Contact distributions of Boolean models*

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Abstract

The local contact distribution of a random closed set Z is essentially the distribution of the nearest point of Z (if it exists uniquely) measured from a fixed reference point and with respect to a structuring element B. For a stationary Boolean model Z with polyconvex grains and a strictly convex compact set B, the contact distribution is well-defined, and we describe how it is related to certain non-negative mean support measures of the associated stationary particle process X. We discuss several special cases, extensions and applications of this result. In particular, we compare the present methods with an approach which works under very general model assumptions and discuss some examples.

1 Introduction

A basic objective of stochastic geometry is to analyse random patterns. Such patterns can be described as realizations of random variables which are defined on an abstract probability space $(\Omega, \mathbb{A}, \mathbb{P})$ with values in a space Y. If Y is the space of closed subsets of Euclidean space $\mathbb{R}^d$, that is $Y = \mathcal{F}(\mathbb{R}^d)$, such a random variable is called a random closed set and denoted by Z in the following. On the other hand, one is often interested in random collections of closed sets and not just in their union set. The appropriate model for such random objects are particle processes, that is point processes in $\mathcal{F}(\mathbb{R}^d)$ (or subspaces thereof), which we will denote by X. The transition from a particle process X to an associated random closed set Z_X is straightforward by forming the union set

$$Z_X := \bigcup_{K \in X} K\,;$$

see [14], [25], [24] for background information.

In principle, the union set Z_X is a directly observable quantity in contrast to the underlying point process X. It is the aim of the present article to show how the (local)

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contact distributions of Z_X, with respect to different structuring elements, can be used to gain information on certain mean values (densities) which are associated with X, in the case where X is a (stationary) Poisson process. Different approaches have recently been used to achieve similar aims; see, for instance, [16], [28], [29], [30], [31]. A basic advantage of the present method is that it can be applied in a non-stationary setting as well and also if X is not Poisson; compare [12], [13], [8].

In order to introduce the contact distribution with respect to a general convex body (nonempty compact convex set) $B \subset \mathbb{R}^d$ with $0 \in B$ as structuring element, we define the distance $d_B(K, x)$ of $x \in \mathbb{R}^d$ from a closed set $K \subset \mathbb{R}^d$ with respect to B by

$$d_B(K, x) := \inf\{r \geq 0 : x \in K + rB\}.$$

This is a finite number if $0 \in \operatorname{int} B$, but in general $d_B(K, x)$ can be infinite. We set $\check{B} := -B$. If $0 < d_B(K, x) < \infty$ and if K and $x + d_B(K, x)\check{B}$ have precisely one point in common, then this unique point is called the B-projection of x onto K and is denoted by $p_B(K, x)$. In this situation

$$u_B(K, x) := d_B(K, x)^{-1}(x - p_B(K, x))$$

is called the direction of the B-projection of x onto K. The connection of these notions to the geometry of Minkowski spaces is outlined in [8] (see also [26] for general information about Minkowski geometry). Now we consider a random closed set Z and a point $x \in \mathbb{R}^d$ for which the (local) volume density satisfies $\bar{p}(x) < 1$, where

$$\bar{p}(x) := \mathbb{P}(x \in Z);$$

in addition, we assume that $p_B(Z, x)$, and hence $u_B(Z, x)$, are defined $\mathbb{P}$ almost surely. Then we can introduce the local contact distribution by

$$H_B(x, r, A) := \mathbb{P}(0 < d_B(Z, x) \leq r, u_B(Z, x) \in A \mid x \notin Z),$$

where $A \subset \mathbb{R}^d$ is Borel measurable. We set $H_B(x, r) := H_B(x, r, \mathbb{R}^d)$, and thus $H_B(x, \cdot)$ is the conditional distribution function of $d_B(Z, x)$ given that $x \notin Z$. In [8], a general investigation of the local contact distribution was carried out under fairly weak distributional assumptions. However, for a random closed set Z_X which is induced by a (stationary) Poisson particle process X more specific and additional results can be obtained by a more direct approach. Indeed, in addition to various minor points, compared with [8] the main progress of the present work is that we are able to treat structuring elements which are non-smooth and lower-dimensional; furthermore, we also discuss Boolean models with polyconvex grains, whereas the investigation in [8] was restricted to convex grains.

Except for [8], previous investigations essentially were restricted to the consideration of contact distribution functions of *stationary* random closed sets and to the *spherical* or *linear* contact distribution function. In the stationary case, $H_B(x, r, A)$ is independent of x and therefore it is sufficient to consider $x = 0$; for notational convenience we then simply write $H_B(r, A)$ and $H_B(r)$. The stationary situation has already received much attention through the recent development of Kaplan-Meier type estimators for $H_B(r)$ and the proof of asymptotic properties of empirical contact distribution functions; see [2], [5], [4], [6], and [17] for a general survey. However, the assumption of stationarity, or even isotropy, is a mathematical convenience which cannot always be justified in practical situations. In fact, the need to study contact distributions for general structuring elements B and without the assumption of stationarity became recently clear; compare the discussion in [1, p. 73], [8], [3], [4].

Let us briefly outline the scope of the paper; definitions and further references to the literature are provided in the following sections. In Sections 2 and 3 we shall consider a stationary Boolean model $Z = Z_X$ with grains in the convex ring, which is associated with a stationary Poisson particle process X. We shall show how the contact distribution $H_B(r, A)$, for $r \geq 0$ and a Borel set $A \subset \mathbb{R}^d$, is related to certain mean (non-negative) support measures which are associated with X and B, where $B \subset \mathbb{R}^d$ is assumed to be strictly convex and to satisfy $0 \in \operatorname{int} B$. In particular, if Z has convex grains, we obtain a probabilistic version of a Steiner formula, that is

$$H_B(r, A) = \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r t^{d-1-j}(1 - H_B(t))dt\, \overline{S}_j(X, B; A)\,, \tag{1}$$

provided that B is strictly convex and $0 \in \operatorname{int} B$. This formula thus connects the local contact distribution of the stationary Boolean model Z_X to the mean surface area measures $\overline{S}_j(X, B; \cdot)$, $j \in \{0, \ldots, d-1\}$, of the underlying stationary Poisson particle process X. Clearly, such a result must be viewed as a particular feature of the stationary Boolean model and cannot be obtained in such a simple form in a more general setting. If $K \subset \mathbb{R}^d$ is a convex body, then the deterministic counterpart of (1) is

$$\begin{aligned}\mathcal{H}^d &\left(\left\{x \in \mathbb{R}^d : 0 < d_B(K, x) \leq r, u_B(K, x) \in A\right\}\right)\\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \frac{r^{d-j}}{d-j}\, S_j(K, B; A)\,,\end{aligned} \tag{2}$$

where $S_j(K, B; \cdot)$, $j \in \{0, \ldots, d-1\}$, are the surface area measures of K with respect to B and $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure with respect to an auxiliary Euclidean norm which is held fixed in the following; see [10] and [8]. (The norm will be denoted by $\|\cdot\|$ and the associated scalar product by $\langle \cdot\,, \cdot \rangle$.) Thus, when compared with the right-hand side of (2), the monomial t^{d-1-j} in (1) has to be weighted with $1 - H_B(t)$. Moreover equation (1) shows that the mean surface area measures $\overline{S}_j(X, B; A)$, $j \in \{0, \ldots, d-1\}$, which appear as coefficients in (1), are determined by iterated derivatives of $H_B(\cdot, A)$ at $r = 0$. For a Boolean model with polyconvex grains it is still possible to determine the special mean mixed volumes $\overline{V}(X[d-1], B)$ from measurements of dilation volumes of Z from a single observation in an increasing sequence of observation windows. As a consequence the mean surface area measure $\overline{S}_{d-1}(X, \cdot)$ is determined as well.

In [18], Rataj determines the mean mixed volume $\overline{V}(Z[d-1], B)$ of ergodic random closed sets which can be described as unions of compact sets of positive reach satisfying some additional hypotheses. Here we show, following a different argument, that stronger conclusions can be obtained for random closed sets in the extended convex ring, and we apply these to the Boolean model in the same way as in [18]. In particular, we establish an infinitesimal version of a local Steiner formula, with respect to an arbitrary gauge body (structuring element), for sets in the extended convex ring.

The results mentioned so far are derived by combinations of probabilistic and integral-geometric arguments. This is also true for an extension of (1) to structuring elements lying in subspaces. To obtain such an extension, we first establish a corresponding deterministic Steiner formula for lower-dimensional structuring elements involving certain mixed surface area measures.

In the final section, we outline in an exemplary manner the transition to the more general non-stationary situation. We show, under some reasonable assumptions and for an underlying Poisson process X of convex particles, that the local surface density $\lambda^+_{d-1}(B, x)$ of Z_X with respect to a structuring element B can be expressed by means of the derivative of $H_B(x, r)$ at $r = 0$ and the volume fraction $\bar{p}(x)$, for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$. This extends Remark 4.15 in [8]. In view of Proposition 4.11 in [8], a similar result immediately follows for the surface density $\lambda_{d-1}(B, x)$ of the particle process X. An intuitive argument for such a relationship in a Euclidean setting was suggested in [15]. Finally, we provide a rigorous derivation of some formulae for the surface density $\lambda^+_{d-1}(B, x)$, which have been used in [15] to generate various plots of surface densities.

2 Contact distributions of stationary Boolean models

Throughout this section, we consider a stationary Boolean model Z with polyconvex grains (if not stated otherwise). Following [11], we call a set $K \subset \mathbb{R}^d$ polyconvex if it is a finite union of convex bodies (that is, an element of the convex ring). We write $\mathcal{R}^d$ for the family of polyconvex sets and $\mathcal{K}^d$ for the subfamily of convex bodies. The superscript d, indicating the dimension in which these sets are considered, is usually omitted. We write $\mathcal{R}_0$ and $\mathcal{K}_0$ for the subfamilies of sets from $\mathcal{R}$ and $\mathcal{K}$, respectively, having the centre of their circumscribed ball at the origin.

Let X denote the stationary Poisson particle process in $\mathcal{R}$ which is associated with Z; compare the discussion in Sections 4.3 and 4.4 in [24], in particular Satz 4.4.3. The intensity measure $\Theta = \mathbb{E}[X(\cdot)]$ of X is always assumed to be locally finite and to satisfy $\Theta \not\equiv 0$ even if this is not explicitly mentioned. The former condition means that $\Theta(\mathcal{F}_C) < \infty$ for all compact sets $C \subset \mathbb{R}^d$, where $\mathcal{F}_C$ denotes the family of all closed sets $F \subset \mathbb{R}^d$ for which $F \cap C \neq \emptyset$. Therefore Θ can be written as

$$\Theta(\cdot) = \gamma \int_{\mathcal{R}_0} \int_{\mathbb{R}^d} \mathbf{1}\{x + K \in \cdot\} \mathcal{H}^d(dx) \mathbb{Q}(dK) \,, \tag{3}$$

where γ is the intensity and $\mathbb{Q}$ is the shape distribution of X. Of course, in order to obtain such a decomposition we do not need the assumption that X is a Poisson process. We refer to the books [23], [21], [24], and the literature cited there, for essential background information and the basic notation used in this paper. The condition that Θ is locally finite means that

$$\int_{\mathcal{R}_0} \mathcal{H}^d(K+C)\mathbb{Q}(dK) < \infty \,, \tag{4}$$

for all compact $C \subset \mathbb{R}^d$; hence,

$$\begin{aligned} \mathbb{P}(0 \notin Z) &= \mathbb{P}(X(\mathcal{F}_{\{0\}}) = 0) \\ &= \exp\left\{-\Theta(\mathcal{F}_{\{0\}})\right\} \\ &= \exp\left\{-\gamma \int_{\mathcal{R}_0} \mathcal{H}^d(K)\mathbb{Q}(dK)\right\} \\ &> 0 \,. \end{aligned}$$

In the following, we shall occasionally use results from [8]. However, we shall deviate from the notation introduced there in some respects. For example, if $K \in \mathcal{R}$, $B \in \mathcal{K}$ is strictly convex and $0 \in \operatorname{int} B$, then we shall write $\Theta_j(K,B;\cdot)$ and $\Theta_j^+(K,B;\cdot)$ instead of $C_j^B(K,\cdot)$ and $C_j^{B,+}(K,\cdot)$ for the additive and the non-negative extension of the Minkowski support measures to polyconvex sets; the precise relationship is

$$\Theta_j(K,B;\cdot) = \frac{(d-j)\kappa_{d-j}}{\binom{d-1}{j}} C_j^{\check{B}}(K,\cdot) \,,$$

and similarly for $\Theta_j^+(K,B;\cdot)$, $j \in \{0,\dots,d-1\}$, where κ_i denotes the Euclidean volume of an i-dimensional unit ball. It was shown in [8], Theorem 3.4, that the measures $\Theta_j^+(K,B;\cdot)$ are the natural non-negative extensions of the support measures for convex bodies, in a Minkowski space with gauge body B. (Note that we always speak of a Minkowski space even if the gauge body B is not centrally symmetric.) Curvature measures in Minkowski spaces have been considered for the first time in [22]. In a Euclidean setting, Schneider [20] and in a special case also Matheron [14] investigated non-negative and also additive extensions of Euclidean support measures. In analogy to the Euclidean case, we define Minkowski surface area measures

$$S_j(K,B;\cdot) := \Theta_j(K,B;\mathbb{R}^d \times \cdot) \,,$$

for $K \in \mathcal{R}$ and $j \in \{0,\dots,d-1\}$, as Borel measures over $\mathbb{R}^d$, which are concentrated on bd B. It is shown in [9] that these measures can be expressed in terms of special mixed surface area measures (see [21] for the definition of these measures):

$$S_j(K,B;\cdot) = \int_{S^{d-1}} \mathbf{1}\{\nabla h_B(u) \in \cdot\} h(B,u) S(K[j], B[d-1-j], du) \,,$$

where $h_B = h(B,\cdot)$ denotes the support function of B and S^{d-1} is the Euclidean unit sphere. In addition to these purely deterministic measures, we introduce certain mean measures of the stationary (Poisson) particle process X in $\mathcal{R}$ such as the mean $(d-1)$st surface area measure

$$\overline{S}_{d-1}(X,B;\cdot) := \gamma \int_{\mathcal{R}_0} S_{d-1}(K,B;\cdot)\mathbb{Q}(dK)\,;$$

clearly, this is a well-defined non-negative Borel measure. For a stationary particle process X in $\mathcal{K}$ we also define

$$\overline{S}_j(X,B;\cdot) := \gamma \int_{\mathcal{K}_0} S_j(K,B;\cdot)\mathbb{Q}(dK)\,.$$

More general definitions, under suitable integrability hypotheses, will be presented in Section 3. The corresponding Euclidean notions are always denoted by omitting the letter B; so we write $S_j(K,\cdot)$ instead of $S_j(K,B^d;\cdot)$, where B^d is the Euclidean unit ball.

In addition to (4), we shall sometimes require that

$$\int_{\mathcal{R}_0} \Theta_j^+(K,B;\mathbb{R}^{2d})\mathbb{Q}(dK) < \infty\,, \tag{5}$$

for $j = 0,\dots,d-1$. For $K \in \mathcal{R}$, let $N(K)$ be the smallest number m for which there exist convex bodies $K_1,\dots,K_m \subset \mathbb{R}^d$ such that $K = K_1 \cup \ldots \cup K_m$. We write $S(m)$, $m \in \mathbb{N}$, for the set of all non-empty subsets of $\{1,\dots,m\}$, and set $K_v := \cap\{K_i : i \in v\}$ for $v \in S(m)$. The cardinality of $v \in S(m)$ is denoted by $|v|$. Finally, $Nor_B(K) \subset \mathbb{R}^{2d}$ denotes the Minkowski normal bundle of K with respect to B. Instead of $Nor_{B^d}(K)$, which is the Euclidean normal bundle of K, we simply write $Nor(K)$. In order to define these normal bundles, we write $\mathrm{exo}_B(K)$ for the set of all $x \in \mathbb{R}^d \setminus K$ for which $p_B(K,x)$ is not defined, and we call this set the exoskeleton of K with respect to B. Then we set

$$Nor_B(K) := \{(p_B(K,x),u_B(K,x)) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d \setminus (K \cup \mathrm{exo}_B(K))\}\,;$$

compare [8]. Using this notation and some basic properties of Minkowski support measures (see Sections 2 and 3 in [8]), we obtain

$$\begin{aligned}
\Theta_j^+(K,B;\mathbb{R}^{2d}) &= \Theta_j(K,B;Nor_B(K)) \\
&= \sum_{v\in S(m)} (-1)^{|v|}\Theta_j(K_v,B;Nor_B(K)) \\
&\le \sum_{v\in S(m)} \Theta_j(K_v,B;\mathbb{R}^{2d}) \\
&= \sum_{v\in S(m)} dV(K_v[j],B[d-j]) \\
&\le c\,2^{N(K)}r(K)^j\,,
\end{aligned}$$

where c is a constant which does not depend on K and $r(K)$ denotes the radius of the circumsphere of K. Hence, the condition

$$\int_{\mathcal{R}_0} 2^{N(K)} r(K)^j \mathbb{Q}(dK) < \infty\,, \tag{6}$$

for $j = 0, \ldots, d-1$, ensures that (5) is satisfied. Furthermore, since

$$\mathcal{H}^d(K + C) \le \bar{c}\, 2^{N(K)}(r(K)+1)^d$$

is fulfilled for every compact set $C \subset \mathbb{R}^d$, with a constant $\bar{c}$ which merely depends on C, we see that the condition

$$\int_{\mathcal{R}_0} 2^{N(K)}(r(K)+1)^d \mathbb{Q}(dK) < \infty$$

implies that (4) and (5) are fulfilled.

The following theorem requires some more notation. We define

$$\delta_B(K, z, b) := \inf\{r > 0 : z + rb \in \operatorname{exo}_B(K)\}$$

if $(z, b) \in Nor_B(K)$, and otherwise we set $\delta_B(K, z, b) = 0$. The function $\delta_B(K, \cdot)$ is Borel measurable and takes values in $[0, \infty]$; compare [8]. In order to obtain a succinct statement of the result in Theorem 2.1 below, we use $\delta_B(K, \cdot)$ to introduce measures over $\mathbb{R}^d \times \mathbb{R}^d \times [0, \infty]$ by

$$\Theta_j^*(K, B; \cdot) := \int \mathbf{1}\{(z, b, \delta_B(K, z, b)) \in \cdot\}\Theta_j^+(K, B; d(z, b))\,,$$

for $K \in \mathcal{R}$, and

$$\overline{\Theta}_j^*(X, B; \cdot) := \gamma \int_{\mathcal{R}_0} \Theta_j^*(K, B; \cdot)\mathbb{Q}(dK)\,,$$

if X is a stationary particle process in $\mathcal{R}$ with shape distribution $\mathbb{Q}$. Note that if $K \in \mathcal{K}$, then $\delta_B(K, z, b) = \infty$ for $(z, b) \in Nor_B(K)$, and thus we have $\Theta_j^*(K, B; \cdot \times (s, \infty]) = \Theta_j(K, B; \cdot)$ for all $s \ge 0$.

Theorem 2.1 *Let Z be a stationary Boolean model with polyconvex grains, and let X be the associated stationary particle process in $\mathcal{R}$. Let $B \subset \mathbb{R}^d$ be strictly convex and $0 \in \operatorname{int} B$. Then*

$$H_B(r) = 1 - \exp\left\{-\sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^r \overline{\Theta}_j^*(X, B; \mathbb{R}^{2d} \times (s, \infty])s^{d-1-j} ds\right\}. \tag{7}$$

Proof. It is easy to see that

$$H_B(r) = 1 - \exp\left\{-\gamma \int_{\mathcal{R}_0} \mathcal{H}^d((K + rB) \setminus K)\mathbb{Q}(dK)\right\}.$$

An application of Theorem 3.3 in [8] yields

$$\mathcal{H}^d((K + rB) \setminus K) = \sum_{j=0}^{d-1} \binom{d-1}{j} \int\int \mathbf{1}\{\delta_B(K, z, b) > s\}$$
$$\times\ \mathbf{1}\{s \leq r\} s^{d-1-j} \Theta_j^+(K, B; d(z, b)) ds\,.$$

Combining these two equations, we obtain

$$H_B(r) = 1 - \exp\left\{-\gamma \sum_{j=0}^{d-1} \binom{d-1}{j} \int_{\mathcal{R}_0} \int \int_0^r \mathbf{1}\{\delta_B(K, z, b) > s\}\right.$$
$$\left.\times\ s^{d-1-j} ds \Theta_j^+(K, B; d(z, b))\mathbb{Q}(dK)\right\},$$

which is the assertion of the theorem. □

From this general result we now deduce a representation of a special mean mixed volume of the stationary particle process X in terms of the contact distribution of the induced stationary Boolean model Z. Recall that the special mixed volume $V(K[d-1], B)$, $K \in \mathcal{R}$, can be expressed in different ways such as

$$dV(K[d-1], B) = \int_{S^{d-1}} h(B, u) S_{d-1}(K, du)$$
$$= S_{d-1}(K, B; \mathbb{R}^d)\,;$$

see [21]. The corresponding mean value with respect to the stationary particle process X is defined by

$$\overline{V}(X[d-1], B) = \gamma \int_{\mathcal{R}_0} V(K[d-1], B)\mathbb{Q}(dK)\,.$$

The following corollary explains how this quantity is related to the contact distribution function.

Corollary 2.2 *Let the assumptions of Theorem 2.1 and condition* (5) *be satisfied. Then*

$$\left.\frac{\partial}{\partial t}\right|_{t=+0} H_B(t) = \int_{S^{d-1}} h(B, u)\overline{S}_{d-1}(X, du)$$
$$= d\overline{V}(X[d-1], B)\,.$$

Proof. The assumption (5) justifies that we can differentiate the right-hand side of equation (7) under the integral with respect to r at $r = 0$. In fact, since $\Theta_j^+(K, B; \cdot)$ is concentrated on $Nor_B(K)$ and $\delta_B(K, z, b) > 0$ for all $(z, b) \in Nor_B(K)$, we finally obtain

$$\frac{\partial}{\partial t}\bigg|_{t=+0} H_B(t) = \gamma \int_{\mathcal{R}_0} \Theta_{d-1}^+(K, B; \mathbb{R}^{2d})\mathbb{Q}(dK) .$$

By Theorem 3.9 and Proposition 3.10 in [8], we deduce

$$\begin{aligned}
\frac{\partial}{\partial t}\bigg|_{t=+0} H_B(t) &= \gamma \int_{\mathcal{R}_0} \int h(B, u)\Theta_{d-1}(K, d(x, u))\mathbb{Q}(dK) \\
&= \gamma \int_{\mathcal{R}_0} \int_{S^{d-1}} h(B, u)S_{d-1}(K, du)\mathbb{Q}(dK) \\
&= \int_{S^{d-1}} h(B, u)\overline{S}_{d-1}(X, du) \\
&= \gamma d \int_{\mathcal{R}_0} V(K[d-1], B)\mathbb{Q}(dK) \\
&= d\overline{V}(X[d-1], B) ,
\end{aligned}$$

which is the desired result. □

Remarks. 1. If X is non-degenerate, that is if the linear hull of the support of $\overline{S}_{d-1}(X, \cdot)$ spans $\mathbb{R}^d$, then by Minkowski's existence theorem we can find a convex body $B(X) \subset \mathbb{R}^d$ such that

$$S_{d-1}(B(X), \cdot) = \overline{S}_{d-1}(X, \cdot) .$$

The set $B(X)$ is uniquely determined up to a translation and it is called the Blaschke body of X. Hence Corollary 2.2 can be restated as

$$\frac{\partial}{\partial t}\bigg|_{t=+0} H_B(t) = dV(B(X)[d-1], B) .$$

2. Let Z be a stationary Boolean model with convex grains, and let X be the associated stationary particle process in $\mathcal{K}$. Further, let $B \subset \mathbb{R}^d$ be strictly convex and $0 \in \text{int}\, B$. Then Theorem 2.1 implies that

$$H_B(r) = 1 - \exp\left\{ - \sum_{j=0}^{d-1} \binom{d}{j} r^{d-j}\overline{V}(X[j], B[d-j]) \right\} ,$$

where

$$\overline{V}(X[j], B[d-j]) := \gamma \int_{\mathcal{K}_0} V(K[j], B[d-j])\mathbb{Q}(dK) .$$

By an approximation argument one can extend this relationship to the case where B is an arbitrary convex body with $0 \in B$. Of course, this can also be derived more directly; see [27].

Our next aim is to establish a local version of Theorem 2.1. As before, Z is a stationary Boolean model with polyconvex grains and X is the associated stationary Poisson particle process in $\mathcal{R}$. For technical reasons we also consider the stationary marked Poisson process $\tilde{X}$ in $\mathbb{R}^d \times \mathcal{R}_0$ which is transformed into X by the map $\mathbb{R}^d \times \mathcal{R}_0 \to \mathcal{R}$, $(x, K) \mapsto x + K$. We denote by $\alpha := \mathbb{E}[\tilde{X}(\cdot)]$ the intensity measure of $\tilde{X}$. Hence, by (3) we obtain that

$$\alpha(d(y, L)) = \gamma\, \mathcal{H}^d(dy)\mathbb{Q}(dL)\,.$$

For the subsequent arguments, it will be essential that we can identify the space $\mathsf{N}_e(\mathbb{R}^d \times \mathcal{R}_0)$ of all locally finite simple counting measures over $\mathbb{R}^d \times \mathcal{R}_0$ with a subset of the space of all sequences

$$\big\{(x_i, K_i)_{i\in\mathbb{N}} : (x_i, K_i) \in \mathbb{R}^d \times \mathcal{R}_0\big\}$$

via a measurable map $\eta \mapsto (x_i, K_i)_{i\in\mathbb{N}}$ such that

$$\eta = \sum_{i=1}^{\infty} \delta_{(x_i, K_i)}\,. \tag{8}$$

We refer to Lemma 3.1.7 in [24] for an explicit construction of such a map. Although the summation in (8) is formally extended over all $i \in \mathbb{N}$, this notation is meant to include the situation where the summation extends over all $i \in \{1, \ldots, \tau\}$ and τ is a random variable taking values in $\mathbb{N}$. In particular, for the marked point process $\tilde{X}$ in $\mathbb{R}^d \times \mathcal{R}_0$ we thus obtain a representation $\tilde{X} = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)}$ with random variables ξ_n, Z_n, $n \in \mathbb{N}$. Of course, similar remarks apply to simple counting measures over $\mathcal{R}$ and the particle process X.

Lemma 2.3 *Let $\tilde{X} = \sum_{n=1}^{\infty} \delta_{(\xi_n, Z_n)}$ be a stationary marked Poisson process in $\mathbb{R}^d \times \mathcal{R}_0$, and let $B \in \mathcal{K}^d$ with $0 \in B$ be given. Let $r \in (0, \infty)$ be fixed. Then, $\mathbb{P}$ almost surely,*

$$d_B(\xi_n + Z_n, 0) \neq r$$

for all $n \in \mathbb{N}$.

Proof. First, we can estimate

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{n=1}^{\infty}\{d_B(\xi_n+Z_n,0)=r\}\right)\\
&\le \mathbb{E}\left[\int \mathbf{1}\{d_B(y+L,0)=r\}\,\tilde{X}(d(y,L))\right]\\
&=\int \mathbf{1}\{d_B(y+L,0)=r\}\,\alpha(d(y,L))\\
&\le \gamma\int_{\mathcal{R}_0}\int_{\mathbb{R}^d}\mathbf{1}\{d_B(y+L,0)=r\}\,\mathcal{H}^d(dy)\mathbb{Q}(dL)\,.
\end{aligned}$$

It is sufficient to show that the inner integral vanishes for any fixed L. Suppose that $L=K_1\cup\ldots\cup K_m$ with $K_i\in\mathcal{K}$. Then $d_B(y+L,0)=r$ implies that

$$0\in y+K_{i_0}+rB\,,$$

for some $i_0\in\{1,\ldots,m\}$, but

$$0\notin y+K_{i_0}+\bar{r}B\,,$$

for all $\bar{r}\in[0,r)$. This shows that

$$y\in \operatorname{bd}(\check{K}_{i_0}+r\check{B})\,.$$

Since the union of the boundaries $\operatorname{bd}(\check{K}_i+r\check{B})$, $i=1,\ldots,m$, has $\mathcal{H}^d$ measure zero, the assertion follows. □

Fix a convex body $B\subset\mathbb{R}^d$ with $0\in B$. Then we define a measurable map $D_B:\mathsf{N}_e(\mathbb{R}^d\times\mathcal{R}_0)\to\mathbb{N}$ in the following way. Let $\eta\in\mathsf{N}_e(\mathbb{R}^d\times\mathcal{R}_0)$ be given with representation $\eta=(x_i,K_i)_{i\in\mathbb{N}}$; we write

$$K:=\bigcup_{i\in\mathbb{N}}(x_i+K_i)\,.$$

If $0<d_B(K,0)<\infty$, then we define $D_B(\eta)$ as the smallest integer $n\in\mathbb{N}$ such that

$$d_B(x_n+K_n,0)\le d_B(x_m+K_m,0)$$

for all $m\in\mathbb{N}$; if $d_B(K,0)=0$, then we define $D_B(\eta)$ as the smallest integer $n\in\mathbb{N}$ such that

$$0\in x_n+K_n\,;$$

and if $d_B(K,0)=\infty$, then we set $D_B(\eta):=1$. The last case is included in view of Theorem 3.10. Note that $0\in K+[0,\infty)B$ is satisfied, for instance, if $0\in\operatorname{int}B$. Clearly, D_B depends on the choice of the identification which was made initially, but this is immaterial. For brevity we write $\tilde{D}_B := D_B \circ \tilde{X} : \Omega \to \mathbb{N}$ which again is a measurable map.

Let $\mathcal{S}$ denote the extended convex ring in $\mathbb{R}^d$. We proceed to define a vector $u_B(K,x)$ if $K \in \mathcal{S}$, $x \in \mathbb{R}^d$, and $B \in \mathcal{K}$ is strictly convex with $0 \in B$. We distinguish two cases:

1. If $d_B(K,x) \in \{0,\infty\}$, or if $d_B(K,x) \in (0,\infty)$ and $x \in \mathrm{exo}_B(K)$, then we set $u_B(K,x) := b_0$ for some arbitrary but fixed $b_0 \in$ bd B.

2. If $d_B(K,x) \in (0,\infty)$ and $x \notin \mathrm{exo}_B(K)$, then there is a uniquely determined point $p_B(K,x) \in K$ such that

$$K \cap (x + d_B(K,x)\check{B}) = \{p_B(K,x)\}\,;$$

compare the introduction. Hence there is also a uniquely determined vector $u_B(K,x) \in \mathrm{bd}\, B$ such that

$$x = p_B(K,x) + d_B(K,x)u_B(K,x)\,.$$

Recall from [8], [10] that d_B and u_B are Borel measurable maps. Also note that in the second case of the definition for $u_B(K,x)$, the point $p_B(K,x)$ is the nearest point of K from the reference point x if the distance is measured in terms of the gauge body B; moreover, $u_B(K,x)$ has the same direction as $x - p_B(K,x)$ and it is a unit vector with respect to the gauge body B.

Let Z be a stationary random closed set in the extended convex ring, and let $B \subset \mathbb{R}^d$ be strictly convex with $0 \in \mathrm{int}\, B$; then

$$\mathbb{P}(0 \in \mathrm{exo}_B(Z)) = 0\,.$$

This follows from Theorem 3.2 in [8] by the stationarity of Z. Hence, if $\omega \in \Omega$ is such that $d_B(Z(\omega),0) \in (0,\infty)$ and $0 \notin \mathrm{exo}_B(Z(\omega))$, then

$$u_B(Z(\omega),0) = u_B(\xi_{\tilde{D}_B}(\omega) + Z_{\tilde{D}_B}(\omega),0)\,.$$

Thus, $\mathbb{P}$ almost surely, knowing $(d_B(Z,0), u_B(Z,0)) \in (0,\infty) \times \mathrm{bd}\, B$ amounts to the same as knowing $p_B(Z,0)$, which is the unique contact point of Z and $d_B(Z,0)\check{B}$.

Theorem 2.4 *Let Z be a stationary Boolean model with polyconvex grains, and let X be the associated stationary particle process. Let $B \in \mathcal{K}$ be strictly convex, $0 \in \mathrm{int}\, B$, and let $A \subset \mathbb{R}^d$ be Borel measurable. Then $H_B(\cdot, A)$ is absolutely continuous with density*

$$t \mapsto \sum_{j=0}^{d-1} \binom{d-1}{j} (1 - H_B(t)) t^{d-1-j} \overline{\Theta}_j^*(X, \mathbb{R}^d \times A \times (t,\infty])\,.$$

Proof. If $\eta \in \mathsf{N}_e(\mathbb{R}^d \times \mathcal{R}_0)$ and $y \in \mathbb{R}^d$, then we write $\eta + y = t_y(\eta)$ for the image measure of η under the map $t_y : \mathbb{R}^d \times \mathcal{R}_0 \to \mathbb{R}^d \times \mathcal{R}_0$, defined by $t_y(x, K) := (x + y, K)$. Let

$$f : \mathbb{R}^d \times \mathcal{R}_0 \times \mathsf{N}_e(\mathbb{R}^d \times \mathcal{R}_0) \to [0, \infty)$$

be Borel measurable. By the refined Campbell theorem for stationary marked point processes we obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{(x,K)\in\tilde{X}} f(x, K, \tilde{X})\right] \qquad (9)\\ &= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{R}_0} \int_{\mathsf{N}_e(\mathbb{R}^d\times\mathcal{R}_0)} f(y, K, \eta + y)\mathbb{P}^{0,K}(d\eta)\mathbb{Q}(dK)\mathcal{H}^d(dy)\,; \end{aligned}$$

see Korollar 3.4.5 in [24]. Moreover, Slivnyak's theorem for marked point processes shows that, for $\mathbb{Q}$ almost all $K \in \mathcal{R}_0$,

$$\mathbb{P}^{0,K}(\cdot) = \mathbb{P}(\tilde{X} + \delta_{(0,K)} \in \cdot)\,; \qquad (10)$$

see Satz 3.4.9 in [24]. Combining (9) and (10), we obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{(x,K)\in\tilde{X}} f(x, K, \tilde{X})\right] \qquad (11)\\ &= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{R}_0} \int_{\Omega} f(x, K, t_x(\tilde{X}(\omega) + \delta_{(0,K)}))\mathbb{P}(d\omega)\mathbb{Q}(dK)\mathcal{H}^d(dx)\,. \end{aligned}$$

We fix a Borel measurable set $A \subset \mathbb{R}^d$, $r > 0$ and define

$$\begin{aligned} f(x, K, \eta) := \mathbf{1}\{&0 < d_B(x + K, 0) \le r, u_B(x + K, 0) \in A,\\ &(x, K) = (x_{D_B(\eta)}, K_{D_B(\eta)})\}\,, \end{aligned}$$

where $x \in \mathbb{R}^d$, $K \in \mathcal{R}_0$, and $\eta \in \mathsf{N}_e(\mathbb{R}^d \times \mathcal{R}_0)$. By (11) and the discussion preceding the statement of the theorem, we find

$$\begin{aligned} &(1 - \bar{p}(0))H_B(r, A)\\ &= \mathbb{P}(0 < d_B(Z, 0) \le r, u_B(Z, 0) \in A)\\ &= \mathbb{E}\left[\sum_{(x,K)\in\tilde{X}} f(x, K, \tilde{X})\right]\\ &= \gamma \int_{\mathcal{R}_0} \int_{\mathbb{R}^d} \int_{\Omega} f(-x, K, t_{-x}\tilde{X}(\omega) + \delta_{(-x,K)})\mathbb{P}(d\omega)\mathcal{H}^d(dx)\mathbb{Q}(dK)\,. \end{aligned}$$

Moreover, by Lemma 2.3 and writing $\tilde{X} = (\xi_i, Z_i)_{i\in\mathbb{N}}$, we see that, $\mathbb{P}$ almost surely,

$$\begin{aligned} &f(-x, K, t_{-x}\tilde{X} + \delta_{(-x,K)}) \\ &= \mathbf{1}\{0 < d_B(-x+K,0) \le r, u_B(-x+K,0) \in A, \\ &\qquad d_B(-x+K,0) < d_B(-x+\xi_i+Z_i,0) \text{ for } i \in \mathbb{N}\} \\ &= \mathbf{1}\{0 < d_B(K,x) \le r, u_B(K,x) \in A, d_B(K,x) < d_B(Z,x)\}\,, \end{aligned}$$

for $x \in \mathbb{R}^d$ and $K \in \mathcal{R}_0$. This shows that

$$\begin{aligned} H_B(r,A) = \gamma \int_{\mathcal{R}_0} \int_{\mathbb{R}^d} &(1 - H_B(d_B(K,x))) \\ &\times \mathbf{1}\{0 < d_B(K,x) \le r, u_B(K,x) \in A\} \mathcal{H}^d(dx)\mathbb{Q}(dK)\,. \end{aligned} \tag{12}$$

For the final step we use Theorem 3.3 in [8] to obtain

$$\begin{aligned} H_B(r,A) = \gamma \int_{\mathcal{R}_0} \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int &\mathbf{1}\{\delta_B(K,z,b) > s\} s^{d-1-j} \\ &\times (1 - H_B(s)) \mathbf{1}\{b \in A\} \Theta_j^+(K, d(z,b)) ds \mathbb{Q}(dK)\,, \end{aligned}$$

which implies the desired result. □

Corollary 2.2 admits a local version as well.

Corollary 2.5 *Let the assumptions of Theorem 2.4 and condition* (5) *be satisfied. Then*

$$\frac{\partial}{\partial t}\bigg|_{t=+0} H_B(t,A) = \int_{(\nabla h_B)^{-1}(A)} h(B,u)\overline{S}_{d-1}(X, du)\,.$$

For a stationary Boolean model with convex grains the situation is much simpler. In this case the particles of X are convex and we define

$$d\overline{V}(X[i], B[d-i]) := \gamma \int_{\mathcal{K}_0} V(K[i], B[d-i]) \mathbb{Q}(dK)\,,$$

for $i \in \{0, \dots, d\}$. If one wishes to define such mean functionals (densities) for a stationary particle process with particles in $\mathcal{R}$, then one has to impose an additional integrability assumption.

Theorem 2.6 *Let Z be a stationary Boolean model with convex grains, and let X be the associated stationary particle process. Let $B \in \mathcal{K}$ be strictly convex, $0 \in \operatorname{int} B$, and let*

$g : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ *be Borel measurable. Then*

$$\begin{aligned}&\mathbb{E}[g(d_B(Z,0), u_B(Z,0))|0 \notin Z]\\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} \int \int_0^\infty g(s,b) s^{d-1-j}\\ &\quad \times \exp\left\{ -\sum_{i=0}^{d-1} \binom{d}{i} s^{d-i} \overline{V}(X[i], B[d-i]) \right\} ds \overline{S}_j(X,B;db)\,.\end{aligned}$$

3 Consequences and extensions

In this section, we discuss some results which are directly related to Theorems 2.4 and 2.6. The first concerns the estimation of the mean surface area measures of the underlying stationary particle process X of a stationary Boolean model Z with convex grains. If Z has polyconvex grains, then we can still estimate $\overline{V}(X[d-1], B)$, for a large class of test bodies B. Thus we find that the mean surface area measure of order $d-1$ of X can be determined by measurements of the union set Z of X for a single realization. Moreover, we compare this result to an approach which was recently developed by Rataj [18]. Another objective is to extend Theorem 2.6 so that structuring elements which lie in a subspace are admitted; the result obtained includes, as a very special case, the linear contact distribution functions.

Let X be a stationary Poisson particle process whose union thus is a stationary Boolean model Z. In practical situations, we can only observe directly the union set and not the individual particles which may overlap. It is a surprising consequence of the Poisson assumption that still all information about the mean surface area measure $\overline{S}_{d-1}(X,\cdot)$ of X can be retrieved, at least in principle, from knowledge of the mean surface area measure $\overline{S}_{d-1}(Z,\cdot)$ of Z; the latter is a non-negative measure which is accessible to direct measurements (compare [27]). However, it is also desirable to determine $\overline{S}_{d-1}(X,\cdot)$, and similarly the remaining lower-dimensional mean surface area measures, by measurements of volumes; this was suggested, for instance, by recent work of Rataj [18]. The local contact distribution function provides an appropriate tool for such purposes. In fact, Theorem 2.6 yields that

$$H_B(r,A) = \sum_{j=0}^{d-1} \binom{d-1}{j} c_{d-1-j}(r) \overline{S}_j(X,B;A)\,,$$

where

$$c_{d-1-j}(r) = \int_0^r t^{d-1-j}(1 - H_B(t))dt\,, \tag{13}$$

for $j \in \{0,\dots,d-1\}$, $r \geq 0$ and $A \subset \mathbb{R}^d$ is Borel measurable. It is easy to check that the functions $c_0,\dots,c_{d-1}$ are linearly independent, and therefore the values $\overline{S}_j(X,B;\cdot)$ are completely determined by the function $H_B(\cdot, A)$. For example, for Borel sets $A \subset S^{d-1}$ we obtain

$$\begin{aligned}\overline{S}_{d-1}(X, B; A) &= \left.\frac{\partial}{\partial t}\right|_{t=+0} H_B(t, A)\,, \\ (d-1)\overline{S}_{d-2}(X, B; A) &= \left.\frac{\partial^2}{\partial t^2}\right|_{t=+0} H_B(t, A) \\ &\quad + \left.\frac{\partial}{\partial t}\right|_{t=+0} H_B(t, A)\, \left.\frac{\partial}{\partial t}\right|_{t=+0} H_B(t)\,, \end{aligned} \tag{14}$$

and the description of the measures $\overline{S}_{d-j}(X, B; A)$, $j \in \{3, \ldots, d\}$, in terms of $H_B(t, A)$ requires derivatives up to order j. On the other hand, by stationarity we have

$$H_B(t, A) = \frac{\mathbb{E}\left[\mathcal{H}^d([(Z + tB) \setminus Z] \cap W \cap u_B(Z, \cdot)^{-1}(A))\right]}{\mathbb{E}\left[\mathcal{H}^d(W \setminus Z)\right]}\,,$$

where $W \subset \mathbb{R}^d$ is any Borel set with positive volume. Thus for the contact distribution function we obtain a ratio of two estimators each of which is unbiased and which is based on measurements of Z. (For the ease of the presentation we neglect edge effects.)

In order to obtain the local result in (14), and similarly for the remaining mean surface area measures, we essentially used that the grains are convex. For the global quantity $\overline{V}(X[d-1], B)$, however, we obtain a more general result. In fact, let Z be a stationary Boolean model with polyconvex grains and with associated stationary particle process X in $\mathcal{R}$. Let $B \in \mathcal{K}$ with $0 \in \operatorname{int} B$. It is known that Z is mixing; see §5.2 in [24] and in particular Satz 5.2.6. Moreover, $Z + tB$ is also a random closed set and mixing; this follows from Satz 5.2.5 in [24] or, alternatively, since $Z + tB$ is a stationary Boolean model as well. In particular, Z and $Z + tB$ are ergodic random closed sets, and therefore

$$H_B(t) = \lim_{r \to \infty} \frac{\mathcal{H}^d([(Z + tB) \setminus Z] \cap rW)}{\mathcal{H}^d(rW \setminus Z)}$$

is satisfied $\mathbb{P}$ almost surely, where $W \in \mathcal{K}$ and $\mathcal{H}^d(W) > 0$. Furthermore, by Corollary 2.2 we obtain the following corollary.

Corollary 3.1 *Let Z be a stationary Boolean model with polyconvex grains, let X be the associated stationary particle process in $\mathcal{R}$, and assume that (5) is satisfied. Let $B \in \mathcal{K}$ be strictly convex and $0 \in \operatorname{int} B$. Further, let $t_i \in (0, \infty)$, $i \in \mathbb{N}$, with $t_i \downarrow 0$ as $i \to \infty$. Then, $\mathbb{P}$ almost surely,*

$$d\overline{V}(X[d-1], B) = \lim_{i \to \infty} \lim_{r \to \infty} \frac{\mathcal{H}^d([(Z + t_i B) \setminus Z] \cap rW)}{t_i\, \mathcal{H}^d(rW \setminus Z)}\,,$$

for any $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$.

Remark. If Z is a stationary Boolean model with convex grains, then the same conclusion can be obtained for any $B \in \mathcal{K}$ with $0 \in B$.

In [18], a related representation was established. There, however, the grains were compact sets of positive reach instead of polyconvex sets and the order of the limits appears in the reversed order. It seems to be appropriate to discuss Rataj's Theorem 6.1 and Corollary 6.1 in [18] in the present framework. We subdivide our analysis into three parts.

Part 1. Following basically the approach in [24] (see also [27]), we first consider the densities of the surface area measures of a stationary random closed set Z in the extended convex ring. Let $C^d := [0,1]^d$ be the d-dimensional unit cube and $\partial^+ C^d$ its upper right boundary (that is, the set of all $(x_1, \dots, x_d) \in C^d$ such that $x_i = 1$ for some $i \in \{1, \dots, d\}$). Let $B \in \mathcal{K}$ be strictly convex with $0 \in \operatorname{int} B$, and let $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$. Then

$$\begin{aligned}\overline{S}_j(Z, B; \cdot) :=& \lim_{r \to \infty} \frac{\mathbb{E}\left[S_j(Z \cap rW, B; \cdot)\right]}{\mathcal{H}^d(rW)} \\ =& \mathbb{E}\left[S_j(Z \cap C^d, B; \cdot) - S_j(Z \cap \partial^+ C^d, B; \cdot)\right] ,\end{aligned}$$

$j \in \{0, \dots, d-1\}$, provided that

$$\mathbb{E}\left[2^{N(Z \cap C^d)}\right] < \infty . \tag{15}$$

This assertion follows, e.g., from Satz 5.1.3 in [24] by setting $\varphi(K) := S_j(K, B; A)$ for $K \in \mathcal{R}$ and a measurable set $A \subset \mathbb{R}^d$; more directly, one can apply Lemma 5.1.2 in [24] to $\overline{\varphi}(K) := \mathbb{E}\left[S_j(Z \cap K, B; A)\right]$. Since we shall be interested mainly in the case $j = d-1$, we define

$$S_{d-1}(K, B; \mathbb{R}^d) := \int_{S^{d-1}} h(B, u) S_{d-1}(K, du) ,$$

where $K \in \mathcal{R}$ and $B \in \mathcal{K}$ with $0 \in B$. Of course, this expression equals $dV(K[d-1], B)$. Also note that this definition is consistent with the previous ones; see [9]. Moreover, for $i = d-1$ we can replace (15) by the much weaker condition

$$\mathbb{E}\left[N(Z \cap C^d)\right] < \infty . \tag{16}$$

To see this, define

$$\overline{\varphi}(K) := \mathbb{E}\left[S_{d-1}(Z \cap K, B; A)\right] ,$$

for $K \in \mathcal{R}$ and a fixed Borel set $A \subset \mathbb{R}^d$. Then $\overline{\varphi}$ is translation invariant, additive and locally bounded (that is bounded on convex bodies which lie in C^d). For the proof of the last assertion, let $K \in \mathcal{K}$ and $K \subset C^d$. Since Z lies in the extended convex ring (for each realization), there exists a representation

$$Z \cap K = K_1 \cup \ldots \cup K_N$$

with $K_i \in \mathcal{K}$ and $N = N(Z \cap K)$. Therefore,

$$
\begin{aligned}
S_{d-1}(Z \cap K, B; A) &= S_{d-1}\left(K_1 \cup \ldots \cup K_N, B; A\right) \\
&\leq c_1\, \Theta_{d-1}(K_1 \cup \ldots \cup K_N, \mathbb{R}^{2d})\,,
\end{aligned}
$$

where c_1 is a constant which merely depends on B. In the last step we used that Θ_{d-1} is non-negative. As in the proof of Theorem 2.2 in [27] it follows that

$$
\Theta_{d-1}(K_1^\rho \cup \ldots \cup K_N^\rho, \mathbb{R}^{2d}) \to \Theta_{d-1}(K_1 \cup \ldots \cup K_N, \mathbb{R}^{2d})\,,
$$

where $\rho \downarrow 0$ and $K_i^\rho := K_i + \rho B^d$. Moreover, since $K_1^\rho \cup \ldots \cup K_N^\rho$ is normal (see [27]), we deduce

$$
\begin{aligned}
\Theta_{d-1}(K_1^\rho \cup \ldots \cup K_N^\rho, \mathbb{R}^{2d}) &\leq \mathcal{H}^{d-1}(\mathrm{bd}(K_1^\rho \cup \ldots \cup K_N^\rho)) \\
&\leq \sum_{i=1}^{N} \mathcal{H}^{d-1}(\mathrm{bd}(K_i^\rho)) \\
&\leq N\, \mathcal{H}^{d-1}(\mathrm{bd}\ (C^d)^\rho)\,,
\end{aligned}
$$

since $K_i \subset K \subset C^d$. But this holds for any $\rho > 0$, and thus

$$
S_{d-1}(Z \cap K, B; A) \leq c_1\, \mathcal{H}^{d-1}(\mathrm{bd}\ C^d)\, N =: c\, N\,.
$$

This finally yields that

$$
|\overline{\varphi}(K)| \leq c\, \mathbb{E}[N(Z \cap K)] \leq c\, \mathbb{E}\big[N(Z \cap C^d)\big] < \infty
$$

for all $K \subset C^d$.

Part 2. A formally different definition for mean surface area measures of certain stationary random closed sets is given in [18]. Let Z be a stationary random closed set in the extended convex ring, and let $B \in \mathcal{K}$ be strictly convex with $0 \in \operatorname{int} B$. Under these assumptions the random measure $\Theta_{d-1}(Z, B; \cdot)$ is well-defined and non-negative. Furthermore, for any $B \in \mathcal{K}$ with $0 \in B$ we set

$$
\Theta_{d-1}(Z, B; \alpha \times \mathbb{R}^d) := \int \mathbf{1}\{x \in \alpha\} h(B, u) \Theta_{d-1}(Z, d(x, u))\,,
$$

which also defines a non-negative measure with respect to its dependence on Borel sets $\alpha \subset \mathbb{R}^d$. Under the additional condition (15) it follows that

$$
\tilde{S}_j(Z, B; \cdot) := \lim_{r \to \infty} \frac{\mathbb{E}\left[\Theta_j(Z, B; rW \times \cdot)\right]}{\mathcal{H}^d(rW)}\,,
$$

$j \in \{0, \ldots, d-2\}$, is a deterministic signed measure over $\mathbb{R}^d$ which is independent of the particular choice of $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$, and for $j = d-1$ the same conclusion is obtained under the assumption (16). In fact, it follows that

$$\overline{S}_j(Z, B; \cdot) = \tilde{S}_j(Z, B; \cdot)\,, \tag{17}$$

for $j \in \{0, \ldots, d-1\}$. In order to prove this, one can essentially repeat the proof of Lemma 5.1.2 in [24], starting with

$$\mathbb{E}\left[\Theta_j(Z, B; rW \times A)\right] = \sum_{z \in \mathbb{Z}^d} \left\{\mathbb{E}\left[\Theta_j(Z \cap (C^d + z), B; rW \times A)\right.\right.$$
$$\left.\left. -\Theta_j(Z \cap (\partial^+ C^d + z), B; rW \times A)\right]\right\}\,,$$

where $A \subset \mathbb{R}^d$ is a fixed measurable set, $r > 0$, and $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$ is arbitrary; see also [33], Theorem 3.2.4. In the special case $j = d-1$ more can be said. Indeed, $\mathbb{E}[\Theta_{d-1}(Z, B; \cdot \times A)]$ defines a locally finite, translation invariant, non-negative Borel measure over $\mathbb{R}^d$, which hence is a multiple $\lambda_A \mathcal{H}^d(\cdot)$ of Lebesgue measure, for any fixed Borel set $A \subset \mathbb{R}^d$. Therefore we can conclude that

$$\lambda_A = \frac{\mathbb{E}\left[\Theta_{d-1}(Z, B; W \times A)\right]}{\mathcal{H}^d(W)} = \tilde{S}_{d-1}(Z, B; A) = \overline{S}_{d-1}(Z, B; A)\,,$$

for any Borel set $A \subset \mathbb{R}^d$ and an arbitrarily chosen $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$. Thus, in particular, we have proved the following theorem.

Theorem 3.2 *Let Z be a stationary random closed set in the extended convex ring, let $B \in \mathcal{K}$ be strictly convex with $0 \in \operatorname{int} B$. Assume that condition* (16) *is satisfied. Then*

$$\mathbb{E}[\Theta_{d-1}(Z, B; \cdot)] = \mathcal{H}^d(\cdot) \otimes \overline{S}_{d-1}(Z, B; \cdot)\,.$$

For the special case $B = B^d$ and under a stronger integrability assumption, this result was announced as Theorem 6 in [28].

Part 3. We assume that Z is a stationary random closed set in the extended convex ring which satisfies (16) and is ergodic. Let $B \in \mathcal{K}$ with $0 \in B$. Further, we define

$$\xi_K(Z) := \Theta_{d-1}(Z, B; K \times \mathbb{R}^d)\,, \quad K \in \mathcal{R}\,.$$

Then $(\xi_K)_{K \in \mathcal{R}}$ is a spatial process which is additive, $\mathcal{T}$-covariant and locally bounded; see [24] for the terminology. Let $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$ be given. Using Part 1 and Part 2, the assumption that Z is ergodic and applying Satz 5.2.1 in [24] to $\xi_K(Z)$, we deduce

$$\begin{aligned}
d\overline{V}(Z[d-1],B) &:= \int_{S^{d-1}} h(B,u)\overline{S}_{d-1}(Z,du) \\
&= \overline{S}_{d-1}(Z,B;\mathbb{R}^d) \\
&= \lim_{r\to\infty} \frac{\mathbb{E}\left[\Theta_{d-1}(Z,B;rW\times\mathbb{R}^d)\right]}{\mathcal{H}^d(rW)} \\
&= \lim_{r\to\infty} \frac{1}{\mathcal{H}^d(rW)} \int \mathbf{1}\{x\in rW\} h(B,u)\Theta_{d-1}(Z,d(x,u))\,,
\end{aligned} \tag{18}$$

$\mathbb{P}$ almost surely.

In order to proceed, we need the following theorem. A special case of it was proved in [8], a related version was obtained by Rataj [18]. The present result is obtained by combining the methods of these two contributions and by adapting an idea of R. Schneider.

Theorem 3.3 *Let $K\subset\mathbb{R}^d$ be in the extended convex ring, let $B\in\mathcal{K}$ with $0\in\operatorname{relint} B$, and let $C\subset\mathbb{R}^d$ be bounded, measurable and such that $C_{d-1}(K,\operatorname{bd} C)=0$. Then*

$$\lim_{t\downarrow 0} \frac{\mathcal{H}^d([(K+tB)\setminus K]\cap C)}{t} = \int \mathbf{1}\{x\in C\} h(B,u)\Theta_{d-1}(K,d(x,u))\,.$$

The restriction to sets of the extended convex ring thus has the advantage of admitting the statement of a common generalization (in the present setting) of Theorems 4.1, 4.2 and 4.3 in [18]. In particular, we do not have to adopt any assumption concerning the existence of a suitable representation of K as a union of compact convex sets for which the mutual intersections of the boundaries have $(d-1)$-dimensional Hausdorff measure zero.

We postpone the proof of Theorem 3.3 for a moment in order to continue the argument.

By an application of Fubini's theorem we find a measurable set $\mathbb{R}^*\subset[0,\infty)$ with $\mathcal{H}^1(\mathbb{R}^*)=0$ and such that if $r\in[0,\infty)\setminus\mathbb{R}^*$, then $C_{d-1}(Z(\omega),\partial rW)=0$ for $\mathbb{P}$ almost all $\omega\in\Omega$ (the set of $\mathbb{P}$ measure zero which has to be excluded may depend on r). Thus, combining (18) and Theorem 3.3, we obtain the following theorem.

Theorem 3.4 *Let $Z\subset\mathbb{R}^d$ be an ergodic stationary random closed set in the extended convex ring which satisfies (16). Let $B\in\mathcal{K}$ with $0\in\operatorname{relint} B$, and let $W\in\mathcal{K}$ with $\mathcal{H}^d(W)>0$. Further, let $r_i\in[0,\infty)\setminus\mathbb{R}^*$, $i\in\mathbb{N}$, with $r_i\to\infty$ as $i\to\infty$. Then, $\mathbb{P}$ almost surely,*

$$d\overline{V}(Z[d-1],B) = \lim_{i\to\infty}\lim_{\epsilon\downarrow 0} \frac{\mathcal{H}^d([(Z+\epsilon B)\setminus Z]\cap r_iW)}{\epsilon\,\mathcal{H}^d(r_iW)}\,.$$

We can apply this result to a stationary Boolean model Z with underlying stationary particle process X by using Theorem 4.1 in [27] to obtain the next corollary.

Corollary 3.5 *Let Z be a stationary Boolean model with convex grains, and let X be the associated stationary particle process. Let $B \in \mathcal{K}$ with $0 \in \text{relint } B$, and let $W \in \mathcal{K}$ with $\mathcal{H}^d(W) > 0$. Further, let $r_i \in [0,\infty) \setminus \mathbb{R}^*$, $i \in \mathbb{N}$, with $r_i \to \infty$ as $i \to \infty$. Then, $\mathbb{P}$ almost surely,*

$$d\overline{V}(X[d-1], B) = \lim_{i\to\infty} \lim_{\epsilon\downarrow 0} \frac{\mathcal{H}^d([(Z+\epsilon B)\setminus Z] \cap r_i W)}{\epsilon\, \mathcal{H}^d(r_i W \setminus Z)} .$$

This should be carefully compared to Corollary 3.1 and the subsequent remark.

Now we provide the proof for Theorem 3.3.

Proof of Theorem 3.3. The proof is divided into three steps. First, we remark that the assertion was proved in [8], Remark 4.8, for strictly convex $B \in \mathcal{K}$ with $0 \in \text{int } B$.

(i) The assertion is true if $\dim B = d$.

Choose a sequence $B_n \in \mathcal{K}$, $n \in \mathbb{N}$, with B_n strictly convex, $0 \in \text{int } B_n$, and $B_n \downarrow B$ as $n \to \infty$. Then

$$\begin{aligned} &\overline{\lim_{t\downarrow 0}} \frac{\mathcal{H}^d([(K+tB)\setminus K]\cap C)}{t} \\ &\quad \leq \lim_{t\downarrow 0} \frac{\mathcal{H}^d([(K+tB_n)\setminus K]\cap C)}{t} \\ &\quad = \int \mathbf{1}\{x \in C\} h(B_n, u)\Theta_{d-1}(K, d(x,u)) \\ &\quad \to \int \mathbf{1}\{x \in C\} h(B, u)\Theta_{d-1}(K, d(x,u)) \end{aligned}$$

as $n \to \infty$, and hence

$$\begin{aligned} &\overline{\lim_{t\downarrow 0}} \frac{\mathcal{H}^d([(K+tB)\setminus K]\cap C)}{t} \\ &\quad \leq \int \mathbf{1}\{x \in C\} h(B, u)\Theta_{d-1}(K, d(x,u)) . \end{aligned} \tag{19}$$

In fact, (19) thus follows for all $B \in \mathcal{K}$ with $0 \in B$.

Similarly, choose a sequence $\bar{B}_n \in \mathcal{K}$, $n \in \mathbb{N}$, with $\bar{B}_n$ strictly convex, $0 \in \text{int } \bar{B}_n$, and $\bar{B}_n \uparrow B$ as $n \to \infty$. Then we obtain by an analogous argument as before that

$$\underline{\lim_{t\downarrow 0}} \frac{\mathcal{H}^d([(K+tB)\setminus K]\cap C)}{t} \tag{20}$$

$$\geq \int \mathbf{1}\{x \in C\} h(B, u)\Theta_{d-1}(K, d(x,u)) . \tag{21}$$

The assertion follows from (19) and (20).

(ii) The assertion is true if $j := \dim B \in \{2, \dots, d-1\}$.

Let L be an arbitrary j-dimensional linear subspace of $\mathbb{R}^d$. Then one can apply the result of (i) to $K \cap (L+y)$ and $C \cap (L+y)$, for $\mathcal{H}^{d-j}$ almost all $y \in L^\perp$, use a translative Crofton formula (see Corollary 3.3 in [19]) and Fubini's theorem; compare Case 2 of the proof of Theorem 4.3 in [18].

(iii) The assertion is true if $\dim B = 1$.

Suppose that $K = K_1 \cup \ldots \cup K_n$ with $K_i \in \mathcal{K}$ (this is possible since C is bounded). For the proof, we can assume that $C \cap \operatorname{int} K = \emptyset$. (Here we use that $\operatorname{bd} K \cap \operatorname{bd}(C \setminus \operatorname{int} K) \subset \operatorname{bd} C$.) Then

$$\begin{aligned} &\mathcal{H}^d([(K+tB)\setminus K]\cap C) \qquad (22)\\ &= \sum_{v\in S(n)} (-1)^{|v|} \left[\mathcal{H}^d\left(\bigcap_{i\in v}(K_i+tB)\cap C\right) - \mathcal{H}^d(K_v\cap C)\right] . \end{aligned}$$

We show that, for $v \in S(n)$,

$$\begin{aligned} &\mathcal{H}^d([(K_v+tB)\setminus K_v]\cap C) \qquad (23)\\ &= t\int \mathbf{1}\{x\in C\} h(B,u)\Theta_{d-1}(K_v, d(x,u)) + o(t)\,, \end{aligned}$$

as $t \downarrow 0$, and hence

$$\begin{aligned} &\sum_{v\in S(n)} (-1)^{|v|}\mathcal{H}^d([(K_v+tB)\setminus K_v]\cap C) \qquad (24)\\ &= t\int \mathbf{1}\{x\in C\} h(B,u)\Theta_{d-1}(K, d(x,u)) + o(t)\,. \end{aligned}$$

Moreover, for $v \in S(n)$, we prove that

$$\mathcal{H}^d\left(\left(\left[\bigcap_{i\in v}(K_i+tB)\right] \setminus (K_v+tB)\right)\cap C\right) = o(t)\,, \qquad (25)$$

as $t \downarrow 0$.

The assertion then follows from (22), (24) and (25). For the clarity of the exposition, we subsequently state three lemmas which together complete the proof of the theorem. □

Lemma 3.6 *Let $K = K_1 \cup \ldots \cup K_n$ with $K_i \in \mathcal{K}$, let $C \subset \mathbb{R}^d$ be measurable and such that $C \cap \operatorname{int} K = \emptyset$. Furthermore, assume that $C_{d-1}(K, \operatorname{bd} C) = 0$. Then $C_{d-1}(K_v, \operatorname{bd} C) = 0$ for all $v \in S(n)$.*

Proof. We fix $v \in S(n)$ and define

$$G := \left[(\text{bd } C \cap \text{bd } K_v) \times S^{d-1}\right] \cap Nor(K) ;$$

then by Theorem 3.9 and Lemma 3.1 in [8] we obtain

$$\begin{aligned} 0 = C_{d-1}(K, \text{bd } C) &= \Theta_{d-1}(K, \text{bd } C \times S^{d-1}) \\ &= \Theta_{d-1}(K, (\text{bd } C \times S^{d-1}) \cap Nor(K)) \\ &= \Theta_{d-1}(K, G) = \Theta_{d-1}(K_v, G) . \end{aligned}$$

For $L \in \mathcal{K}$ and $x \in L$, we write $N(L, x)$ for the normal cone of L at x, and reg L denotes the set of regular boundary points of L, that is the set of all $x \in \text{bd } L$ for which $\dim N(L, x) = 1$ (compare [21]). We define

$$R := \bigcup_{v \in S(n)} (\text{bd } K_v \setminus \text{reg } K_v) \cap \text{bd } K ,$$

and hence $\mathcal{H}^{d-1}(R) = 0$. Therefore it remains to prove that $C_{d-1}(K_v, (\text{bd } C) \setminus R) = 0$. Choose any $x \in \text{bd } C \cap (\text{bd } K_v) \setminus R$ and $(x, u) \in Nor(K_v)$. Then for all $m \in \{1, \dots, n\}$ with $x \in \text{bd } K_m$, we have $N(K_m, x) \subset \text{lin}\{u\}$, since $x \notin R$. If $(x, u) \notin Nor(K)$, then there is some $i \in \{1, \dots, n\}$ with $x \in \text{bd } K_i$ and $K_i \not\subset H^- := \{y \in \mathbb{R}^d : \langle y - x, u\rangle \leq 0\}$. Set $H := \text{bd } H^-$ and $H^+ := \text{clos}(\mathbb{R}^d \setminus H^-)$. Now we have $x \in (K_i \cap K_v) \setminus R$ and therefore $x \in \text{int}_H(K_i \cap H)$, $K_v \subset H$, and $x \in \text{int}_H K_v$. But then $K_j \subset H^+$ for all $j \in \{1, \dots, n\}$ such that $x \in \text{bd } K_j$, since $x \in (\text{bd } K) \setminus R$. (Note that $x \in \text{bd } K_v \cap \text{bd } C \subset K \setminus \text{int } K = \text{bd } K$.) This yields that $(x, -u) \in Nor(K)$. Therefore in any case

$$\begin{aligned} 0 &= 2\Theta_{d-1}(K_v, [((\text{bd } C) \setminus R) \times S^{d-1}] \cap Nor(K)) \\ &\geq C_{d-1}(K_v, (\text{bd } C) \setminus R) \geq 0 . \end{aligned}$$

This proves the lemma. □

Lemma 3.7 *Let $L \in \mathcal{K}$, let B be a non-degenerate segment, and let $C \subset \mathbb{R}^d$ be measurable and such that $C_{d-1}(L, \text{bd } C) = 0$. Then*

$$\lim_{t \downarrow 0} \frac{\mathcal{H}^d([(L + tB) \setminus L] \cap C)}{t} = \int \mathbf{1}\{x \in C\} h(B, u) \Theta_{d-1}(L, d(x, u)) .$$

Proof. It is sufficient to consider the case where $\dim L \geq d - 1$. We can write $B := [-\alpha e_1, \beta e_1]$ with $\alpha, \beta > 0$ and define $T : \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}^d$ by $T(x, u, \lambda) := x + \lambda e_1$. We also set

$$h_t(x, u, \lambda) := \mathbf{1}\{\langle u, e_1\rangle \neq 0\} \mathbf{1}\left\{x + \lambda \frac{\langle u, e_1\rangle}{|\langle u, e_1\rangle|} e_1 \in (K + tB) \cap C\right\}$$

for $(x,u,\lambda) \in \mathbb{R}^{2d} \times \mathbb{R}$ and $t > 0$. Then the coarea formula yields

$$
\begin{aligned}
&\mathcal{H}^d([(L+tB)\setminus L]\cap C)\\
&= \int_{\mathbb{R}^d} \int_{T^{-1}(\{z\})\cap(Nor(L)\times(0,\infty))} h_t(x,u,\lambda)\mathcal{H}^0(d(x,u,\lambda))\mathcal{H}^d(dz)\\
&= \int_{Nor(L)\times(0,\infty)} \mathrm{ap}J_dT(x,u,\lambda)h_t(x,u,\lambda)\mathcal{H}^d(d(x,u,\lambda))\\
&= \int_{Nor(L)} \int_0^\infty |\langle u,e_1\rangle| h_t(x,u,\lambda)d\lambda\Theta_{d-1}(L,d(x,u))\,;
\end{aligned}
$$

see, e.g., [7] for similar calculations of Jacobians. It is easy to check that, for $(x,u) \in Nor(L)$,

$$
\frac{|\langle x,u\rangle|}{t}\int_0^\infty h_t(x,u,\lambda)d\lambda \to \begin{cases} h(B,u)\,, & x \in \operatorname{int} C\,,\\ 0\,, & x \notin \operatorname{clos} C\,. \end{cases}
$$

Moreover, for all $(x,u) \in Nor(L)$,

$$
\frac{|\langle x,u\rangle|}{t}\int_0^\infty h_t(x,u,\lambda)d\lambda \le h(B,u)\,.
$$

Since $\Theta_{d-1}(L,(\mathrm{bd}\, C \times S^{d-1}) \cap Nor(L)) = 0$ by assumption, the assertion is implied by the dominated convergence theorem. $\square$

Hence Lemmas 3.6 and 3.7 together yield (23). The following lemma is essentially due to R. Schneider.

Lemma 3.8 *Let $K_1,\dots,K_n \in \mathcal{K}$, let $B = [-\alpha e_1, \beta e_1]$, $\alpha,\beta > 0$, and let $t > 0$. Then*

$$
\mathcal{H}^d\left(\left[\bigcap_{i=1}^n (K_i+tB)\right]\setminus\left(\bigcap_{i=1}^n K_i + tB\right)\right) = o(t)
$$

as $t \downarrow 0$.

Proof. We set

$$
M(t) := \left[\bigcap_{i=1}^n (K_i+tB)\right]\setminus\left(\bigcap_{i=1}^n K_i + tB\right).
$$

Let $x \in M(t)$ and $v := \{1,\dots,n\}$; then $x \notin K_v$, since $0 \in B$, and therefore $x \notin K_k$, for some $k \in v$, but $x \in K_k + tB$. We set

$$
d_B(K_k,x) := \min\{\lambda \ge 0 : x \in K_k + \lambda B\} \in (0,t]\,,
$$

and hence we obtain a uniquely defined point

$$\{y_k\} := K_k \cap (x + d_B(K_k, x)\check{B}) \in \text{bd } K_k .$$

Suppose that $y_k \in \cap_{i\neq k} K_i$ and thus $y_k \in K_v$; hence $x \in K_v + tB$, a contradiction. This shows that there is some $m \in v \setminus \{k\}$ such that $y_k \notin K_m$. Since $x \in K_m + tB$, there is some $y_m \in K_m$ such that $x - y_m \in tB$. In addition, $y_k - x \in t\check{B} \subset c_1 tB$, where c_1 is a constant which is independent of t. We deduce that $y_k - y_m \in c_2 tB$, and hence

$$y_k \in [(K_m + c_2 tB) \setminus K_m] \cap \text{bd } K_k =: \beta_{k,m}(t) .$$

(Note that c_2, and also c_3 below, is independent of t.) From this we can conclude that

$$\mathcal{H}^d(M(t)) \leq \sum_{\substack{k,m=1 \\ k\neq m}}^{n} c_3\, \mathcal{H}^{d-1}(\beta_{k,m}(t))\, t .$$

Since $\beta_{k,m}(t) \downarrow \emptyset$ as $t \downarrow 0$ and $\beta_{k,m}(t) \subset \text{bd } K_k$, we obtain

$$\mathcal{H}^{d-1}(\beta_{k,m}(t)) \downarrow 0$$

as $t \downarrow 0$, for all $k, m \in v$ with $k \neq m$. □

We now turn our attention to the second subject of this section. In the first part of the proof of Theorem 2.6 the strict convexity of B was not used. This assumption was only relevant for the application of the local Steiner formula in Minkowski spaces. Therefore we now extend this formula in a suitable way, which will enable us to treat lower-dimensional structuring elements also in a stochastic setting.

Theorem 3.9 *Let $K, B \in \mathcal{K}^d$, and let $L \subset \mathbb{R}^d$ be a k-dimensional linear subspace, $1 \leq k \leq d$, which contains B. Assume that with respect to L, $B \in \mathcal{K}$ is strictly convex and contains 0 in its interior. Let $g : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ be Borel measurable. Then*

$$\begin{aligned} &\int_{\mathbb{R}^d\setminus K} g(d_B(K,x), u_B(K,x))\mathcal{H}^d(dx) \\ &= \sum_{j=0}^{k-1} \binom{d-1}{j} \int_{S^{d-1}\setminus L^\perp} \int_0^\infty g(s, \nabla h_B(u)) s^j\, ds \\ &\qquad\qquad \times\ h(B,u) S(K[d-1-j], B[j], du) . \end{aligned}$$

Remarks. 1. The integral on the left-hand side of the asserted equation effectively extends only over the set $(K + L) \setminus K$. Here we use the convention that $g(d_B(K,x), u_B(K,x)) = 0$ if $d_B(K,x) = \infty$, although $u_B(K,x)$ is not defined then; this is consistent with the case where g is an indicator function. Moreover, if $u \in L$, then $h(B,u) = 0$; hence, if we adopt the convention that the integrand on the right-hand side of the preceding formula is defined to be zero even if $\nabla h_B(u)$ is not defined uniquely, then we can extend the integration formally over S^{d-1} instead of $S^{d-1} \setminus L^\perp$.

2. If $k = 1$, then it is sufficient to assume that $0 \in B$ and B is non-degenerate. Indeed, in this particular case the approximation argument in the proof of Theorem 3.9 is not needed.

Proof. Both sides of the asserted equation vanish if $\dim(K + L) < d$. For the right-hand side this follows, since for any Borel set $A \subset S^{d-1}$ and $j \in \{0, \dots, d-1\}$,

$$\begin{aligned} 0 &\leq \int_{S^{d-1}\setminus L^\perp} \mathbf{1}\{\nabla h_B(u) \in A\} h(B,u) S(K[d-1-j], B[j], du) \\ &\leq dV(K[d-1-j], B[j+1]) = 0 \,. \end{aligned} \tag{26}$$

Therefore we can assume that $\dim(K + L) = d$.

First, we assume that $h_B|L$ is of class C^2. For $k = 1$ this assumption is automatically satisfied and the subsequent approximation argument is not necessary. Observe that

$$\mathcal{H}^d(\mathrm{relbd}(K|L^\perp) + L) = 0$$

if $k \leq d - 1$; hence we obtain

$$\begin{aligned} &\mathcal{H}^d(\{x \in \mathbb{R}^d : d_B(K,x) \in (r_1, r_2], u_B(K,x) \in A\}) \\ &= \mathcal{H}^d(\{x + \lambda \nabla h_B(u) : (x,u) \in Nor(K), u \notin L^\perp, \\ &\qquad \lambda \in (r_1, r_2], \nabla h_B(u) \in A\}) \,, \end{aligned} \tag{27}$$

where $r_1, r_2 \geq 0$ and $A \subset \mathbb{R}^d$ is Borel measurable. Now we apply the coarea formula to the map

$$F : Nor(K) \times [0, \infty) \to \mathbb{R}^d, \quad (x, u, \lambda) \mapsto x + \lambda \nabla h_B(u) \,.$$

Using the fact that, for $\mathcal{H}^d$ almost all $y \in \mathbb{R}^d$, the cardinality of the preimage $F^{-1}(\{y\})$ is at most one (compare [9]), we finally obtain that (27) is equal to

$$\begin{aligned} &\sum_{j=0}^{d-1} (r_2^{j+1} - r_1^{j+1}) \frac{1}{j+1} \binom{d-1}{j} \int_{S^{d-1}\setminus L^\perp} \mathbf{1}\{\nabla h_B(u) \in A\} h(B,u) \\ &\qquad \times S(K[d-1-j], B[j], du) \,; \end{aligned}$$

compare the arguments in [9]. Moreover, (26) implies that all summands vanish which correspond to indices $j \geq k$.

Now general measure theoretic arguments imply that the asserted equation is true if $h_B|L$ is of class C^2.

In order to obtain the general result, we choose a decreasing sequence of convex bodies $B_i \subset L$, $i \in \mathbb{N}$, such that $h_{B_i}|L$ is of class C^2 and $B_i \to B$ as $i \to \infty$. It is sufficient to consider a function g which is continuous and has compact support. We first verify that

$$\int_{\mathbb{R}^d \setminus K} g(d_{B_i}(K,x), u_{B_i}(K,x))\mathcal{H}^d(dx)$$
$$\to \int_{\mathbb{R}^d \setminus K} g(d_B(K,x), u_B(K,x))\mathcal{H}^d(dx)$$

as $i \to \infty$. By assumption we have $K + [0,\infty)B = K + L = K + [0,\infty)B_i$ for all $i \in \mathbb{N}$. Moreover, $\mathcal{H}^d(\mathrm{bd}(K+L)) = 0$ and

$$d_{B_i}(K,x) \to d_B(K,x) \qquad \text{and} \qquad u_{B_i}(K,x) \to u_B(K,x)$$

as $i \to \infty$, for all $x \in \mathrm{int}(K+L) \setminus K$. Thus the dominated convergence theorem implies the desired convergence.

Next, for any $j \in \{0, \dots, d-1\}$, we show that

$$\int_{S^{d-1} \setminus L^\perp} g(s, \nabla h_{B_i}(u)) h(B_i, u) S(K[d-1-j], B_i[j], du)$$
$$\to \int_{S^{d-1} \setminus L^\perp} g(s, \nabla h_B(u)) h(B, u) S(K[d-1-j], B[j], du) \tag{28}$$

as $i \to \infty$, for all $s \in [0,\infty)$. Once this has been shown, the required convergence of the right-hand side follows again by an application of the dominated convergence theorem.

We fix some $s \in [0,\infty)$. In order to prove (28), we set

$$\mu_i(du) := h(B_i, u) S(K[d-1-j], B_i[j], du)\,,$$

for $i \in \mathbb{N}$, and

$$\mu(du) := h(B, u) S(K[d-1-j], B[j], du)\,.$$

These measures are bounded by a constant C_1. Further, let $|g|$ be bounded by C_2. We define

$$\mathrm{I}(i) := \int_{S^{d-1} \setminus L^\perp} |g(s, \nabla h_{B_i}(u)) - g(s, \nabla h_B(u))| \mu_i(du)$$

and

$$\mathrm{II}(i) := \int_{S^{d-1} \setminus L^\perp} g(s, \nabla h_B(u)) \mu_i(du) - \int_{S^{d-1} \setminus L^\perp} g(s, \nabla h_B(u)) \mu(du)\,,$$

for $i \in \mathbb{N}$. It is sufficient to show that

$$\mathrm{I}(i) \to 0 \qquad \text{and} \qquad \mathrm{II}(i) \to 0$$

as $i \to \infty$.

To prove the first assertion, we fix some $\epsilon > 0$. Then we choose an open neighbourhood U_ϵ of $S^{d-1} \cap L^\perp$ such that $\mu_i(U_\epsilon) \leq \epsilon$ for all $i \in \mathbb{N}$; recall that h_{B_i} and h_B vanish on $L^\perp$ and $B_i \to B$. Since g is uniformly continuous, there is some $\delta > 0$ such that

$$|g(s,v) - g(s,w)| \leq \epsilon$$

if $v, w \in \mathbb{R}^d$ and $\|v - w\| \leq \delta$. Further, since $S^{d-1} \setminus U(\epsilon)$ is compact, there is some $i(\epsilon) \in \mathbb{N}$ such that

$$|\nabla h_{B_i}(u) - \nabla h_B(u)| \leq \delta$$

for $i \geq i(\epsilon)$ and all $u \in S^{d-1} \setminus U(\epsilon)$. Hence

$$\begin{aligned} \mathrm{I}(i) &\leq 2C_2\mu_i(U_\epsilon) + \epsilon\mu_i(S^{d-1} \setminus U(\epsilon)) \\ &\leq (C_1 + 2C_2)\epsilon \end{aligned}$$

if $i \geq i(\epsilon)$.

For the second assertion, we remark that $\mu_i \to \mu$ weakly on S^{d-1} as $i \to \infty$. We define a function d on S^{d-1} by setting $d(u) := g(s, \nabla h_B(u))$ if $u \in S^{d-1} \setminus L^\perp$, and $d(u) := 0$ if $u \in S^{d-1} \cap L^\perp$. Then d is bounded, continuous on $S^{d-1} \setminus L^\perp$, and $\mu(S^{d-1} \cap L^\perp) = 0$ since $h(B,u) = 0$ for $u \in L^\perp$. The assertion now follows from the classical Portmanteau theorem. □

By applying Theorem 3.9 to equation (12) in the proof of Theorem 2.4, we arrive at the following result.

Theorem 3.10 *Let Z be a stationary Boolean model with convex grains, and let X be the associated stationary particle process. Let $L \subset \mathbb{R}^d$ be a k-dimensional linear subspace, $1 \leq k \leq d$, which contains $B \in \mathcal{K}^d$. Assume that with respect to L, B is strictly convex and contains 0 in its interior. Let $A \subset \mathbb{R}^d$ be Borel measurable and $r \geq 0$. Then*

$$\begin{aligned} H_B(r, A) = \gamma \sum_{j=0}^{k-1} \binom{d-1}{j} &\int_0^r s^j (1 - H_B(s)) ds \\ &\times \int_{\mathcal{K}_0} \int_{S^{d-1} \setminus L^\perp} \mathbf{1}\{\nabla h_B(u) \in A\} h(B,u) \\ &\times S(K[d-1-j], B[j], du) \mathbb{Q}(dK) \,. \end{aligned} \tag{29}$$

Proof. It only remains to remark that $\mathbb{P}(0 \in \mathrm{exo}_B(Z)) = 0$. In fact, $\mathcal{H}^d(\mathrm{exo}_B(Z(\omega))) = 0$, for each $\omega \in \Omega$, follows by applying Fubini's theorem with respect to the decomposition $\mathbb{R}^d = L \times L^\perp$ und by using the corresponding known assertion in the section planes parallel to L. □

This theorem contains as a very special case an expression for the linear contact distribution function of a Boolean model; see Theorems 4.2 and 4.3 in [27]. As noted in Remark 2 above, we can choose $k = 1$ and $B = [0, v]$ for a unit vector v. In this situation, Theorem 3.10 yields

$$\frac{H_{[0,v]}(r)}{\displaystyle\int_0^r \big(1 - H_{[0,v]}(s)\big)\, ds} = \int_{\mathcal{K}_0} h(\Pi K, v)\mathbb{Q}(dK)$$

$$= \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle| S_{d-1}(X, du)\,,$$

where ΠK denotes the projection body of K. In view of Theorem 4.1 in [27], a straightforward integration now immediately implies Theorems 4.2 and 4.3 in [27] (observe the different notation used in [27]).

4 Contact distributions of non-stationary Boolean models

The objective of this section is to comment on some general results of [8], to indicate some improvements, and to explain, in the case of a random closed set which is induced by an inhomogeneous marked Poisson process, how the investigation of contact distributions of random closed sets can be extended to a non-stationary (inhomogeneous) setting. More precisely, we consider the following framework. Let Z be a random closed set in the extended convex ring. By the result in [32] (see also Satz 4.4.2 in [24]) there exists a marked point process $\tilde{X} = \sum_{i=1}^{\infty} \delta_{(\xi_i, Z_i)}$ in $\mathbb{R}^d \times \mathcal{K}_0$ such that

$$Z = \bigcup_{i=1}^{\infty} (\xi_i + Z_i)\,.$$

We denote by α the intensity measure of $\tilde{X}$. In the following, we restrict our attention to the case where Z is such that $\tilde{X}$ can be chosen as a Poisson process for which the conditions

(a)

$$\int \mathbf{1}\{(x + K) \cap C \neq \emptyset\}\alpha(d(x, K)) < \infty$$

for every compact set $C \subset \mathbb{R}^d$.

(b)

$$\alpha(d(x, K)) = f(x, K)\mathcal{H}^d(dx)\mathbb{Q}(dK)$$

for a measurable function $f : \mathbb{R}^d \times \mathcal{K}_0 \to [0, \infty)$.

are satisfied.

The subsequent result was established in [8]. We mention that Theorem 4.1 and the following discussion can be extended to localized notions and to other point processes as well. This might form the subject of a different paper (compare also [8]).

Theorem 4.1 *Let Z be a random closed set which is derived from a marked Poisson process $\tilde{X}$ as described above. Let $B \in \mathcal{K}$ be strictly convex with $0 \in B$, and let $x \in \mathbb{R}^d$, $r \geq 0$. Then*

$$H_B(x,r) = 1 - \exp\left\{ - \int_0^r \rho_B(x,s)ds \right\}$$

and

$$\rho_B(x,s) = \sum_{j=0}^{d-1} \binom{d-1}{j} s^{d-1-j}$$
$$\times \int_{\mathcal{K}_0} \int f(x - z - sb, K)\Theta_j(K, B; d(z,b))\mathbb{Q}(dK)\,.$$

Related to the contact distribution function are the intensity measures

$$\mathbb{E}\left[\Theta_j^+(Z, B; \cdot \times \mathbb{R}^d)\right]\,,$$

$j \in \{0, \ldots, d-1\}$. In [8], these random measures as well as their local generalizations were investigated. It was shown under quite general assumptions that they are absolutely continuous with respect to $\mathcal{H}^d$. Let us denote the corresponding densities by $\lambda_j^+(B, \cdot)$, where B is the underlying gauge body, which in [8] was assumed to be strictly convex and smooth (of class C^1). Actually, the assumption of smoothness can be dispensed with by a number of additional arguments which will be presented elsewhere. This fact will be indicated in the following by including the hypothesis of smoothness in brackets. The function $\lambda_{d-1}^+(B, \cdot)$ is called the local surface density. For a Poisson process $\tilde{X}$ it was proved in [8] that

$$\lambda_{d-1}^+(B,x) = (1 - \bar{p}(x)) \int_{\mathcal{K}_0} \int f(x-y,K)\Theta_{d-1}(K,B; dy \times \mathbb{R}^d)\mathbb{Q}(dK)\,, \qquad (30)$$

for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$. A comparison of (30) with Theorem 4.1 suggests the following theorem for which a local version can be established as well; compare Propositions 4.10, 4.11 and 4.26 in [8]. But first we introduce the integrability condition

$$\int_{\mathcal{K}_0} V_j(K)\mathbb{Q}(dK) < \infty\,, \qquad (31)$$

$j = 0, \ldots, d-1$, for the intrinsic volumes V_j. Note that if f is bounded and (31) is also assumed for $j = d$, then condition (a) is implied. The following result provides a rigorous version (in a generalized form) of Theorem 4.3 in [15].

Theorem 4.2 *Let Z be a random closed set which is derived from a marked Poisson process $\tilde{X}$ as described above. Let $B \in \mathcal{K}$ be strictly convex (and smooth) with $0 \in \operatorname{int} B$. Assume that f is bounded,* (31) *is satisfied, and, for $\mathbb{Q}$ almost all $K \in \mathcal{K}_0$, the set of points of discontinuity of $f(\cdot, K)$ has $\mathcal{H}^d$ measure zero. Then*

$$(1 - \bar{p}(x)) \left. \frac{\partial}{\partial t} \right|_{t=+0} H_B(x,t) = \lambda^+_{d-1}(B,x)\,,$$

for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$.

Proof. The assumptions ensure that the function $\rho_B(x, \cdot)$ is continuous at $s = 0$, for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$. In fact, Fubini's theorem implies that, for $\mathcal{H}^d$ almost all $x \in \mathbb{R}^d$, $x - z$ is not a point of discontinuity of $f(\cdot, K)$, for $\Theta_{d-1}(K, B; d(z, b))\mathbb{Q}(dK)$ almost all (z, b, K). Hence the dominated convergence theorem yields the assertion, since f is assumed to be bounded. The theorem now follows immediately from Theorem 4.1 and (30).□

In order to illustrate the representation (30) for the surface density, we consider the special case where the shape distribution is concentrated on homothets of the gauge body B and the radius distribution of the homothets centred at $y \in \mathbb{R}^d$ is absolutely continuous with density $g(y, \cdot)$; in addition, we assume that the density of the underlying ordinary Poisson point process is absolutely continuous as well with density $l(\cdot)$, that is, $l(\cdot)$ is the spatial density. Hence condition (b) can now be written in the form

$$f(y,R)\mathcal{H}^d(dy)\mu(dR) = l(y)g(y,R)\mathcal{H}^d(dy)\mathcal{H}^1(dR)\,,$$

where $\mu(\cdot) = \mathbb{Q}(\{tB : t \in \cdot\})$. Furthermore, we write

$$G_y((-\infty, R]) := \int_{-\infty}^{R} g(y,t)\mathcal{H}^1(dt)$$

for the radius distribution at y and define

$$\|x\|_B := \min\{\lambda \geq 0 : x \in \lambda B\}\,,$$

for $x \in \mathbb{R}^d$. Using this notation, we deduce

$$\begin{aligned}
1 - \bar{p}(x) &= \exp\left\{-\int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}\{x - y \in RB\} f(y,R)\mu(dR)\mathcal{H}^d(dy)\right\} \\
&= \exp\left\{-\int_{\mathbb{R}^d} l(y)\int_{\|x-y\|_B}^\infty g(y,R)\mathcal{H}^1(dR)\mathcal{H}^d(dy)\right\} \\
&= \exp\left\{-\int_{\mathbb{R}^d} l(y)(1 - G_y(\|x-y\|_B))\mathcal{H}^d(dy)\right\}\,.
\end{aligned}$$

Moreover, by Corollary 2.6 in [8]

$$
\begin{aligned}
&\int_0^\infty \int f(x-y,R)\Theta_{d-1}(RB,B;dy\times\mathbb{R}^d)\mu(dR)\\
&= \int_0^\infty \int f(x-y,\|y\|_B)C_{d-1}(tB,B;dy)dt\\
&= \int_{\mathbb{R}^d} l(x-y)g(x-y,\|y\|_B)\mathcal{H}^d(dy)\\
&= \int_{\mathbb{R}^d} l(z)g(z,\|x-z\|_B)\mathcal{H}^d(dz)\,.
\end{aligned}
$$

We summarize the preceding calculations in the following corollary.

Corollary 4.3 *Let the assumptions of the preceding discussion be satisfied. Let $B\in\mathcal{K}$ be strictly convex (and smooth) with $0\in\operatorname{int}B$. Then, for $\mathcal{H}^d$ almost all $x\in\mathbb{R}^d$,*

$$
\begin{aligned}
\lambda^+_{d-1}(B,x) = &\int_{\mathbb{R}^d} l(y)g(y,\|x-y\|_B)\mathcal{H}^d(dy)\\
&\times\exp\left\{-\int_{\mathbb{R}^d} l(y)(1-G_y(\|x-y\|_B))\mathcal{H}^d(dy)\right\}.
\end{aligned}
$$

In particular, if $g(y,\cdot)$, and hence also G_y, is independent of y, then we write

$$
\tilde{g}(\cdot):=g\circ\|\cdot\|_B \qquad\text{and}\qquad \tilde{G}(\cdot)=1-G\circ\|\cdot\|_B\,.
$$

Using this notation and writing f_1*f_2 for the convolution of two functions $f_1,f_2:\mathbb{R}^d\to[0,\infty)$, we obtain the following special case of Corollary 4.3.

Corollary 4.4 *Let $B\in\mathcal{K}$ be strictly convex (and smooth) with $0\in\operatorname{int}B$. Let Z be a Boolean model of homothets of B, having spatial density l and radius distribution G with density g (as described above). Then, for $\mathcal{H}^d$ almost all $x\in\mathbb{R}^d$,*

$$
\lambda^+_{d-1}(B,x)=(l*\tilde{g})(x)\exp\left\{-(l*\tilde{G})(x)\right\}.
$$

It should be pointed out that we follow [24] in defining the notion of a Boolean model in the non-stationary setting (our assumptions imply that the marks are independent of the grains). In the special Euclidean case $B=B^d$, the preceding considerations leading to Corollary 4.4 provide a rigorous derivation of Theorem 5.2 in [15]. By Theorem 3.9 in [8], Corollary 4.4 also extends the second equation of Theorem 7 in [31] to general dimensions.

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References

[1] A.J. Baddeley, Spatial sampling and censoring, Stochastic Geometry: Likelihood and Computation, ed. by O. Barndorff-Nielsen, W. Kendall, and M.N.M. van Lieshout, Chapman & Hall/CRC, Boca Raton, 37–78, 1999.

[2] A.J. Baddeley, R.D. Gill, The empty space hazard of a spatial pattern, Preprint 845, University Utrecht, Department of Mathematics, (1994).

[3] U. Hahn, D. Stoyan, Unbiased stereological estimation of surface area density of gradient surface processes, *Adv. in Appl. Probab.* **30** (1998), 904–920.

[4] M.B. Hansen, A.J. Baddeley, R.D. Gill, First contact distributions for spatial patterns: regularity and estimation, *Adv. in Appl. Probab.* **31** (1999), 15–33.

[5] M.B. Hansen, R.D. Gill, A.J. Baddeley, Kaplan-Meier type estimators for linear contact distributions, *Scand. J. Statist.* **23** (1998), 129–155.

[6] L. Heinrich, Asymptotic properties of minimum contrast estimators for parameters of Boolean models, *Metrika* **40** (1993), 67–94.

[7] D. Hug, On the mean number of normals through a point in the interior of a convex body, *Geom. Dedicata* **55** (1995), 319–340.

[8] D. Hug and G. Last, On support measures in Minkowski spaces and contact distributions in stochastic geometry, preprint.

[9] D. Hug, Habilitationsschrift, in preparation.

[10] M. Kiderlen and W. Weil, Measure-valued valuations and mixed curvature measures of convex bodies, *Geom. Dedicata* **76** (1999), 291–329.

[11] D.A. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, Cambridge, 1997.

[12] G. Last and M. Holtmann, On the empty space function of some germ-grain models, to appear in *Pattern Recognition.*

[13] G. Last and R. Schassberger, On the distribution of the spherical contact vector of stationary germ-grain models, *Adv. in Appl. Probab.* **30** (1998), 36–52.

[14] G. Matheron, *Random Sets and Integral Geometry*, Wiley, New York, 1975.

[15] A. Micheletti and D. Stoyan, Volume fraction and surface density for inhomogeneous random sets, Preprint 17/1998, Universitá degli studi di Milano, dipartimento di Matematica “F. Enriques”.

[16] I.S. Molchanov, Statistics of the Boolean model: from the estimation of means to the estimation of distributions, *Adv. in Appl. Probab.* **27** (1995), 63–86.

[17] I.S. Molchanov, *Statistics of the Boolean model for practitioners and mathematicians*, Wiley, Chichester, 1997.

[18] J. Rataj, Determination of spherical area measures by means of dilation volumes, preprint.

[19] J. Rataj, Translative and kinematic formulae for curvature measures of flat sections, *Math. Nachr.* **197** (1999), 89–101.

[20] R. Schneider, Parallelmengen mit Vielfachheit und Steiner-Formeln, *Geom. Dedicata* **9** (1980), 111–127.

[21] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1993.

[22] R. Schneider, An extension of the principal kinematic formula of integral geometry, *Rend. Circ. Mat. Palermo*, II. Ser., **35** (1994), 275–290.

[23] R. Schneider and W. Weil, *Integralgeometrie*, Teubner, Stuttgart, 1992.

[24] R. Schneider and W. Weil, *Stochastische Geometrie*, Teubner, Stuttgart, 2000, in preparation.

[25] D. Stoyan, W.S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications,* Second Edition, Wiley, Chichester, 1995.

[26] A.C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and its Applications 63, Cambridge University Press, Cambridge, 1996.

[27] W. Weil, Mean bodies associated with random closed sets, *Rend. Circ. Mat. Palermo*, II. Ser., Suppl. **50** (1997), 387–412.

[28] W. Weil, The mean normal distribution of stationary random sets and particle processes, Advances in Theory and Applications of Random Sets, Proc. Int. Symposium, Fontainebleau, France, 9-11 Oct. 1996, ed. by D. Jeulin, World Scientific Publ. Singapore, 21–33, 1997.

[29] W. Weil, On the mean shape of particle processes, *Adv. in Appl. Probab.* **29** (1997), 890–908.

[30] W. Weil, Intensity analysis of Boolean models, *Pattern Recognition* **32** (1999), 1675–1684.

[31] W. Weil, Densities of mixed volumes for Boolean models, Preprint, 1999.

[32] W. Weil and J. Wieacker, A representation theorem for random sets, *Probab. Math. Statist.* **9** (1988), 147–151.

[33] M. Zähle, Curvature measures and random sets II, *Probab. Theory Related Fields* **71** (1986), 37–58.

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