^\alpha := \left\{ \left( x, \alpha(1-x^2) \left[ 1 - \frac{(z-2)^2}{(|x|-1)^2(|x|+2)^2} \right], z \right) \in \mathbb{R}^3 : \\ x \in (-1,1),\ |x| + x^2 \leq z \leq 2 \right\} .$$

The convexity of $F_2$ is clear, since $F_2$ is obtained by rotating the strictly monotone convex curve

$$x \mapsto \left( x, 0, x + \frac{4}{5}x^2 \right) , \quad x \in [0,1] ,$$

around the $e_3$-axis. The convexity of $F_1$ and $F_4^\alpha$ can be proved with the help of Tietze's theorem; see, for example, Theorem 4.10 of Valentine's book [48]. For the smooth boundary points of $F_1$ and $F_4^\alpha$ the local supporting property, which is required for the application of Tietze's theorem, can be checked by verifying that the Gauss-Kronecker curvature is positive. For the non-smooth boundary points the local supporting property can be seen directly.

Now, let $E^\alpha$ be the union $E_1 \cup E_2 \cup E_3^\alpha$ of the epigraphs $E_1$, $E_2$, $E_3^\alpha$ of $F_1$, $F_2$, $F_3^\alpha$, respectively. Then $E^\alpha$ is a closed convex set if $\alpha \in (0,1]$ is sufficiently small. To see this check the local supporting property for the points which belong to the curves

$$x \mapsto \left(x, 0, |x| + \frac{4}{5}x^2\right) , \quad x \in [-1,1] ,$$

and

$$x \mapsto \left(x, 1, |x| + x^2\right) , \quad x \in [-1,1] .$$

In fact, it can be shown that any $\alpha \in (0,1]$ is suitable. But this requires some calculations. Let us denote by $E$ one such suitable set. Finally, set $K_1 := E \cap H^-(e_3, 1)$, and let $K_2$ be the reflection of $K_1$ at the hyperplane $H(e_3,1)$. Here, $H^-(e_3,1) := \{x \in \mathbb{R}^d : \langle x, e_3\rangle \leq 1\}$, $H(e_3,1)$ is the bounding affine hyperplane, and $\langle\cdot\,,\cdot\rangle$ denotes the Euclidean scalar product. Then $K := K_1 \cup K_2$ is the required convex body. For a visualization of $K$, see Fig. 1 below*. Again convexity can be proved by verifying the local supporting property for the points in bd $K \cap H(e_3,1)$.

The absolute continuity of $S_2(K,\cdot)$ can be seen from Theorem 3.7, since $K$ has been constructed in such a way that $H_2(K,x) > 0$ for $\mathcal{H}^2$ almost all $x \in$ bd $K$. This follows from explicit calculations. The first surface area measure, $S_1(K,\cdot)$, however, is not absolutely continuous. To see this, consider the set $\mathcal{N}_1$ which is defined by

$$\mathcal{N}_1 := \{((0,y,0),u) \in \mathcal{N}(K) : y \in (0,1)\} .$$

Again by construction we have $\mathcal{H}^2(\mathcal{N}_1) > 0$, and for $\mathcal{H}^2$ almost all $v \in \mathcal{N}_1$ we also have

$$k_1(v) = 0 \quad \text{and} \quad k_2(v) = \infty ,$$

since the straight edge $\{(0,y,0) \in \mathbb{R}^3 : y \in (0,1)\}$ consists of ridge points of order one. Hence, the previous statement immediately follows from Theorem 3.6. Alternatively, this can be proved from the representation of the surface area measures as coefficients of a local Steiner formula.

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* I am obliged to Dr. Alfred Schmidt for producing the figure with the GRAPE package developed at Bonn and Freiburg.

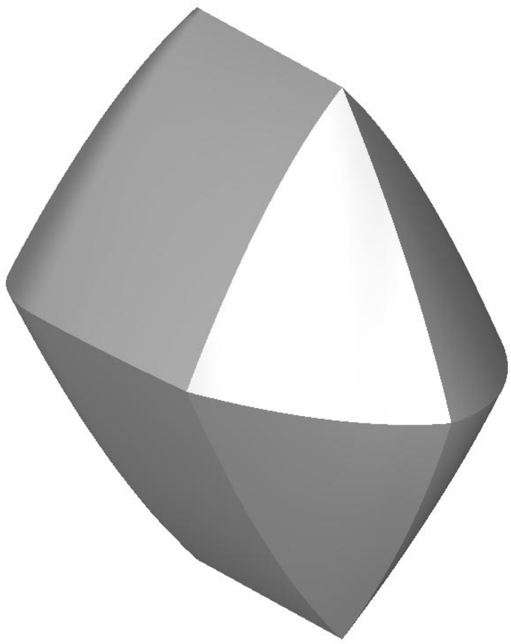

**Fig. 1** Example of a convex body $K \in \mathcal{K}^3_o$ for which $S_2(K,\cdot)$ is absolutely continuous, but $S_1(K,\cdot)$ is not absolutely continuous with respect to $S_0(K,\cdot)$

A dual example for curvature measures follows by using the polar body of $K$ with respect to a suitable choice of the origin. This kind of argument is investigated more thoroughly in a subsequent paper [25].

We conclude this section by stating a sufficient condition for the absolute continuity of *mixed surface area measures*. In fact, Corollary 3.8 below improves a remark in Aleksandrov's fundamental paper [1]. For a definition of and results on mixed surface area measures we refer to Schneider [41]. In addition, for a convex body $K \in \mathcal{K}^d$, let $S^{d-1}(K)$ be the set of all unit vectors $u \in S^{d-1}$ for which there is a point $x \in \operatorname{bd} K$ and some $R > 0$ such that $K$ is contained in the closed ball of radius $R$ centred at $x - Ru$.

**Corollary 3.8.** *Let $K_i \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and suppose that $\omega \subseteq S^{d-1}(K_i)$ for all $i \in \{1,\ldots,d-1\}$. Then $S(K_1,\ldots,K_{d-1},\cdot) \llcorner \omega \ll \mathcal{H}^{d-1} \llcorner \omega$.*

*Proof.* Use Theorem 3.7 and Eq. (5.1.17) from Schneider [41]. □

## 4. Integral-geometric results: surface area measures

The principal aim of this section is the derivation of an integral-geometric extension of Theorem 3.7, which treats the case of surface area measures of any order. Actually, we prove two such extensions in Theorems 4.5 and 4.6. The basic idea underlying the proofs of these results is to use integral-geometric *projection formulae* for surface area measures. Such formulae relate the $i$-th surface area measure of a convex body in $\mathbb{R}^d$ to the $i$-th surface area measure of projections of $K$ onto $j$-dimensional subspaces ($i < j$) by averaging the latter with respect to a Haar measure on the Grassmann manifold of $j$-dimensional linear subspaces of $\mathbb{R}^d$. The projection formulae and additional integral-geometric transformations (Lemmas 4.1 and 4.3) lead to an integral-geometric characterization of absolute continuity for surface area measures which is stated as Theorem 4.4. From this and Theorem 3.7 we deduce Theorem 4.5. Variants and further applications of Theorem 4.4 will be given in [25].

The second characterization, Theorem 4.6, is stated in terms of touching affine subspaces and supporting orthogonal spherical cylinders (definitions will be given later in this section). This result can in turn be used to make precise (in Theorem 4.7) the intuitive feeling that a convex body one of whose surface area measures is rectifiable (absolutely continuous) should not deviate too much from a *strictly convex* body.

Before we can go further, some additional notation is needed. For $j \in \{1, \dots, d\}$, let $\mathbf{G}(d,j) := \mathbf{G}(\mathbb{R}^d, j)$ be the Grassmann manifold of $j$-dimensional linear subspaces of $\mathbb{R}^d$, let

$$\mathbf{G}^u(d,j) := \{V \in \mathbf{G}(d,j) : u \in V\}$$

if $u \in S^{d-1}$, and define the flag manifold

$$\mathbf{G}_0(d,j,1) := \{(u,V) \in S^{d-1} \times \mathbf{G}(d,j) : u \in V\} \,.$$

It is well known that $\mathbf{G}(d,j)$ and $\mathbf{G}_0(d,j,1)$ together with the natural operation of the orthogonal group $\mathbf{O}(d)$ of the Euclidean space $\mathbb{R}^d$ are homogeneous $\mathbf{O}(d)$-spaces, and that $\mathbf{G}^u(d,j)$, for each $u \in S^{d-1}$, is a homogeneous $\mathbf{O}(u^\perp)$-space with respect to the canonical operation of the subgroup

$$\mathbf{O}(u^\perp) := \{\rho \in \mathbf{O}(d) : \rho u = u\} \,.$$

Let us denote by $\nu_j$, $\nu_j^u$, and $\nu_{j,1}$ the corresponding normalized Haar measures of $\mathbf{G}(d,j)$, $\mathbf{G}^u(d,j)$, and $\mathbf{G}_0(d,j,1)$, respectively. Recall that $\omega_j := \mathcal{H}^{j-1}(S^{j-1})$, for $j \in \{1, \dots, d\}$, and denote by $K|V$ the orthogonal projection of the convex body $K$ onto the linear subspace $V \in \mathbf{G}(d,j)$. Furthermore, note that $h_{K|V} = h_K|V$ if $h_K = h(K,\cdot)$ is the support function of $K \in \mathcal{K}^d$ and $h_{K|V}$ is considered as a function defined on $V \in \mathbf{G}(d,j)$. Finally, observe that

$$D_i^U h(K|U,u) = \det\Big(d^2 h_{K|U}(u)|(u^\perp \cap U)\Big) \,,$$

$i \in \{1, \dots, d-1\}$, holds for all $(u, U) \in \mathbf{G}_0(d, i+1, 1)$ for which $h_K|U$ is *second order differentiable* (sod) at $u$. We write $D_i^U h(L, \cdot)$ if $L$ is a convex body which is contained in the $j$-dimensional linear subspace $U$ and $j \in \{i+1, \dots, d-1\}$, in order to indicate that this expression has to be calculated with respect to the linear subspace $U$. The same convention is used for surface area measures such as $S_i^U(L, \cdot)$.

The next two lemmas are required to justify the repeated interchange of the order of integration in the proof of Theorem 4.4 below.

**Lemma 4.1.** *Let $j \in \{2, \dots, d-1\}$, and let $f : \mathbf{G}_0(d, j, 1) \to [0, \infty]$ be Borel measurable. Then*

$$\begin{aligned} &\frac{\omega_j}{\omega_d} \int_{S^{d-1}} \int_{\mathbf{G}^u(d,j)} f(u, V)\, \nu_j^u(dV)\, \mathcal{H}^{d-1}(du) \\ &= \int_{\mathbf{G}(d,j)} \int_{S^{d-1}\cap V} f(u, V)\, \mathcal{H}^{j-1}(du)\, \nu_j(dV) \\ &= \omega_j \int_{\mathbf{G}_0(d,j,1)} f(u, V)\, \nu_{j,1}(d(u, V)) \ . \end{aligned}$$

*Proof.* This can be proved in a similar way to Satz 6.1.1 in Schneider and Weil [44]. □

**Lemma 4.2.** *Let $K \in \mathcal{K}^d$, $i \in \{1, \dots, d-2\}$, and $j \in \{i+1, \dots, d-1\}$. Then the following three statements hold:*

(1) $D_1 := \{(u, V) \in \mathbf{G}_0(d, j, 1) : h_{K|V}$ is (sod) at $u\}$ *is a Borel set;*
(2) $(u, V) \mapsto D_i^V h(K|V, u)$ *is Borel measurable on $D_1$;*
(3) $\nu_{j,1}(\mathbf{G}_0(d, j, 1) \setminus D_1) = 0$.

*Proof.* Let $\{L_m : m \in \mathbb{N}\}$ be a dense set of linear functionals on $\mathbb{R}^d$, and let $\{B_n : n \in \mathbb{N}\}$ be a dense set of bilinear functionals on $\mathbb{R}^d \times \mathbb{R}^d$. For $m, k, l \in \mathbb{N}$, we define $W_{mkl}$ as the set of all $(u, V) \in \mathbf{G}_0(d, j, 1)$ for which the implication

$$|x| < \frac{1}{k} \Rightarrow |h_K(u + x) - h_K(u) - L_m(x)| \le \frac{1}{l}|x|$$

holds for all $x \in V$, and then we set

$$D_0 := \bigcap_{l \in \mathbb{N}} \ \bigcup_{m,\, k \in \mathbb{N}} W_{mkl} \ .$$

Thus $D_0$ is equal to the set of all $(u, V) \in \mathbf{G}_0(d, j, 1)$ for which $h_{K|V}$ is differentiable at $u$. This implies that $D_0$ is a Borel set, since $W_{mkl}$ is a closed set.

Furthermore, for $n, k, l \in \mathbb{N}$, we define $U_{nkl}$ as the set of all $(u, V) \in D_0$ for which the implication

$$|x| < \frac{1}{k} \Rightarrow \left| h_K(u+x) - h_K(u) - \left\langle D^V h_{K|V}(u), x \right\rangle - \frac{1}{2} B_n(x,x) \right| \leq \frac{1}{l}|x|^2$$

is true for all $x \in V$, and thus we obtain that

$$D_1 = \bigcap_{l \in \mathbb{N}} \bigcup_{n,\, k \in \mathbb{N}} U_{nkl} \ .$$

Now we can complete the proof as follows. Consider $D_0$ as a topological subspace of $\mathbf{G}_0(d, j, 1)$. In the subspace topology of $D_0$, the set $U_{nkl}$ is closed, since the map

$$D_0 \to \mathbb{R}^d \ , \quad (u, V) \mapsto D^V h_{K|V}(u) \ ,$$

is continuous. But then $D_1 \in \mathfrak{B}(D_0) = D_0 \cap \mathfrak{B}(\mathbf{G}_0(d, j, 1))$; see [19, Satz 1.2.10]. By the definition of the trace $\sigma$-algebra $D_0 \cap \mathfrak{B}(\mathbf{G}_0(d, j, 1))$, this completes the proof, since we have already shown that $D_0 \in \mathfrak{B}(\mathbf{G}_0(d, j, 1))$.

The second statement is easy to see, and the third statement follows from the first one and from Lemma 4.1 if the second order differentiability almost everywhere of a convex function is used. □

*Remark 4.* By essentially the same proof it follows that

$$D_2 := \{(u, V) \in \mathbf{G}_0(d, j, 1) : h_K \text{ is (sod) at } u\}$$

is a Borel set. Although one has the obvious inclusion $D_2 \subseteq D_1$, it is still true that $\nu_{j,1}(\mathbf{G}_0(d, j, 1) \setminus D_2) = 0$.

The following lemma expresses a result which is known in the special case $j = i + 1$. For this case, it is mentioned without a proof in [9, §19.3.5], and, for $j = i + 1 = d - 1$, the recent paper by Barvinok [7], Lemma 2.3 and Theorem 2.4, contains a proof which is different from the subsequent argument. Obviously, Lemma 4.3 can be extended to a relation between mixed discriminants by the usual method of polynomial expansion. It should also be emphasized that Lemma 4.3 can be viewed as an algebraic version (for quadratic forms) of integral-geometric projection formulae for surface area measures. In fact, for convex bodies with support functions of class $C^2$, the lemma is implied by such integral-geometric formulae. In the general case, we prefer to proceed in a different way.

**Lemma 4.3.** *Let $K \in \mathcal{K}^d$, $i \in \{1, \dots, d-2\}$, $j \in \{i+1, \dots, d-1\}$, and assume that $h_K$ is second order differentiable at $u \in S^{d-1}$. Then*

$$D_i h(K, u) = \int_{\mathbf{G}^u(d,j)} D_i^V h(K|V, u)\, \nu_j^u(dV) \ .$$

*Proof.* First, let us assume that $j = i+1$. Set $u_d := u$, and let $(u_1, \dots, u_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that $u_1, \dots, u_{d-1}$ are eigenvectors of $d^2 h_K(u)|u^\perp$ with corresponding eigenvalues $r_1, \dots, r_{d-1}$. Let $(e_1, \dots, e_{d-1})$ denote the standard basis of $\mathbb{R}^{d-1}$. Then define the $(d-1) \times i$-matrix $P$ and the diagonal matrix $D$ by

$$P := (e_1 \dots e_i) \quad \text{and} \quad D := \begin{pmatrix} \sqrt{r_1} & & O \\ & \ddots & \\ O & & \sqrt{r_{d-1}} \end{pmatrix} \ .$$

Finally, for $\rho \in \mathbf{O}(u^\perp)$, set

$$S_\rho := \left(s_{jl}^\rho\right)_{j,l=1}^{d-1} := (\langle \rho u_l, u_j \rangle)_{j,l=1}^{d-1} \ .$$

Then we obtain that

$$\left(d^2 h_K(u)(\rho u_l, \rho u_k)\right)_{l,k=1}^i = P^\top S_\rho^\top D^\top D S_\rho P \ ,$$

and from this we infer that

$$\begin{aligned} &\det\left(\left(d^2 h_K(u)(\rho u_l, \rho u_k)\right)_{l,k=1}^i\right) \\ &= \sum_{1 \le j_1 < \dots < j_i \le d-1} r_{j_1} \cdots r_{j_i} \left[\det\left((\langle u_{j_l}, \rho u_k \rangle)_{l,k=1}^i\right)\right]^2 \ . \end{aligned}$$

Denote by $\nu^u$ the normalized Haar measure on $\mathbf{O}(u^\perp)$. Then

$$\begin{aligned} &\int_{\mathbf{G}^u(d,i+1)} D_i^U h(K|U, u)\, \nu_{i+1}^u(dU) \\ &= \int_{\mathbf{O}(u^\perp)} D_i^{\mathrm{lin}\{\dots\}} h(K|\mathrm{lin}\{\rho u_1, \dots, \rho u_i, u\}, u)\, \nu^u(d\rho) \\ &= \int_{\mathbf{O}(u^\perp)} \det\left(\left(d^2 h_K(u)(\rho u_l, \rho u_k)\right)_{l,k=1}^i\right)\, \nu^u(d\rho) \\ &= \sum_{1 \le j_1 < \dots < j_i \le d-1} r_{j_1} \cdots r_{j_i} \\ &\quad \times \int_{\mathbf{O}(u^\perp)} \left[\det\left((\langle u_{j_l}, \rho u_k \rangle)_{l,k=1}^i\right)\right]^2\, \nu^u(d\rho) \ . \end{aligned} \tag{18}$$

The last integral is a constant $c$ which depends neither on $K$, nor on the indices $j_1, \dots, j_i$, nor on the special choice of the orthonormal vectors $u_1, \dots, u_{d-1} \perp u$. This follows from the invariance of $\nu^u$ with respect to $\mathbf{O}(u^\perp)$. By evaluating relation (18) for the unit ball, we conclude that $c = 1/\binom{d-1}{i}$, and this yields the statement of the lemma for $j = i+1$.

The general case now follows by applying an integral-geometric identity which is essentially equivalent to Satz 6.1.1 in [44] and by using twice the special case which has been established in the first part of the proof. □

The following theorem plays a central rôle in the context of characterizations of absolute continuity for surface area measures. There is also an analogous result involving the additional assumption of bounded densities, but a precise description and a proof of this statement will be postponed to [25].

**Theorem 4.4.** *Let $K \in \mathcal{K}^d$, $i \in \{1, \dots, d-2\}$, $j \in \{i+1, \dots, d-1\}$, and let $\omega \in \mathfrak{B}(S^{d-1})$. Then*

$$S_i(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega$$

*if and only if*

$$S_i^V(K|V, \cdot) \llcorner (\omega \cap V) \ll S_0^V(K|V, \cdot) \llcorner (\omega \cap V) \ ,$$

*for $\nu_j$ almost all linear subspaces $V \in \mathbf{G}(d,j)$.*

*Proof.* First, let us assume that

$$S_i^V(K|V, \cdot) \llcorner (\omega \cap V) \ll S_0^V(K|V, \cdot) \llcorner (\omega \cap V) \ ,$$

for $\nu_j$ almost all $V \in \mathbf{G}(d,j)$. But then, for $\nu_j$ almost all $V \in \mathbf{G}(d,j)$, the equation

$$S_i^V(K|V, \alpha \cap V) = \int_{\alpha \cap V} D_i^V h(K|V, u)\, \mathcal{H}^{j-1}(du) \tag{19}$$

holds for any Borel set $\alpha \subseteq \omega$.

On the other hand, it is known that the projection formula

$$S_i(K, \alpha) = \frac{\omega_d}{\omega_j} \int_{\mathbf{G}(d,j)} S_i^V(K|V, \alpha \cap V)\, \nu_j(dV) \tag{20}$$

holds for all $\alpha \in \mathfrak{B}(S^{d-1})$; see relation (4.5.26) in [41]. Inserting Eq. (19) into Eq. (20), we obtain from Lemma 4.1 and Lemma 4.3 that

$$\begin{aligned} S_i(K, \alpha) &= \frac{\omega_d}{\omega_j} \int_{\mathbf{G}(d,j)} \int_{S^{d-1} \cap V} \mathbf{1}_\alpha(u) D_i^V h(K|V, u)\, \mathcal{H}^{j-1}(du)\, \nu_j(dV) \\ &= \int_\alpha \int_{\mathbf{G}^u(d,j)} D_i^V h(K|V, u)\, \nu_j^u(dV)\, \mathcal{H}^{d-1}(du) \\ &= \int_\alpha D_i h(K, u)\, \mathcal{H}^{d-1}(du) \ , \end{aligned}$$

where $\alpha \subseteq \omega$ is an arbitrary Borel set. This shows that $S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$.

Conversely, assume now that $S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$. Employing successively Lemma 4.1, Lemma 4.3, Eq. (2.8) from [24], Eq. (20), and the Lebesgue decomposition theorem for $S_i^V(K|V,\cdot)$, we obtain that

$$\begin{aligned}
&\int_{\mathbf{G}(d,\,j)} \int_{S^{d-1}\cap V} \mathbf{1}_\omega(u) D_i^V h(K|V,u)\, \mathcal{H}^{j-1}(u)\, \nu_j(dV) \\
&= \frac{\omega_j}{\omega_d} \int_\omega \int_{\mathbf{G}^u(d,\,j)} D_i^V h(K|V,u)\, \nu_j^u(dV)\, \mathcal{H}^{d-1}(du) \\
&= \frac{\omega_j}{\omega_d} \int_\omega D_i h(K,u)\, \mathcal{H}^{d-1}(du) \\
&= \frac{\omega_j}{\omega_d} S_i^a(K,\omega) = \frac{\omega_j}{\omega_d} S_i(K,\omega) \\
&= \int_{\mathbf{G}(d,\,j)} S_i^V(K|V,\omega\cap V)\, \nu_j(dV) \\
&= \int_{\mathbf{G}(d,\,j)} \int_{S^{d-1}\cap V} \mathbf{1}_\omega(u) D_i^V h(K|V,u)\, \mathcal{H}^{j-1}(du)\, \nu_j(dV) \\
&\quad + \int_{\mathbf{G}(d,\,j)} \left(S_i^V\right)^s (K|V,\omega\cap V)\, \nu_j(dV)\ .
\end{aligned}$$

This yields

$$\int_{\mathbf{G}(d,\,j)} \left(S_i^V\right)^s (K|V,\omega\cap V)\, \nu_j(dV) = 0\ .$$

Hence, for $\nu_j$ almost all $V \in \mathbf{G}(d,\,j)$, we obtain

$$\left(S_i^V\right)^s (K|V,\omega\cap V) = 0\ ,$$

that is,

$$S_i^V(K|V,\cdot) \llcorner (\omega\cap V) \ll S_0^V(K|V,\cdot) \llcorner (\omega\cap V)\ ,$$

and this completes the proof. □

As an immediate consequence we obtain:

**Theorem 4.5.** *Let* $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, *and* $i \in \{1,\dots,d-2\}$. *Then*

$$S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

*if and only if, for* $\nu_{i+1}$ *almost all* $U \in \mathbf{G}(d,i+1)$, *the projection* $K|U$ *is supported from outside by an* $(i+1)$*-dimensional ball at* $\mathcal{H}^i$ *almost all points of the set* $\tau(K|U,\omega\cap U)$.

*Proof.* This immediately follows from Theorem 3.7 and from a special case of Theorem 4.4. □

In the remaining part of this section, we establish a characterization for the absolute continuity of surface area measures which involves touching planes and supporting orthogonal spherical cylinders. This also leads to a regularity result. To achieve this aim we introduce some terminology.

For a convex body $K \in \mathcal{K}^d$ and some $r \in \{0, \dots, d-1\}$, the set $\mathrm{A}(K, d, r)$ of $r$-dimensional affine subspaces of $\mathbb{R}^d$ which touch $K$ has been defined in Sect. 2. A parametrization of this rectifiable set is provided in [54] and [35]. We say that $K$ is *supported from outside by an orthogonal spherical cylinder at* $E \in \mathrm{A}(K, d, r)$ if there is some $R > 0$ and some $u \in S^{d-1}$ with $E \subseteq H(K, u)$ such that $K \subseteq E + B(-Ru, R)$.

Weil [51] defines a natural measure on $\mathrm{A}(K, d, r)$ in the following way. Let $B \in \mathfrak{B}(\mathrm{A}(d, r))$ and $U \in \mathbf{G}(d, d-r)$. Then

$$T(B, U) := \left\{ x \in U : x + U^{\perp} \in B \right\}$$

is a Borel set, and we can define the measure

$$\mu_r(K, B) := \int_{\mathbf{G}(d,\, d-r)} C^U_{d-1-r}(K|U, T(B, U))\, \nu_{d-r}(dU)\ . \tag{21}$$

The measurability of the integrand was proved by Weil [51]; see also §5.3 in [44]. Although $\mu_r(K, \cdot)$ is defined on $\mathfrak{B}(\mathbf{A}(d, r))$, the measure is concentrated on the subset $\mathbf{A}(K, d, r)$. Henceforth, we shall replace the measure spaces $(\mathfrak{B}(\mathbf{A}(d, r)), \mu_r(K, \cdot))$ and $(\mathbf{G}(d, r), \nu_r)$ by their completions without changing our notation. The members of the extended $\sigma$-algebras will be called $\mu_r(K, \cdot)$ and $\nu_r$ measurable sets, respectively. It was shown in [51], for $K \in \mathcal{K}^d$, $r \in \{0, \dots, d-1\}$ and $\omega \in \mathfrak{B}(S^{d-1})$, that

$$\tau_r(K, \omega) := \{E \in \mathbf{A}(K, d, r) : E \subseteq H(K, u) \text{ for some } u \in \omega\}$$

is $\mu_r(K, \cdot)$ measurable and

$$S_{d-1-r}(K, \omega) = \frac{\omega_d}{\omega_{d-r}}\, \mu_r(K, \tau_r(K, \omega))\ . \tag{22}$$

The set $\tau_r(K, \omega)$ will be called the *reverse spherical image of order* $r$ *of* $K$ *at* $\omega$. Thus the reverse spherical image of order $r = 0$ is just the ordinary reverse spherical image.

Equation (22) has previously been used as an integral-geometric interpretation for the intermediate surface area measures. In the present context, it shows that it is natural to state a characterization of absolute continuity for surface area measures by using touching planes.

**Theorem 4.6.** *Let $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and $i \in \{1, \ldots, d-2\}$. Then*

$$S_i(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega$$

*if and only if $K$ is supported from outside by an orthogonal spherical cylinder at $\mu_{d-1-i}(K, \cdot)$ almost all $E \in \tau_{d-1-i}(K, \omega)$.*

*Remark 5.* In contrast to the two-step procedure of Theorem 4.5, Theorem 4.6 provides a one-step procedure for verifying the absolute continuity of surface area measures of convex bodies. With regard to Eq. (22), this characterization connects the measure theoretic and geometric aspects of the problem in a natural way. Furthermore, note that the equivalence of conditions (a) and (c) of Theorem 3.7 can be viewed as the statement of Theorem 4.6 in the case $i = d - 1$ if properly interpreted.

*Proof of Theorem 4.6.* The set $B_{d-1-i}(K, \omega)$ of all $E \in \tau_{d-1-i}(K, \omega)$ such that $K$ is supported from outside by an orthogonal spherical cylinder at $E$ is $\mu_{d-1-i}(K, \cdot)$ measurable. In fact, this set is equal to the set of all $E \in \tau_{d-1-i}(K, \omega)$ for which there is some $n \in \mathbb{N}$ and some $u \in S^{d-1}$ such that

$$E \subseteq H(K, u) \quad \text{and} \quad K \subseteq E + B(-nu, n) \; . \tag{23}$$

Therefore it remains to prove that, for each $n \in \mathbb{N}$, the set of all touching affine subspaces $E \in \mathbf{A}(K, d, d-1-i)$ for which there is some $u \in S^{d-1}$ such that condition (23) is satisfied, is closed in $\mathbf{A}(d, d-1-i)$. But this can easily be checked.

Now, let us denote by

$$B^c_{d-1-i}(K, \omega) := \tau_{d-1-i}(K, \omega) \setminus B_{d-1-i}(K, \omega)$$

the set of all $E \in \tau_{d-1-i}(K, \omega)$ such that $K$ is not supported from outside by an orthogonal spherical cylinder at $E$. From relation (5.2) in Weil [51, p. 97] it can be inferred that

$$\mu_{d-1-i}(K, B^c_{d-1-i}(K, \omega)) = 0$$

if and only if

$$\mathcal{H}^i \left( \mathrm{bd}_U(K|U) \cap T_{d-1-i} \left( B^c_{d-1-i}(K, \omega), U \right) \right) = 0 \; ,$$

for $\nu_{i+1}$ almost all $U \in \mathbf{G}(d, i+1)$. Moreover, we can write

$$\begin{aligned}
&\mathrm{bd}_U(K|U) \cap T_{d-1-i}\left(B^c_{d-1-i}(K,\omega), U\right)\\
&= \left\{x \in \mathrm{bd}_U(K|U) : x + U^\perp \in B^c_{d-1-i}(K,\omega)\right\}\\
&= \Big\{x \in \mathrm{bd}_U(K|U) : \big(x + U^\perp \subseteq H(K,v) \text{ for some } v \in \omega\big) \text{ and}\\
&\qquad \big(x + U^\perp \subseteq H(K,u) \Rightarrow\\
&\qquad\quad K \not\subseteq x + U^\perp + B(-Ru, R)\big)\\
&\qquad \text{holds for all } R > 0 \text{ and all } u \in S^{d-1}\Big\}\\
&= \Big\{x \in \tau(K|U, \omega \cap U) : K|U \text{ is not supported from outside}\\
&\qquad \text{by an } (i+1)\text{-dimensional ball at } x\Big\} .
\end{aligned}$$

An application of Theorem 4.5 then completes the proof. □

The next theorem demonstrates that the rectifiability of some surface area measure of a convex body $K$ leads to a certain degree of strict convexity for $K$. Another precise statement in this direction was established in [24, Theorem 4.8]. Recall that a support plane $H(K,u)$, $u \in S^{d-1}$, of a convex body $K$ is said to be regular if $u$ is a regular normal vector of $K$.

**Theorem 4.7.** *Let* $K \in \mathcal{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, $i \in \{1, \dots, d-2\}$, *and assume that*

$$S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega .$$

*Then, for* $\mu_{d-1-i}(K,\cdot)$ *almost all* $E \in \tau_{d-1-i}(K,\omega)$, *every support plane of* $K$ *which contains* $E$ *is regular.*

*Proof.* Denote by $\mathcal{E}_1$ the set of all $E \in \tau_{d-1-i}(K,\omega)$ for which

$$\text{card}(E \cap K) > 1 \quad \text{or} \quad \text{card}\left\{u \in S^{d-1} : E \subseteq H(K,u)\right\} > 1 .$$

By a result of Zalgaller [56] (see also Schneider [41, §2.3]), the proof of which is based on methods of Ewald, Larman & Rogers [15], and using Lemma 5.5 of Weil [51], we deduce that $\mathcal{E}_1$ has $\mu_{d-1-i}(K,\cdot)$ measure zero. Further, let $\mathcal{E}_2$ be the set of all $E \in \tau_{d-1-i}(K,\omega)$ such that $K$ is not supported from outside by an orthogonal spherical cylinder at $E$. Theorem 4.6 implies that $\mathcal{E}_2$ has $\mu_{d-1-i}(K,\cdot)$ measure zero as well.

Now choose any $E \in \tau_{d-1-i}(K,\omega) \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$. Then $K \subseteq E + B(-Ru, R)$ holds for some $R > 0$ and for a uniquely determined vector $u \in \omega$ with $E \subseteq H(K,u)$. Therefore,

$$\begin{aligned}
1 &\le \text{card } F(K,u) = \text{card}(H(K,u) \cap K)\\
&= \text{card}(H(K,u) \cap K \cap (E + B(-Ru, R))) = \text{card}(E \cap K) = 1 ,
\end{aligned}$$

which proves the assertion of the Theorem. □

## 5. Integral-geometric results: curvature measures

In this final section, our first aim is to deduce Theorems 2.5 and 2.4 from a sequence of auxiliary results. Then we prove Theorems 2.6 and 2.7. Essentially, the basic approach for curvature measures is dual to the one for surface area measures. Instead of projections onto linear subspaces, which have been essential for surface area measures in Sect. 4, we now consider intersections of convex bodies with affine subspaces. Moreover, principal radii of curvature (as functions which are defined almost everywhere on the unit sphere) are replaced by principal curvatures which are defined (almost everywhere) on the boundary of a given convex body.

However, for curvature measures the situation is more complicated. For example, Lemma 5.4 below cannot be obtained by using invariance properties of suitably defined Haar measures, at least not in an obvious way. This is in contrast to the proof of Lemma 4.1. Instead one uses Federer's coarea formula and the alternating calculus of multilinear algebra to establish the required integral-geometric transformation. A similar remark applies to the proof of Proposition 5.11, for which no analogue is required in Sect. 4. It is a special feature of the present work that both results about Haar measures and basic arguments from geometric measure theory are combined. A second complication arises, since it is not sufficient to consider affine subspaces which intersect the boundary of a given convex body orthogonally at a prescribed boundary point. As a consequence, even for a smooth convex body $K \in \mathcal{K}_o^d$ (of class $C^2$) the principal curvatures of the intersections $K \cap E$ of $K$ with affine subspaces $E$ passing through a fixed boundary point $x \in \text{bd}\, K$ are not uniformly bounded. In fact, these curvatures approach infinity (provided they are not zero) as the section plane approaches a tangential position. For smooth convex bodies this is implied by Meusnier's theorem. Lemma 5.2 extends this classical result in the present setting.

We introduce some additional notation. Let $\mathbf{G}_d$ be the motion group of $\mathbb{R}^d$. Denote by $\mathbf{A}(d,k)$, for $k \in \{1, \dots, d-1\}$, the homogeneous $\mathbf{G}_d$-space of $k$-dimensional affine subspaces of $\mathbb{R}^d$, and let $\mu_k$ be the corresponding Haar measure which is normalized as in Schneider [41]. Also from [41, pp. 230–231] we adopt the number $[L, L']$ in the special case where $L = e^\perp$, $e \in S^{d-1}$, $L' = U \in \mathbf{G}(d,s)$, $s \in \{2, \dots, d-1\}$, and $\text{lin}\{L, L'\} = \mathbb{R}^d$. In this situation one has $[e^\perp, U] = |\langle e, u\rangle|$ if $u \in S^{d-1} \cap U \cap V^\perp$ and $V := e^\perp \cap U \in \mathbf{G}(d, s-1)$. In particular, the subspace $e^\perp$ will be the $(d-1)$-dimensional linear tangent space $T_x K$ of a convex set $K$ at a regular boundary point $x$.

By $\mathcal{M}(K)$ we denote the set of all *normal boundary points* of $K \in \mathcal{C}_0^d$. The definition of a normal boundary point in Schneider [41], §2.5, involves the notion of convergence in the sense of Hausdorff closed limits; see also [38]. This concept is, for example, described in §§1.1–1.4 of Matheron's book [31] or in Hausdorff's classical treatise [20]. Lemma 5.1 below, which is used for the proof of Lemma 5.2, provides equivalent conditions in the present special situation for convergence in the sense of Hausdorff closed limits.

**Lemma 5.1.** *Let $M_i$, $i \in \mathbb{N}$, and $M$ be non-empty closed convex subsets of $\mathbb{R}^n$, $n \geq 1$, with $o \in M_i$ for all $i \in \mathbb{N}$. Then the following conditions are equivalent for $i \to \infty$:*

(a) $M_i \to M$ *in the sense of Hausdorff closed limits;*
(b) $M_i \cap B(o, \rho) \to M \cap B(o, \rho)$ *in the sense of Hausdorff closed limits for all* $\rho > 0$*;*
(c) $M_i \cap B(o, \rho) \to M \cap B(o, \rho)$ *with respect to the Hausdorff metric for all* $\rho > 0$*.*

*Proof.* $(\mathrm{a}) \Leftrightarrow (\mathrm{b})$ immediately follows, for example, from the definitions and from Proposition 1-2-3 in Matheron [31]. Note that for the proof of $(\mathrm{a}) \Rightarrow (\mathrm{b})$ one uses the fact that $M_i$ is star-shaped with respect to $o$ for all $i \in \mathbb{N}$. Further, $(\mathrm{b}) \Leftrightarrow (\mathrm{c})$ is a consequence of Proposition 1-4-1 and Proposition 1-4-4 in [31]. □

In the following, we shall occasionally attach a prime ' $'$ ' to certain quantities in order to indicate that they have to be calculated with respect to an affine subspace. For example, the quantity $H'_{r-1}(K \cap (x+U), x)$ in Lemma 5.2 is the normalized elementary symmetric function of order $r-1$ of the principal curvatures of the convex body $K \cap (x+U)$ at $x$ with respect to the $s$-dimensional affine subspace $x+U$. See Lemma 5.2 for the precise assumptions. This lemma represents a generalization of Meusnier's theorem from classical differential geometry in the non-smooth setting of convex geometry; compare Spivak [47, vol. III, p. 276 (7$'$)].

**Lemma 5.2.** *Let $K \in \mathcal{C}^d_o$, $r \in \{2, \dots, d-1\}$, and $s \in \{r, \dots, d-1\}$. Furthermore, assume that $x \in \mathcal{M}(K)$ and $U \in \mathbf{G}(d, s)$ satisfy $U \not\subseteq T_x K$. Then $x \in \mathcal{M}'(K \cap (x+U))$. Moreover, if $U_0 := \operatorname{lin}\{\sigma_K(x), U \cap T_x K\}$, then the principal curvatures of the intersections $K \cap (x+U)$ and $K \cap (x+U_0)$ at $x$ are related by*

$$k'_i(K \cap (x+U), x) = [T_x K, U]^{-1} k'_i(K \cap (x+U_0), x) \ ,$$

*for $i \in \{1, \dots, s-1\}$, and they correspond to the same directions of the common tangent space $T_x K \cap U$. In particular,*

$$H'_{r-1}(K \cap (x+U), x) = [T_x K, U]^{1-r} H'_{r-1}(K \cap (x+U_0), x) \ .$$

*Proof.* All limits in the proof are meant in the sense of Hausdorff closed limits. We can assume that $x = o$. Let $e_d := -\sigma_K(x)$ and $V := U \cap T_x K$.

Further, choose $\lambda > 0$, $e_s(\lambda) \in U \cap V^\perp \cap S^{d-1}$ and $e_s \in S^{d-1} \cap e_d^\perp \cap V^\perp$ such that

$$U = \operatorname{lin}\{V, e_s(\lambda)\} \quad \text{and} \quad e_s(\lambda) = \frac{e_d + \lambda e_s}{\sqrt{1+\lambda^2}} \; .$$

We set $U_\lambda := U$ and define

$$S(h) := K \cap \left(e_d^\perp + he_d\right) - he_d \; .$$

Since $x \in \mathcal{M}(K)$, there exists a closed set $D \subseteq e_d^\perp$ such that

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{2h}} S(h) = D \; , \tag{24}$$

and the boundary (if any) of $D$ is a quadric.

Now set

$$S_\lambda(h) := K \cap U_\lambda \cap (V + he_s(\lambda)) - he_s(\lambda) \; .$$

From Eq. (24), Lemma 5.1 and Theorem 1.8.8 in Schneider [41] we conclude that

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{2h}} S_0(h) = D \cap V \; ,$$

and the boundary (if any) of $D \cap V$ is a quadric.

Observe that $[T_x K, U_\lambda] = \langle e_d, e_s(\lambda)\rangle$ and

$$S_\lambda\left(\langle e_d, e_s(\lambda)\rangle^{-1} h\right) = \left[K \cap \left(e_d^\perp + he_d\right) - he_d\right] \cap [V + \lambda h e_s] - \lambda h e_s \; .$$

Hence,

$$\begin{aligned} &\frac{1}{\sqrt{2h}} S_\lambda\left(\langle e_d, e_s(\lambda)\rangle^{-1} h\right) \\ &= \left[\frac{1}{\sqrt{2h}}\left(K \cap \left(e_d^\perp + he_d\right) - he_d\right)\right] \cap \left(V + \lambda\sqrt{\frac{h}{2}} e_s\right) - \lambda\sqrt{\frac{h}{2}} e_s \; . \end{aligned}$$

Again Eq. (24), Lemma 5.1 and Theorem 1.8.8 in [41] imply that

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{2h}} S_\lambda\left(\langle e_d, e_s(\lambda)\rangle^{-1} h\right) = \lim_{h \downarrow 0} \frac{1}{\sqrt{2h}} S_0(h) = D \cap V \; .$$

Thus, we obtain that

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{2h\langle e_d, e_s(\lambda)\rangle^{-1}}} S_\lambda\left(\langle e_d, e_s(\lambda)\rangle^{-1} h\right) = \sqrt{[T_x K, U]}(D \cap V) \; ,$$

and the boundary (if any) of $\sqrt{[T_x K, U]}(D \cap V)$ is a quadric. This yields the statement of the lemma. □

The next two lemmas will be needed to justify the application of Fubini's theorem and to perform certain integral-geometric transformations in the course of the proofs of Proposition 5.8 and Theorem 2.5.

**Lemma 5.3.** *Let* $K \in \mathcal{C}_o^d$*,* $r \in \{2, \dots, d-1\}$*, and* $s \in \{r, \dots, d-1\}$*. Then the following statements hold:*

(1) $D_2 := \{(x, U) \in \mathrm{bd}\, K \times \mathbf{G}(d, s) : x \in \mathcal{M}(K),\ (x+U) \cap \mathrm{int}\, K \neq \emptyset\}$ *is a Borel set;*
(2) $(x, U) \mapsto H'_{r-1}(K \cap (x+U), x)$ *is Borel measurable on* $D_2$*;*
(3) $\nu_s(\{U \in \mathbf{G}(d, s) : (x+U) \cap \mathrm{int}\, K = \emptyset\}) = 0$ *if* $x \in \mathrm{reg}\, K$.

*Proof.* The proof follows from standard methods of measure theory and convex geometry; compare also the proof of Lemma 4.2. For the proof of the second statement one can use Lemma 5.2. □

**Lemma 5.4.** *Let* $K \in \mathcal{C}_o^d$*,* $s \in \{2, \dots, d-1\}$*, and* $f : \mathrm{bd}\, K \times \mathbf{G}(d, s) \to [0, \infty]$ *be Borel measurable. Then*

$$\int_{\mathrm{bd}\, K} \int_{\mathbf{G}(d,\, s)} [T_x K, U] f(x, U)\, \nu_s(dU)\, \mathcal{H}^{d-1}(dx)$$
$$= \int_{\mathbf{A}(d,\, s)} \int_{\mathrm{bd}\, K \cap E} f(x, U(E))\, \mathcal{H}^{s-1}(dx)\, \mu_s(dE)\ ,$$

*where* $U(E) \in \mathbf{G}(d, s)$ *is the unique linear subspace which is parallel to* $E$.

*Proof.* This is a special case of Theorem 1 in Zähle [55]. Observe that $\mu_s$ almost all $s$-dimensional affine subspaces $E \in \mathbf{A}(d, s)$ which meet $K$ also meet $\mathrm{int}\, K$. □

The following three lemmas, which will be essential for the proof of Proposition 5.8, are based on integral-geometric transformations. In order to state and prove these lemmas, we introduce some further definitions.

Let $s \in \{2, \dots, d-1\}$ and $W \in \mathbf{G}(d, d-1)$. Then we set

$$\mathbf{G}(W, s-1) := \{V \in \mathbf{G}(d, s-1) : V \subseteq W\}$$

and denote by $\nu_{s-1}^W$ the corresponding normalized Haar measure of $\mathbf{G}(W, s-1)$ which is invariant with respect to $\mathbf{O}(W)$. Moreover, if $j \in \{s, \dots, d-1\}$ and $V \in \mathbf{G}(d, s-1)$, then

$$\mathbf{G}^V(d, j) := \{U \in \mathbf{G}(d, j) : V \subseteq U\}\ ,$$

and $\nu_j^V$ is the corresponding normalized Haar measure of $\mathbf{G}^V(d, j)$ which is invariant with respect to all rotations $\rho \in \mathbf{O}(d)$ for which $\rho(v) = v$ holds for all $v \in V$.

**Lemma 5.5.** *Let $K \in \mathcal{C}_o^d$, $r \in \{2, \ldots, d-1\}$, $s \in \{r, \ldots, d-1\}$, and assume that $x \in \mathcal{M}(K)$. In addition, set $e := \sigma_K(x) \in S^{d-1}$. Then*

$$H_{r-1}(K,x) = \int_{\mathbf{G}(e^\perp,\, s-1)} H'_{r-1}(K \cap (x + \mathrm{lin}\{e, V\}), x) \times \nu_{s-1}^{e^\perp}(dV) \ . \tag{25}$$

*Proof.* The proof is essentially the same as the one for Lemma 4.3. □

**Lemma 5.6.** *Let $e \in S^{d-1}$, $s \in \{2, \ldots, d-1\}$, and let $h : \mathbf{G}(d,s) \to [0,\infty]$ be Borel measurable. Then*

$$\frac{2\,\omega_d}{\omega_s\,\omega_{d-s+1}} \int_{\mathbf{G}(d,\,s)} h(U)\,\nu_s(dU) = \int_{\mathbf{G}(e^\perp,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} [e^\perp, U]^{s-1} h(U)\,\nu_s^V(dU)\,\nu_{s-1}^{e^\perp}(dV) \ .$$

*Proof.* The proof will be accomplished by applying Satz 6.1.9 from Schneider & Weil [44]. Let $h : \mathbf{G}(d,s) \to [0,\infty)$ and $g : \mathbf{G}(d,d-1) \to [0,\infty)$ be arbitrary continuous functions. Furthermore, set $f(U,W) := h(U)g(W)$, for any $U \in \mathbf{G}(d,s)$ and $W \in \mathbf{G}(d,d-1)$.

In the following, we shall repeatedly apply Fubini's theorem. The required measurability can be established in the same way as in the proof of Hilfssatz 7.2.4 of [44]. Then Satz 6.1.9 and Satz 6.1.1 from [44] imply that

$$\begin{aligned}
&\left(\bar{c}_{d\,s\,(d-1)}\right)^{-1} \int_{\mathbf{G}(d,\,s)} h(U)\,\nu_s(dU) \int_{\mathbf{G}(d,\,d-1)} g(W)\,\nu_{d-1}(dW) \\
&= \int_{\mathbf{G}(d,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} \int_{\mathbf{G}^V(d,\,d-1)} [U,W]^{s-1} h(U) g(W) \\
&\qquad\qquad \times \nu_{d-1}^V(dW)\,\nu_s^V(dU)\,\nu_{s-1}(dV) \\
&= \int_{\mathbf{G}(d,\,d-1)} \int_{\mathbf{G}(W,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} [U,W]^{s-1} h(U) g(W) \\
&\qquad\qquad \times \nu_s^V(dU)\,\nu_{s-1}^W(dV)\,\nu_{d-1}(dW) \\
&= \int_{\mathbf{G}(d,\,d-1)} g(W) H(W)\,\nu_{d-1}(dW) \ .
\end{aligned}$$

The function $H : \mathbf{G}(d,d-1) \to [0,\infty)$ is defined by

$$H(W) := \int_{\mathbf{G}(W,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} [U,W]^{s-1} h(U)\,\nu_s^V(dU)\,\nu_{s-1}^W(dV) \ .$$

It can be shown that $H$ is continuous. This follows by applying twice an argument which is similar to the one used to verify Hilfssatz 7.2.4 in Schneider

& Weil [44]. In fact, one defines

$$\mathbf{G}(d, d-1, s-1) := \{(V, W) \in \mathbf{G}(d, s-1) \times \mathbf{G}(d, d-1) : V \subseteq W\}$$

and starts by proving that the map

$$\mathbf{G}(d, d-1, s-1) \to [0, \infty), \qquad (V, W) \mapsto \int_{\mathbf{G}^V(d,s)} [U, W]^{s-1} h(U)\, \nu_s^V(dU) ,$$

is continuous.

Since $g$ was arbitrarily chosen and $H$ is continuous, we thus conclude that the relation

$$\int_{\mathbf{G}(d,s)} h(U)\, \nu_s(dU) = \bar{c}_{d\,s\,(d-1)} H(W)$$

holds for an arbitrary $W \in \mathbf{G}(d, d-1)$. Choosing $W := e^\perp$ and noting that

$$\bar{c}_{d\,s\,(d-1)} = \frac{\omega_{d-s+1}\, \omega_s}{\omega_d\, \omega_1} ,$$

we obtain the statement of the lemma for a continuous function $h$. But then the general result follows by standard approximation arguments. □

*Remark 6.* Lemma 5.6 can also be proved by applying the coarea formula to the map

$$T : \mathbf{G}(d, s)^* \to \mathbf{G}(e^\perp, s-1) , \quad U \mapsto e^\perp \cap U ,$$

where

$$\mathbf{G}(d, s)^* := \{U \in \mathbf{G}(d, s) : U \not\subseteq e^\perp\} .$$

For this approach one has to check that $T$ is differentiable and that

$$J_{(s-1)(d-s)} T(U) = [e^\perp, U]^{1-s}$$

for all $U \in \mathbf{G}(d, s)^*$.

In Lemma 5.7 and subsequently we write $\kappa_n$ for the volume of the $n$-dimensional unit ball, $n \geq 0$, that is, $\kappa_n = \pi^{n/2}/\Gamma(1 + n/2)$.

**Lemma 5.7.** *Let* $e \in S^{d-1}$, $r \in \{2, \ldots, d-1\}$, $s \in \{r, \ldots, d-1\}$, *and choose some* $V \in \mathbf{G}(e^\perp, s-1)$. *Then*

$$\int_{\mathbf{G}^V(d,s)} [e^\perp, U]^{s-r+1}\, \nu_s^V(dU) = \frac{2}{\omega_{d-s+1}} \frac{\kappa_{d-r}}{\kappa_{s-r}} .$$

*Proof.* Let the assumptions of the lemma be fulfilled. Then, using the introductory remarks of Chap. 6 in Schneider & Weil [44], we obtain

$$\begin{aligned}
&\int_{\mathbf{G}^V(d,\,s)} [e^\perp, U]^{s-r+1}\, \nu_s^V(dU) \\
&= \int_{\mathbf{G}(V^\perp,\,1)} [e^\perp, \operatorname{lin}\{V, L\}]^{s-r+1}\, \nu_1^{V^\perp}(dL) \\
&= \int_{S^{d-1}\cap V^\perp} |\langle e, u\rangle|^{s-r+1} (\omega_{d-s+1})^{-1}\, \mathcal{H}^{d-s}(du) \\
&= \frac{1}{\omega_{d-s+1}} \int_{S^{d-1-s}} \int_0^\pi |\cos\varphi|^{s-r+1}\, |\sin\varphi|^{d-1-s}\, d\varphi\, \mathcal{H}^{d-1-s}(du) \\
&= \frac{\omega_{d-s}}{\omega_{d-s+1}} 2 \int_0^{\pi/2} (\cos\varphi)^{s-r+1} (\sin\varphi)^{d-1-s}\, d\varphi \\
&= \frac{2}{\omega_{d-s+1}} \frac{\kappa_{d-r}}{\kappa_{s-r}}\ ,
\end{aligned}$$

and this completes the proof. □

The following proposition represents the main tool for establishing Theorem 2.5.

**Proposition 5.8.** *Let* $K \in \mathcal{C}_o^d$, $r \in \{2, \dots, d-1\}$, $s \in \{r, \dots, d-1\}$, *and assume that* $x \in \mathcal{M}(K)$. *Then*

$$\begin{aligned}
H_{r-1}(K, x) = a_{dsr} \int_{\mathbf{G}(d,\,s)} &[T_x K, U] \\
&\times H'_{r-1}(K \cap (x+U), x)\, \nu_s(dU)\ ,
\end{aligned} \tag{26}$$

*where*

$$a_{dsr} := \frac{\kappa_{s-r}\, \omega_d}{\kappa_{d-r}\, \omega_s}\ .$$

*Proof.* Let $e \in (T_x K)^\perp \cap S^{d-1}$. By successively applying Lemma 5.6, Lemma 5.2, and finally Lemma 5.7 as well as Lemma 5.5, we obtain that

$$\begin{aligned}
&\frac{2\,\omega_d}{\omega_s\, \omega_{d-s+1}} \int_{\mathbf{G}(d,\,s)} [T_x K, U] H'_{r-1}(K \cap (x+U), x)\, \nu_s(dU) \\
&= \int_{\mathbf{G}(e^\perp,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} [e^\perp, U]^s \\
&\quad \times H'_{r-1}(K \cap (x+U), x)\, \nu_s^V(dU)\, \nu_{s-1}^{e^\perp}(dV) \\
&= \int_{\mathbf{G}(e^\perp,\,s-1)} \int_{\mathbf{G}^V(d,\,s)} [e^\perp, U]^{s-r+1}
\end{aligned}$$

$$
\begin{aligned}
&\times H'_{r-1}(K \cap (x + \operatorname{lin}\{e, V\}), x)\, \nu_s^V(dU)\, \nu_{s-1}^{e^\perp}(dV) \\
&= \int_{\mathbf{G}(e^\perp,\, s-1)} H'_{r-1}(K \cap (x + \operatorname{lin}\{e, V\}), x) \\
&\quad \times \int_{\mathbf{G}^V(d,\, s)} [e^\perp, U]^{s-r+1}\, \nu_s^V(dU)\, \nu_{s-1}^{e^\perp}(dV) \\
&= \frac{2}{\omega_{d-s+1}} \frac{\kappa_{d-r}}{\kappa_{s-r}} H_{r-1}(K, x) \ .
\end{aligned}
$$

This yields the desired result. □

Now we have completed the preparations for the proofs of Theorems 2.5 and 2.4.

*Proof of Theorem 2.5.* It is sufficient to assume that $K \in \mathcal{K}_o^d$, since the curvature measures are locally defined. Moreover, we shall repeatedly use Fubini's theorem without further mentioning it. The required measurability is guaranteed by Lemma 5.3.

First, we assume that for $\mu_s$ almost all $E \in \mathbf{A}(d, s)$ such that $E \cap \operatorname{int} K \neq \emptyset$ the relation

$$
C'_{s-r}(K \cap E, \cdot) \llcorner (\beta \cap E) \ll C'_{s-1}(K \cap E, \cdot) \llcorner (\beta \cap E)
$$

is satisfied. Let $\gamma \subseteq \beta$ be an arbitrary Borel set. Then we obtain from the Crofton intersection formula, Theorem 4.5.5 in Schneider [41, p. 235], from the assumption and Eq. (2.7) of [24] applied in $s$-dimensional affine subspaces $E$, and from Lemma 5.4 that

$$
\begin{aligned}
&C_{d-r}(K, \gamma) \\
&= a_{dsr} \int_{\mathbf{A}(d,\, s)} C'_{s-r}(K \cap E, \gamma \cap E)\, \mu_s(dE) \\
&= a_{dsr} \int_{\mathbf{A}(d,\, s)} \int_{\operatorname{bd} K \cap E} \mathbf{1}_\gamma(x) H'_{r-1}(K \cap E, x)\, \mathcal{H}^{s-1}(dx)\, \mu_s(dE) \\
&= a_{dsr} \int_{\operatorname{bd} K} \mathbf{1}_\gamma(x) \int_{\mathbf{G}(d,\, s)} [T_x K, U] \\
&\quad \times H'_{r-1}(K \cap (x + U), x)\, \nu_s(dU)\, \mathcal{H}^{d-1}(dx) \\
&= \int_{\operatorname{bd} K \cap \gamma} H_{r-1}(K, x)\, \mathcal{H}^{d-1}(dx) \ .
\end{aligned}
$$

Note that the last equation is implied by Proposition 5.8. Thus

$$
C_{d-r}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta \ ,
$$

since $\gamma$ was an arbitrary Borel subset of $\beta$.

Now we assume that $C_{d-r}(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$. Using Lemma 5.4, Proposition 5.8, Eq. (2.7) from [24], the assumption of the theorem, Theorem 4.5.5 from Schneider [41], and the Lebesgue decomposition theorem applied to $C'_{s-r}(K \cap E,\cdot)$, we obtain that

$$\begin{aligned}
&\int_{\mathbf{A}(d,\,s)} \int_{\mathrm{bd}\,K\cap E} \mathbf{1}_\beta(x) H'_{r-1}(K\cap E,x)\,\mathcal{H}^{s-1}(dx)\,\mu_s(dE) \\
&= \int_{\mathrm{bd}\,K} \int_{\mathbf{G}(d,\,s)} \mathbf{1}_\beta(x)[T_xK,U] \\
&\quad \times H'_{r-1}(K\cap(x+U),x)\,\nu_s(dU)\,\mathcal{H}^{d-1}(dx) \\
&= \frac{1}{a_{dsr}} \int_{\mathrm{bd}\,K} \mathbf{1}_\beta(x) H_{r-1}(K,x)\,\mathcal{H}^{d-1}(dx) \\
&= \frac{1}{a_{dsr}} C^a_{d-r}(K,\beta) = \frac{1}{a_{dsr}} C_{d-r}(K,\beta) \\
&= \int_{\mathbf{A}(d,\,s)} C'_{s-r}(K\cap E,\beta\cap E)\,\mu_s(dE) \\
&= \int_{\mathbf{A}(d,\,s)} \int_{\mathrm{bd}\,K\cap E} \mathbf{1}_\beta(x) H'_{r-1}(K\cap E,x)\,\mathcal{H}^{s-1}(dx)\,\mu_s(dE) \\
&\quad + \int_{\mathbf{A}(d,\,s)} (C'_{s-r})^s(K\cap E,\beta\cap E)\,\mu_s(dE)\ .
\end{aligned}$$

Hence, for $\mu_s$ almost all $E \in \mathbf{A}(d,s)$ such that $\mathrm{int}\,K \cap E \neq \emptyset$, the singular part of the measure $C'_{s-r}(K\cap E,\cdot) \llcorner (\beta \cap E)$ vanishes. This establishes the converse part of the theorem. □

*Proof of Theorem 2.4.* For $r = d$ the theorem has already been verified. Thus we can assume that $r \in \{2,\ldots,d-1\}$. But then the statement follows from Theorem 2.3 and a special case of Theorem 2.5. □

The following three auxiliary results pave the way to the proof of Theorem 2.6. The first of these is of a purely geometric nature, the other two lemmas are integral-geometric results.

**Lemma 5.9.** *Let $K \in \mathcal{K}^d_o$, $r \in \{2,\ldots,d-1\}$, $E \in \mathbf{A}(K,d,r-1)$, and let $p \in E \cap K$. Then the implication*

$$\begin{aligned}
&\big(E+u^-\big) \cap \mathrm{int}\,K \neq \emptyset \Rightarrow \\
&\qquad\qquad \big(B(p-ru,r) \cap \big(E+u^-\big) \subseteq K \text{ for some } r>0\big)
\end{aligned}$$

*holds for some $u \in S^{d-1} \cap U(E)^\perp$ with $(E+u^-) \cap \mathrm{int}\,K \neq \emptyset$ if and only if the implication holds for all $u \in S^{d-1} \cap U(E)^\perp$.*

*Proof.* It can be assumed that $p = o$ and $E = \text{lin}\{e_1, \dots, e_{r-1}\}$. Let the vectors $u_i \in S^{d-1} \cap U(E)^\perp$, $i \in \{1,2\}$, be linearly independent and such that

$$\left(E + u_i^-\right) \cap \text{int}\, K \neq \emptyset\,, \quad \text{for } i \in \{1,2\}\,.$$

Furthermore, suppose that

$$B(-ru_1, r) \cap \left(E + u_1^-\right) \subseteq K$$

for some $r > 0$. Let $y \in \left(E + u_2^-\right) \cap \text{int}\, K$. In particular, $y$ can be chosen such that $y \notin E$. Then, if $\epsilon > 0$ is sufficiently small, we obtain that

$$x := y + \epsilon(y + ru_1) \in \text{int}\, K\,.$$

Hence we have

$$\text{conv}\left\{x, B(-ru_1, r) \cap \left(E + u_1^-\right)\right\} \cap \left(E + u_2^-\right) \subseteq K\,, \tag{27}$$

and it is sufficient to show that the set on the left-hand side of (27) is an ellipsoid, since a ball of a suitably small radius will roll freely inside any given ellipsoid.

In order to prove this assertion, let $e_r \in \text{lin}\{u_1, u_2, E\} \cap S^{d-1} \cap \text{lin}\{u_1, E\}^\perp$ be such that $\langle x, e_r\rangle > 0$. Further, let $\alpha$ be a linear map of $\text{lin}\{u_1, u_2, E\}$ onto itself which leaves $\text{lin}\{u_1, E\}$ invariant and which satisfies

$$\alpha(x) = -ru_1 + \langle x, e_r\rangle e_r\,.$$

This yields that $\alpha(y) = -ru_1 + \langle y, e_r\rangle e_r \neq o$. In addition, we know that $y = e - \lambda_0 u_2$ with some $e \in E$ and some positive constant $\lambda_0$. Therefore,

$$\begin{aligned} &\alpha\left(\text{conv}\left\{x, B(-ru_1, r) \cap \left(E + u_1^-\right)\right\} \cap \left(E + u_2^-\right)\right) \\ &\quad = \text{conv}\left\{\alpha(x), B(-ru_1, r) \cap \left(E + u_1^-\right)\right\} \cap \left(E + (-\alpha(y))^-\right)\,, \end{aligned} \tag{28}$$

since $\alpha(E + u_2^-) = E + (-\alpha(y))^-$. It is a well-known fact of elementary geometry that the set on the right-hand side of (28) is an ellipsoid. Thus, by applying $\alpha^{-1}$ to Eq. (28) the assertion follows. □

**Lemma 5.10.** *Let $r \in \{2, \dots, d-1\}$, and let $f : \mathbf{G}(d, r-1) \times S^{d-1} \to \mathbb{R}$ be a non-negative Borel measurable function. Then*

$$\begin{aligned} &\int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap U^\perp} f(U, u)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU) \\ &\quad = \frac{\omega_{d-r+1}}{\omega_r} \int_{\mathbf{G}(d,r)} \int_{S^{d-1}\cap V} f(V \cap u^\perp, u)\, \mathcal{H}^{r-1}(du)\, \nu_r(dV)\,. \end{aligned}$$

*Proof.* The set

$$\mathbf{G}^* := \left\{(U,u) \in \mathbf{G}(d,r-1) \times S^{d-1} : u \in U^\perp\right\}$$

together with the operation

$$\mathbf{O}(d) \times \mathbf{G}^* \to \mathbf{G}^* , \quad (\rho,(U,u)) \mapsto (\rho U, \rho u) ,$$

is a homogeneous $\mathbf{O}(d)$-space. Using the fact that the map

$$\{(V,u) \in \mathbf{G}(d,r) \times S^{d-1} : u \in V\} \to \mathbf{G}^* , \quad (V,u) \mapsto (V \cap u^\perp, u) ,$$

is Borel measurable, we can define two measures on $\mathbf{G}^*$ by setting

$$\mu_1(A) := \int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap U^\perp} \mathbf{1}_A(U,u)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU)$$

and

$$\mu_2(A) := \int_{\mathbf{G}(d,r)} \int_{S^{d-1}\cap V} \mathbf{1}_A(V \cap u^\perp, u)\, \mathcal{H}^{r-1}(du)\, \nu_r(dV)$$

for $A \in \mathfrak{B}(\mathbf{G}^*)$. These two measures are $\mathbf{O}(d)$-invariant. In fact, for any $\theta \in \mathbf{O}(d)$ we deduce from the $\mathbf{O}(d)$-invariance of $\nu_{r-1}$ and $\mathcal{H}^{d-r}$ that

$$\begin{aligned}\mu_1(A) &= \int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap U^\perp} \mathbf{1}_A(U,u)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU)\\ &= \int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap (\theta U)^\perp} \mathbf{1}_A(\theta U,u)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU)\\ &= \int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap U^\perp} \mathbf{1}_A(\theta U,\theta u)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU)\\ &= \mu_1(\theta^{-1}A) ,\end{aligned}$$

and a similar argument can be given for $\mu_2$. Hence, by the uniqueness theorem for Haar measures, we conclude that $\mu_1 = c\,\mu_2$ with a positive constant $c$. The explicit value of $c$ follows from substituting $A = \mathbf{G}^*$. □

For the statement of the following proposition, which plays a crucial rôle in the proof of Theorem 2.6, two further definitions will be needed.

Let $K \in \mathcal{K}^d_o$ and $r \in \{2, \ldots, d-1\}$. Then we set

$$\mathbf{A}_K(d,r)^* := \{F \in \mathbf{A}(d,r) : F \cap \text{int}\, K \neq \emptyset\}$$

and

$$\begin{aligned}\mathbf{A}(K,d,r-1,1)^* := \Big\{&(E,u) \in \mathbf{A}(K,d,r-1) \times S^{d-1} :\\ &u \in U(E)^\perp, (E+u^-) \cap \text{int}\, K \neq \emptyset\Big\} .\end{aligned}$$

In a certain sense, the next result, Proposition 5.11, provides a tool for translating statements about $(r-1)$-dimensional touching affine subspaces into statements about $r$-dimensional intersecting affine subspaces, and vice versa.

**Proposition 5.11.** *Let $K \in \mathcal{K}_o^d$, $r \in \{2, \ldots, d-1\}$, and assume that the function $f : \mathbf{A}(K, d, r-1) \times S^{d-1} \to [0, \infty]$ is Borel measurable. Then*

$$\begin{aligned}
&\int_{\mathbf{A}(K,d,r-1)} \int_{S^{d-1} \cap U(E)^\perp} \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(E,u) f(E,u) \\
&\times \mathcal{H}^{d-r}(du)\, \mu_{r-1}(K, dE) \\
&= \frac{\omega_{d-r+1}}{\omega_r} \int_{\mathbf{A}_K(d,r)^*} \int_{S^{d-1} \cap U(F)} J(F,u) f(H^F(K \cap F, u), u) \\
&\quad \times \mathcal{H}^{r-1}(du)\, \mu_r(dF)\ ,
\end{aligned}$$

*where*

$$J(F,u) := \left\langle \sigma_{K|\mathrm{lin}\left\{u, U(F)^\perp\right\}} \left( H^F(K \cap F, u) \cap \mathrm{lin}\left\{u, U(F)^\perp\right\}\right), u \right\rangle^{-1}$$

*is well-defined for $\mu_r$ almost all $F \in \mathbf{A}_K(d,r)^*$ and $\mathcal{H}^{r-1}$ almost all unit vectors $u \in S^{d-1} \cap U(F)$.*

*Proof.* The proof is accomplished by a sequence of integral-geometric transformations. First, note that $\mathbf{A}_K(d,r)^*$ and $\mathbf{A}(K,d,r-1,1)^*$ can be written as countable unions of closed sets. This follows from choosing $K_n \in \mathcal{K}^d$, for $n \in \mathbb{N}$, with $K_n \subset K_{n+1}$ and int $K = \cup_{n \geq 1} K_n$. Then, using the definition of $\mu_{r-1}(K, \cdot)$, the fact that $\nu_{d-r+1}$ is the image measure of $\nu_{r-1}$ under the map $\mathbf{G}(d, r-1) \to \mathbf{G}(d, d-r+1)$, $U \mapsto U^\perp$, Fubini's theorem, and

Lemma 5.10, we obtain

$$\begin{aligned}
&\int_{\mathbf{A}(K,d,r-1)} \int_{S^{d-1}\cap U(E)^\perp} \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(E,u) f(E,u) \\
&\times \mathcal{H}^{d-r}(du)\, \mu_{r-1}(K,dE) \\
&= \int_{\mathbf{G}(d,d-r+1)} \int_{\mathrm{bd}_U(K|U)} \int_{S^{d-1}\cap U} \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(x+U^\perp,u) \\
&\quad \times f(x+U^\perp,u) \mathcal{H}^{d-r}(du)\, \mathcal{H}^{d-r}(dx)\, \nu_{d-r+1}(dU) \\
&= \int_{\mathbf{G}(d,r-1)} \int_{\mathrm{bd}_{U^\perp}(K|U^\perp)} \int_{S^{d-1}\cap U^\perp} \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(x+U,u) \\
&\quad \times f(x+U,u) \mathcal{H}^{d-r}(du)\, \mathcal{H}^{d-r}(dx)\, \nu_{r-1}(dU) \\
&= \int_{\mathbf{G}(d,r-1)} \int_{S^{d-1}\cap U^\perp} \int_{\mathrm{bd}_{U^\perp}(K|U^\perp)} \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(x+U,u) \\
&\quad \times f(x+U,u) \mathcal{H}^{d-r}(dx)\, \mathcal{H}^{d-r}(du)\, \nu_{r-1}(dU) \\
&= \frac{\omega_{d-r+1}}{\omega_r} \int_{\mathbf{G}(d,r)} \int_{S^{d-1}\cap V} \int_{\mathrm{bd}_{\mathrm{lin}\{u,V^\perp\}}(K|\mathrm{lin}\{u,V^\perp\})} \\
&\quad \times \mathbf{1}_{\mathbf{A}(K,d,r-1,1)^*}(x+V\cap u^\perp,u) f(x+V\cap u^\perp,u) \\
&\quad \times \mathcal{H}^{d-r}(dx)\, \mathcal{H}^{r-1}(du)\, \nu_r(dV)\ .
\end{aligned}$$

For any fixed $V \in \mathbf{G}(d,r)$ and $u \in S^{d-1}\cap V$, we define the map

$$G_K^{V,u} : \mathrm{bd}_{\mathrm{lin}\{u,V^\perp\}}\left(K|\mathrm{lin}\left\{u,V^\perp\right\}\right) \to K|V^\perp\ , \quad y \mapsto y - \langle y,u\rangle u\ .$$

The restriction of $G_K^{V,u}$ to the set of all $x \in \mathrm{bd}_{\mathrm{lin}\{u,V^\perp\}}\left(K|\mathrm{lin}\left\{u,V^\perp\right\}\right)$ such that $\left(x+(V\cap u^\perp)+u^-\right)\cap \mathrm{int}\,K \neq \emptyset$ is a bijection onto $\mathrm{int}_{V^\perp}\big(K|V^\perp\big)$. Moreover, for $\mathcal{H}^{d-r}$ almost all $x \in \mathrm{bd}_{\mathrm{lin}\{u,V^\perp\}}\left(K|\mathrm{lin}\left\{u,V^\perp\right\}\right)$ such that $\left(x+(V\cap u^\perp)+u^-\right)\cap \mathrm{int}\,K \neq \emptyset$, one obtains that

$$J_{d-r}\, G_K^{V,u}(x) = \left|\left\langle \sigma^{\mathrm{lin}\{u,V^\perp\}}_{K|\mathrm{lin}\{u,V^\perp\}}(x), u\right\rangle\right| > 0\ .$$

Therefore an application of the coarea formula shows that the preceding chain of equalities can be continued with

$$\begin{aligned}
&\frac{\omega_{d-r+1}}{\omega_r} \int_{\mathbf{G}(d,r)} \int_{S^{d-1}\cap V} \int_{\mathrm{int}_{V^\perp}(K|V^\perp)} \left\{ J_{d-r}\, G_K^{V,u}\left(\left(G_K^{V,u}\right)^{-1}(y)\right)\right\}^{-1} \\
&\qquad \times f\left(\left(G_K^{V,u}\right)^{-1}(y) + V\cap u^\perp, u\right) \mathcal{H}^{d-r}(dy)\, \mathcal{H}^{d-r}(du)\, \nu_r(dU)\ .
\end{aligned}$$

But

$$\left(G_K^{V,u}\right)^{-1}(y)+V\cap u^{\perp}=H^{(y+V)}(K\cap(y+V),u)\ ,$$

and for $\mathcal{H}^{d-r}$ almost all $y\in \mathrm{int}_{V^\perp}\left(K|V^\perp\right)$ the $(d-r)$-dimensional Jacobian

$$J_{d-r}\,G_K^{V,u}\left(\left(G_K^{V,u}\right)^{-1}(y)\right)$$

is given by

$$\left|\left\langle \sigma_{K|\mathrm{lin}\left\{u,U(y+V)^\perp\right\}}^{\mathrm{lin}\left\{u,U(y+V)^\perp\right\}}\left(H^{(y+V)}(K\cap(y+V),u)\right.\right.\right.$$
$$\left.\left.\left.\cap\ \left(\mathrm{lin}\left\{u,U(y+V)^\perp\right\}\right)\right),u\right\rangle\right|^{-1}.$$

It is convenient to write the argument of the spherical image map as a set which consists of precisely one point. This slight abuse of notation should not lead to any misunderstanding. Hence, the proof is completed by using once again Fubini's theorem and the representation of $\mu_r$ given in §4.5 of [41]. □

*Proof of Theorem 2.6.* Let $K$, $\beta$, and $r$ be chosen as in the assumptions of Theorem 2.6. In [51], it was shown that there are sets $A_1,A_2\in\mathfrak{B}(\mathbf{A}(d,r))$ with $\mu_{r-1}(K,A_2)=0$ such that

$$A_1\subseteq\sigma_{r-1}(K,\beta)\subseteq A_1\cup A_2\ .$$

By $A^c_{r-1}(K,\beta)$ we denote the set of all $E\in\sigma_{r-1}(K,\beta)$ such that $K$ is not supported from inside by an $r$-dimensional ball at $E$. Moreover, we write $A^c_{r-1,1}(K,\beta)$ for the set of all $(E,u)\in\sigma_{r-1}(K,\beta)\times S^{d-1}$ such that $u\in U(E)^\perp$, $(E+u^-)\cap\mathrm{int}\,K\neq\emptyset$, and such that $B(p-\rho u,\rho)\cap(E+u^-)\not\subseteq K$ holds for all $p\in K\cap E$ and all $\rho>0$. With these definitions we see that the inclusions

$$\begin{aligned}A_1\cap A^c_{r-1}(K,\mathbb{R}^d)&\subseteq A^c_{r-1}(K,\beta)\\&\subseteq\left(A_1\cap A^c_{r-1}(K,\mathbb{R}^d)\right)\cup\left(A_2\cap A^c_{r-1}(K,\mathbb{R}^d)\right),\end{aligned}$$

$$(A_1\times S^{d-1})\cap A^c_{r-1,1}(K,\mathbb{R}^d)\subseteq A^c_{r-1,1}(K,\beta)$$

and

$$\begin{aligned}A^c_{r-1,1}(K,\beta)\subseteq&\left((A_1\times S^{d-1})\cap A^c_{r-1,1}(K,\mathbb{R}^d)\right)\\&\cup\left((A_2\times S^{d-1})\cap A^c_{r-1,1}(K,\mathbb{R}^d)\right)\end{aligned}$$

are satisfied. Observe that $A^c_{r-1}(K,\mathbb{R}^d)$ and $A^c_{r-1,1}(K,\mathbb{R}^d)$ are Borel measurable sets. Since $\mu_{r-1}(K,A_2)=0$, we obtain that

$$\mu_{r-1}(K,A^c_{r-1}(K,\beta))=0$$

if and only if

$$\mu_{r-1}(K,A_1\cap A^c_{r-1}(K,\mathbb{R}^d))=0\ . \tag{29}$$

Lemma 5.9 yields that Eq. (29) is equivalent to

$$0=\int_{\mathbf{A}(K,d,r-1)}\int_{S^{d-1}\cap U(E)^\perp}\mathbf{1}_{(A_1\times S^{d-1})\cap A^c_{r-1,1}(K,\mathbb{R}^d)}(E,u)\\ \times\mathcal{H}^{d-r}(du)\,\mu_{r-1}(K,dE)\ . \tag{30}$$

From Proposition 5.11 and the fact that $J(F,u)\in(0,\infty)$, for $\mu_r$ almost all $F\in\mathbf{A}_K(d,r)^*$ and for $\mathcal{H}^{r-1}$ almost all $u\in S^{d-1}\cap U(F)$, we obtain that Eq. (30) holds if and only if

$$0=\int_{\mathbf{A}_K(d,r)^*}\int_{S^{d-1}\cap U(F)}\mathbf{1}_{(A_1\times S^{d-1})\cap A^c_{r-1,1}(K,\mathbb{R}^d)}(H^F(K\cap F,u),u)\\ \times\mathcal{H}^{r-1}(du)\,\mu_r(dF)\ . \tag{31}$$

The corresponding integral with $A_1$ replaced by $A_2$ also vanishes, since $\mu_{r-1}(K,A_2)=0$. Therefore Eq. (31) is equivalent to

$$\left[\begin{array}{c}\text{For } \mu_r \text{ almost all } F\in\mathbf{A}_K(d,r)^*\\ \text{and for } \mathcal{H}^{r-1} \text{ almost all } u\in S^{d-1}\cap U(F),\\ \big(H^F(K\cap F,u),u\big)\notin A^c_{r-1,1}(K,\beta).\end{array}\right] \tag{32}$$

But obviously condition (32) is equivalent to

$$\left[\begin{array}{c}\text{For } \mu_r \text{ almost all } F\in\mathbf{A}(d,r) \text{ such that } F\cap\operatorname{int}K\neq\emptyset\\ \text{and for } \mathcal{H}^{r-1} \text{ almost all } u\in\sigma'(K\cap F,\beta\cap F)\\ \text{the intersection } K\cap F \text{ is supported from inside}\\ \text{by an } r\text{-dimensional ball in direction } u.\end{array}\right] \tag{33}$$

Finally, an application of Theorem 2.4 shows that condition (33) is equivalent to

$$C_{d-r}(K,\cdot)\llcorner\beta\ll C_{d-1}(K,\cdot)\llcorner\beta\ ,$$

which was to be proved. □

*Proof of Theorem 2.7.* Denote by $\mathcal{E}_3$ the set of all $E\in\sigma_{r-1}(K,\beta)$ such that $\operatorname{card}(E\cap K)>1$, let $\mathcal{E}_4$ be the set of all $E\in\sigma_{r-1}(K,\beta)$ such that $K$ is not supported from inside by an $r$-dimensional ball at $E$, and let $\mathcal{E}_5$ be the set of all $E \in \sigma_{r-1}(K, \beta)$ such that the point $p$ which is defined by

$$\{p\} = E \cap \mathrm{bd}_{U(E)^\perp} \left( K|U(E)^\perp \right)$$

is not a regular boundary point of $K|U(E)^\perp$. Then the result of Zalgaller [56], Theorem 2.6, and Theorem 2.2.4 from Schneider [41] together with the definition of $\mu_{r-1}(K, \cdot)$ in (21) imply that

$$\mu_{r-1}(K, \mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_5) = 0 \ .$$

Now, choose $E \in \sigma_{r-1}(K, \beta) \setminus (\mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_5)$, and let $x$ be defined by $\{x\} = E \cap K$. Let $S(K, x)$ denote the support cone of $K$ at $x$; see [41, p. 70] for a definition. Since $E \notin \mathcal{E}_4$, we deduce that $U(E) \subseteq S(K, x)$, and hence $N(K, x) \subseteq U(E)^\perp$. Furthermore, $E \notin \mathcal{E}_5$ finally implies that $\dim N(K, x) = 1$, since otherwise the orthogonal projection of $x$ onto $U(E)^\perp$ is not a regular boundary point of $K|U(E)^\perp$. □

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