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Intersections and translative integral formulas for boundaries of convex bodies

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Abstract. Let $K, L \subset \mathbb{R}^n$ be two convex bodies with non-empty interiors and with boundaries ∂K, ∂L, and let χ denote the Euler characteristic as defined in singular homology theory. We prove two translative integral formulas involving boundaries of convex bodies. It is shown that the integrals of the functions $t \mapsto \chi(\partial K \cap (\partial L + t))$ and $t \mapsto \chi(\partial K \cap (L + t))$, $t \in \mathbb{R}^n$, with respect to an n-dimensional Haar measure of $\mathbb{R}^n$ can be expressed in terms of certain mixed volumes of K and L. In the particular case where K and L are outer parallel bodies of convex bodies at distance $r > 0$, the result will be deduced from a recent (local) translative integral formula for sets with positive reach. The general case follows from this and from the following (global) topological result. Let K_r, L_r denote the outer parallel bodies of K, L at distance $r \geq 0$. Establishing a conjecture of Firey (1978), we show that the homotopy type of $\partial K_r \cap \partial L_r$ and $\partial K_r \cap L_r$, respectively, is independent of $r \geq 0$ if $K^\circ \cap L^\circ \neq \emptyset$ and if ∂K and ∂L intersect almost transversally. As an immediate consequence of our translative integral formulas, we obtain a proof for two kinematic formulas which have also been conjectured by Firey.

1. Introduction

Kinematic formulas play a central rôle in integral and stochastic geometry. In general, a kinematic formula can be described in the following abstract setting. Let X be a topological space, let G be a topological group, and let $\varphi : G \times X \to X$ be a transitive, continuous left action of G on X such that $\varphi(\cdot, x) : G \to X$ is an open map for any $x \in X$. In this situation, (X, φ) is called a homogeneous G-space; confer [28], p. 15. Usually, it is assumed in addition that G and X are locally compact and Hausdorff (see [9], p. 121). Under this additional assumption on G, there always exists a left Haar measure μ on G. Further, let γ be a geometric functional on certain

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subsets of X, and let A, B be suitable subsets of X such that $\gamma(A \cap gB)$ is defined for μ-almost all $g \in G$ and $g \mapsto \gamma(A \cap gB)$ is integrable with respect to μ. In this general setting, a kinematic formula allows one to express the integral

$$\int_G \gamma(A \cap gB)\mu(dg) \tag{1.1}$$

in terms of geometric functionals of A and B. Such kinematic formulas, for various choices of G, X and γ and for different classes of subsets of X, have been the subject of numerous investigations. Moreover, operations different from the formation of the intersection of A and gB or functions γ which are measure-valued have been considered in integral geometry. We refer to the surveys in [25, §4.5], [29], [28, §3], and to [31, V], [21], [10], [20], [17] for a detailed discussion of the extensive literature on this subject.

The basic result in this context is the principal kinematic formula, which is due to Blaschke and Santaló. A differential-geometric version for domains in $\mathbb{R}^n$ which are bounded by imbedded hypersurfaces of class C^2 was proved by Chern and Yien. In the setting of convex geometry, the principal kinematic formula corresponds to the case where X is the Euclidean space $\mathbb{R}^n$ with $n \geq 2$, scalar product $\langle \cdot , \cdot \rangle$ and induced norm $| \cdot |$, G is the group $\mathbf{G}(n)$ of proper rigid motions of $\mathbb{R}^n$, $\varphi(g, x) = g(x)$ for $g \in \mathbf{G}(n)$ and $x \in \mathbb{R}^n$, γ is the Euler characteristic χ, and A, B are convex bodies (compact convex sets with non-empty interiors) $K, L \subset \mathbb{R}^n$. Let $\alpha(i)$ denote the volume of an i-dimensional Euclidean unit ball, and normalize μ as in [25]. To [25] we also refer for all notions of convex geometry which are not explicitly defined here. Then the integral (1.1) can be expressed in terms of the quermassintegrals (Minkowski functionals) $W_i(K), W_j(L)$, $i, j = 0, \ldots, n$, of K and L (see [25], §4.2 and (5.3.19)), namely

$$\int_{\mathbf{G}(n)} \chi(K \cap gL)\mu(dg) = \frac{1}{\alpha(n)} \sum_{k=0}^{n} \binom{n}{k} W_{n-k}(K) W_k(L). \tag{1.2}$$

In translative integral geometry, again X is the Euclidean space $\mathbb{R}^n$ but now G is the translation group of $\mathbb{R}^n$ (which will be identified with $\mathbb{R}^n$) and $\varphi(t, x) = x + t$ for $t, x \in \mathbb{R}^n$. First contributions to translative integral geometry are due to Blaschke [1] and Berwald and Varga [2]. But only within the last 15 years translative integral formulas have been investigated systematically. Progress in this field was accompanied by, and in fact required for, a corresponding development of stochastic geometry, where non-isotropic models naturally appear in applications. Main recent contributions are due to Schneider and Weil [27], Weil [34], [35], Rataj and M. Zähle [19], Rataj [18]. A particular feature of such translative formulas is that certain mixed functionals such as mixed volumes or mixed curvature measures of the sets involved cannot be avoided. The mixed volume of convex bodies $K_1, \ldots, K_n$ in $\mathbb{R}^n$ is denoted by $V(K_1, \ldots, K_n)$. In particular, we write $V(K[k], L[n-k])$ if $K_1 = \ldots = K_k = K$ and $K_{k+1} = \ldots = K_n = L$, where $k \in \{0, \ldots, n\}$ and K, L are convex bodies. The defining relation for these special mixed volumes is

$$\mathcal{H}^n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k} V(K[k], L[n-k]),$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure and $\lambda, \mu \geq 0$. In particular, for $k = 0, \ldots, n$ and the Euclidean unit ball B one obtains

$$W_{n-k}(K) = V(K[k], B[n-k]).$$

All basic properties of mixed volumes which we need subsequently are described in Section 5 of SCHNEIDER's book [25]. Now the translative counterpart to formula (1.2) is given by

$$\int_{\mathbb{R}^n} \chi(K \cap (L+t))\mathcal{H}^n(dt) = \sum_{k=0}^{n} \binom{n}{k} V(K[k], -L[n-k]); \tag{1.3}$$

see §5.3 in [25]. Extensions of formulas (1.2) and (1.3) to finite unions of convex bodies follow immediately from well-known properties of valuations on convex bodies. More explicit information and extensions to certain finite unions of sets with positive reach have been obtained by means of certain index functions which take account of multiplicities; see, e.g., SCHNEIDER [23] and M. ZÄHLE [37]. In general, however, boundaries of arbitrary convex bodies cannot be represented as locally finite unions of sets with positive reach. It is one purpose of the present paper to establish the following analogues of formula (1.3) involving boundaries of convex bodies. Moreover, we study related topological questions concerning intersections of boundaries of convex bodies. One difficulty in this context is that the Euler characteristic of the intersections $\partial K \cap (\partial L + t)$ and $\partial K \cap (L+t)$ need not be defined for all translation vectors $t \in \mathbb{R}^n$ (see the discussion below). This indicates that for the proof of the following two theorems topological and measure theoretic arguments have to be combined.

Theorem 1.1. *Let $K, L \subset \mathbb{R}^n$ be convex bodies. Then the map $t \mapsto \chi(\partial K \cap (\partial L + t))$ is integrable with respect to $\mathcal{H}^n$ on $\mathbb{R}^n$ and*

$$\begin{aligned} &\int_{\mathbb{R}^n} \chi(\partial K \cap (\partial L + t))\mathcal{H}^n(dt) \\ &= (1 + (-1)^n) \sum_{k=1}^{n-1} \binom{n}{k} \left\{ V(K[k], -L[n-k]) + (-1)^{k-1} V(K[k], L[n-k]) \right\}. \end{aligned} \tag{1.4}$$

Theorem 1.2. *Let $K, L \subset \mathbb{R}^n$ be convex bodies. Then the map $t \mapsto \chi(\partial K \cap (L + t))$ is integrable with respect to $\mathcal{H}^n$ on $\mathbb{R}^n$ and*

$$\begin{aligned} &\int_{\mathbb{R}^n} \chi(\partial K \cap (L + t))\mathcal{H}^n(dt) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} \left\{ V(K[k], -L[n-k]) + (-1)^{n-1-k} V(K[k], L[n-k]) \right\}. \end{aligned} \tag{1.5}$$

Kinematic formulas, concerning the group of proper rigid motions, can be deduced from the preceding two theorems by first applying the corresponding translative formulas to K and ρL, where ρ is an arbitrary proper rotation, and by subsequently carrying out the integration with respect to the Haar probability measure on the group of proper rotations $SO(n)$ of $\mathbb{R}^n$. The desired results then follow by an application of equation (5.3.25) in SCHNEIDER's book [25]. As an immediate consequence we thus obtain for convex bodies $K, L \subset \mathbb{R}^n$ the following kinematic formulas which have been conjectured by FIREY (1978):

$$\int_{\mathbf{G}(n)} \chi(\partial K \cap g\partial L)\mu(dg) = \frac{2}{\alpha(n)} \sum_{k=1}^{n-1} \binom{n}{k} \left(1 - (-1)^k\right) W_{n-k}(K)W_k(L) \tag{1.6}$$

if n is even, and

$$\int_{\mathbf{G}(n)} \chi(\partial K \cap g\partial L)\mu(dg) = 0 \tag{1.7}$$

if n is odd; moreover,

$$\int_{\mathbf{G}(n)} \chi(\partial K \cap gL)\mu(dg) = \frac{1}{\alpha(n)} \sum_{k=0}^{n-1} \binom{n}{k} \left(1 - (-1)^{n-k}\right) W_{n-k}(K)W_k(L) \tag{1.8}$$

for all $n \geq 2$.

Alternatively, formulas (1.6) – (1.8) can also be deduced from Theorem 2.2.1 (see also Corollary 2.2.2) of FU in [11], but the theory developed in [10], [11] does not cover the present more general results of translative integral geometry.

The basic idea for the proof of Theorems 1.1 and 1.2 goes back to FIREY's suggestion for a proof of (1.6) – (1.8). To describe the idea, first in this special case, let again B denote the Euclidean unit ball. For a convex body $C \subset \mathbb{R}^n$ and $r \geq 0$, we write $C_r := C + rB$ and call C_r a parallel body of C at distance r. It is easy to see that for convex bodies $D, E \subset \mathbb{R}^n$, $r > 0$, $K := D_r$ and $L := E_r$ the boundaries ∂K and ∂L are sets of positive reach. It can be deduced from a special case of Federer's kinematic formula for sets with positive reach [8], Theorem 6.11, that the kinematic formulas (1.6) – (1.8) hold for such a special choice of K and L. Then one passes to the limit $r \searrow 0$ in these formulas. The right-hand sides behave continuously with respect to this limit. However, for the integrals this is not at all obvious. The same problem arises in the proofs of the corresponding translative integral formulas. Moreover, in the general case, FEDERER's kinematic formula cannot be applied; instead we use a recent translative integral formula, due to RATAJ and M. ZÄHLE [19].

As a first step towards a proof of Theorems 1.1 and 1.2, one has to ensure that $\chi(\partial K \cap (\partial L + t))$ and $\chi(\partial K \cap (L + t))$ are well-defined, that is, the homology of the respective topological spaces is finitely generated, at least for $\mathcal{H}^n$-almost all $t \in \mathbb{R}^n$. In Section 2, it will be shown that $\partial K \cap (\partial L + t)$ is an $(n-2)$-dimensional compact Lipschitz manifold without boundary and $\partial K \cap (L + t)$ is an $(n-1)$-dimensional compact Lipschitz manifold with boundary, provided that the boundaries of K and $L + t$ intersect almost transversally (see below) and $K^\circ \cap (L + t)^\circ \neq \emptyset$. Then it follows from Propositions IV.8.10 and V.4.11 in [5] that the homology of these intersections is finitely generated; confer also Corollary VIII.1.4 in [5] for the case of a topological manifold without boundary.

In order to give a precise definition for the notion of transversality employed in this paper, let us denote by $N(K, x)$ the closed convex cone of exterior normal vectors of a convex body K at the boundary point $x \in \partial K$. For convex bodies $K, L \subset \mathbb{R}^n$, we say that ∂K and ∂L intersect almost transversally if

$$N(K,x) \cap N(L,x) = \{0\} \tag{1.9}$$

and

$$N(K,x) \cap (-N(L,x)) = \{0\} \tag{1.10}$$

hold for all $x \in \partial K \cap \partial L$. A classical result of Federer [8] or, alternatively, a recent result of Schneider [26] then imply that ∂K and $g\partial L$ intersect almost transversally for μ-almost all $g \in \mathbf{G}(n)$. In fact, Schneider's result even yields that the boundaries intersect transversally for μ-almost all $g \in \mathbf{G}(n)$. Here we say that the boundaries of two convex bodies K and L intersect transversally if for all common boundary points $x \in \partial K \cap \partial L$ the intersection of the linear hulls of the normal cones $N(K,x)$ and $N(L,x)$ equals $\{0\}$, that is

$$\text{lin } N(K,x) \cap \text{lin } N(L,x) = \{0\} \tag{1.11}$$

for all $x \in \partial K \cap \partial L$. Certainly, (1.11) is a much stronger condition than (1.9) and (1.10), since we do not adopt any smoothness assumptions for ∂K and ∂L. In order to prove translative integral formulas, however, we have to ensure that ∂K and $\partial L + t$ intersect almost transversally for $\mathcal{H}^n$-almost all $t \in \mathbb{R}^n$. In fact, such a result is implicitly contained in a paper by Ewald, Larman and Rogers [7]. Some assumptions of convexity seem to be essential in this context. We also wish to emphasize that the proof of an integral formula often requires some delicate argument which allows to exclude a set of measure zero; confer for instance Federer [8], Schneider [22], [24], Fu [11], Glasauer [12], [13], [14], Rataj and Zähle [19], Zähle [38].

As a consequence of the preceding discussion we can conclude that the Euler characteristics of the intersections $\partial K \cap (\partial L + t)$ and $\partial K \cap (L + t)$ are well-defined for $\mathcal{H}^n$-almost all $t \in \mathbb{R}^n$.

For a proof of formulas (1.6) – (1.8), Firey further suggested to show that the Euler characteristics of the sets $\partial K_r \cap g\partial L_r$ and $\partial K_r \cap gL_r$, respectively, are independent of $r > 0$, at least for μ-almost all $g \in \mathbf{G}(n)$. It turns out, however, that in order to follow this strategy of proof it is necessary to establish the corresponding statement for all $r \geq 0$. Actually, the main problem is to include the case $r = 0$. More generally, the following two theorems will be proved. In the first of these we consider the intersections of the boundaries of the parallel bodies of K and L.

Theorem 1.3. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two convex bodies with common interior and whose boundaries intersect almost transversally, that is $K^\circ \cap L^\circ \neq \emptyset$,* (1.9) *and* (1.10) *are satisfied. Define the intersections*

$$S_r := \partial K_r \cap \partial L_r, \quad r \geq 0.$$

Then all S_r have the same homotopy type, that is

$$S_r \simeq S_s \quad \textit{for all } r, s \geq 0.$$

Moreover, the Euler characteristic $\chi(S_r)$ *is well-defined for* $r \geq 0$ *and*

$$\chi(S_r) = \chi(S_s) \quad \textit{for all } r, s \geq 0.$$

For the intersections of boundaries and bodies, we shall prove:

Theorem 1.4. *Let* $K, L \subset \mathbb{R}^n$, $n \geq 2$, *be two convex bodies with* $K^\circ \cap L^\circ \neq \emptyset$ *and for which* (1.9) *and* (1.10) *are satisfied. Define the intersections*

$$H_r := \partial K_r \cap L_r, \quad r \geq 0.$$

Then all H_r *have the same homotopy type, that is*

$$H_r \simeq H_s \quad \textit{for all } r, s \geq 0.$$

Moreover, the Euler characteristic $\chi(H_r)$ *is well-defined for* $r \geq 0$ *and*

$$\chi(H_r) = \chi(H_s) \quad \textit{for all } r, s \geq 0.$$

Theorems 1.3 and 1.4 are proved in Sections 2.1 – 2.4. These two theorems are established in three steps. In Section 2.1, we show with an implicit function theorem for Lipschitz functions that the union

$$M := \bigcup_{0 \leq r \leq 1} S_r \tag{1.12}$$

is an $(n-1)$-dimensional topological, actually a compact Lipschitz, manifold with boundary

$$\partial M = S_0 \cup S_1.$$

Outside $K \cup L$ one knows that the distance functions $d(K, \cdot)$ and $d(L, \cdot)$ have continuous gradients, which yields that the intersections S_r, for $r > 0$, are $(n-2)$-dimensional C^1-manifolds whose normal space is spanned by $\nabla d(K, \cdot)$ and $\nabla d(L, \cdot)$.

In a Morse type lemma, using a flow induced by these normals, we conclude in Section 2.2 that the sets S_r are homeomorphic for $r > 0$. This flow can be extended to $r = 0$, and thus we get a surjective, continuous map

$$\varphi : S_1 \to S_0.$$

As the normals need not be unique for the given convex bodies, φ need not be injective and induces, being a quotient map, a possibly non trivial equivalence relation on S_1. The quotient map φ enables us to show that S_0 is a strong deformation retract of M, hence

$$S_0 \simeq M. \tag{1.13}$$

This can also be seen observing that M is the mapping cylinder of φ; see [30, §1.4].

In Section 2.3, we use a result of Brown on collarings for boundaries of topological manifolds (see [3]), which yields that the inclusion

$$i : M - \partial M \hookrightarrow M$$

is a homotopy equivalence, that is, $M - \partial M$ is a weak deformation retract of M. As clearly the Morse type lemma shows that $S_{\frac{1}{2}}$ is a strong deformation retract of $M - \partial M$, we obtain

$$S_{\frac{1}{2}} \simeq M - \partial M \simeq M.$$

Together with (1.13), this will prove Theorem 1.3.

The proof of Theorem 1.4 follows the same line. In Section 2.1, we prove that the union

$$N := \bigcup_{0 \le r \le 1} H_r \tag{1.14}$$

is an n-dimensional compact Lipschitz manifold with boundary

$$\partial N = M \cup H_0 \cup H_1.$$

In Section 2.4, we modify the flow used in Section 2.2 in order to prove that the sets H_r are homeomorphic for $r > 0$ and that H_0 is a strong deformation retract of N, in particular

$$H_0 \simeq N. \tag{1.15}$$

Finally, we apply once again Brown's Theorem to conclude the proof of Theorem 1.4.

2. Invariance of the homotopy type

In this section, we consider two convex bodies $K, L \subset \mathbb{R}^n$, $n \ge 2$, which have common interior points, that is

$$K^\circ \cap L^\circ \neq \emptyset, \tag{2.1}$$

and which intersect almost transversally. Let $d(C, \cdot) : \mathbb{R}^n \to [0, \infty)$ denote the distance function of a non-empty set $C \subseteq \mathbb{R}^n$. In particular, we consider the parallel bodies

$$K_r := \{x \in \mathbb{R}^n \mid d(K, x) \le r\}, \quad L_r := \{x \in \mathbb{R}^n \mid d(L, x) \le r\},$$

for $r \ge 0$, which are again convex bodies and which can also be described as $K + rB$, $L + rB$. Note that (2.1) implies that K_r and L_s have common interior points, too, and hence

$$N(K_r, x) \cap (-N(L_s, x)) = \{0\} \tag{2.2}$$

holds for all $r, s \ge 0$ and all $x \in \partial K_r \cap \partial L_s$.

In the following, we write $A - B$, for $A, B \subseteq \mathbb{R}^n$, to denote the set of all points which are in A and not in B. The Minkowski sum of A and $-B$ (the reflection of B in the origin) is therefore denoted by $A + (-B)$. Moreover, if f is any function which is defined on a subset $\Omega \subseteq \mathbb{R}^n$ and takes its values in $\mathbb{R}$, then we write $[f = r]$, $r \in \mathbb{R}$, for the set of all $x \in \Omega$ for which $f(x) = r$, etc.

Subsequently, we shall use the notion of a Lipschitz manifold with or without a boundary. For the definition of a Lipschitz manifold (without boundary) we refer to

[32] or [33]. Actually, what we shall call a Lipschitz manifold will even be a locally Lipschitz graph in $\mathbb{R}^n$, and hence a strong Lipschitz submanifold of $\mathbb{R}^n$ in the sense of Walter. The notion of a Lipschitz manifold with boundary is then defined in the obvious way.

2.1. Lipschitz manifolds

In this subsection, we assume that K, L are convex bodies which satisfy (2.1), (1.9) and (1.10). The following lemma will be needed.

Lemma 2.1. *Assume that $N(K,x) \cap N(L,x) = \{0\}$ for all $x \in \partial K \cap \partial L$. Then $N(K_r,x) \cap N(L_r,x) = \{0\}$, for all $r \geq 0$ and $x \in \partial K_r \cap \partial L_r$.*

Proof. Suppose there is some $r > 0$ and some $x \in \partial K_r \cap \partial L_r$ such that

$$u \in N(K_r,x) \cap N(L_r,x) \qquad \text{and} \qquad |u| = 1.$$

Let $B(x,r)$, for $x \in \mathbb{R}^n$ and $r \geq 0$, denote the closed ball of radius r centred at x. Then

$$B(x,r) \cap K = \{x - ru\} = B(x,r) \cap L.$$

In fact, $B(x,r) \cap K \neq \emptyset$ follows from $x \in K_r$. Moreover, $(B(x,r) \cap K) - \{x - ru\} \neq \emptyset$ implies that $x \in K_r^\circ$ or $u \notin N(K_r,x)$.

Therefore, $x - ru \in \partial K \cap \partial L$ and $u \in N(K, x-ru) \cap N(L, x-ru)$, a contradiction. □

The main two results of this subsection are Propositions 2.2 and 2.3.

Proposition 2.2. *Under the assumptions (2.1), (1.9) and (1.10), the set M, defined in (1.12), is an $(n-1)$-dimensional compact Lipschitz manifold with boundary*

$$\partial M = S_0 \cup S_1,$$

and S_r, for any $r \geq 0$, is an $(n-2)$-dimensional compact Lipschitz manifold without boundary.

Proof. Let $z_0 \in M$ and choose $p_0 \in K^\circ \cap L^\circ$. By an appropriate choice of a coordinate system, we can ensure that $p_0 - z_0 \in (0,\infty)e_n$, where $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$. Set $r_0 := d(K,z_0) = d(L,z_0) \in [0,1]$. Lemma 2.1 yields that

$$N(K_{r_0}, z_0) \cap N(L_{r_0}, z_0) = \{0\}. \tag{2.3}$$

Let $\delta > 0$ be chosen such that $B(p_0, 3\delta) \subseteq K^\circ \cap L^\circ$. Further, we write $p_0 = (x_0, t_1)$ and $z_0 = (x_0, t_0)$, where $x_0 \in \mathbb{R}^{n-1}$, $t_0, t_1 \in \mathbb{R}$ and $t_0 < t_1$. Then, for any $r \geq 0$, there is a uniquely determined convex function $f(\cdot, r)$ such that

$$\partial K_r \cap (B(p_0, 3\delta) + (-\infty, 0)e_n) = \text{graph } f(\cdot, r) = \{(x, f(x,r)) \mid x \in B'(x_0, 3\delta)\},$$

where $B'(x_0, 3\delta) \subseteq \mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1} \times \{0\}$ denotes a closed ball of radius 3δ centred at x_0 in $\mathbb{R}^{n-1}$. Similarly, ∂L_r is the graph of a convex function $g(\cdot, r)$ on $B'(x_0, 3\delta)$ for all $r \geq 0$. Obviously, we have $f(x_0, r_0) = t_0 = g(x_0, r_0)$.

Claim 1. Let $c > 0$. Then $f \in C^{0,1}(B'(x_0, \delta) \times [0, c])$.

Proof of Claim 1. Since K_c and L_c are compact, there is a number $a > 0$ such that $|f(x, r)| \leq a$ holds for $x \in B'(x_0, 3\delta)$ and $r \in [0, c]$. Then, as in the proof of Theorem 1.5.1 in [25], we obtain that

$$|f(x_1, r) - f(x_2, r)| \leq \frac{2a}{\delta}|x_2 - x_1|, \tag{2.4}$$

for $x_1, x_2 \in B'(x_0, 2\delta)$ and $r \in [0, c]$. Fix some $x \in B'(x_0, \delta)$ for the moment, and choose $r_1, r_2 \in [0, c]$ with $0 < r_2 - r_1 < \delta$. For $(x, f(x, r_2)) \in \partial K_{r_2}$ there is some $q \in \partial K_{r_1}$ such that $|(x, f(x, r_2)) - q| = r_2 - r_1$. From $|r_2 - r_1| < \delta$ and $x \in B'(x_0, \delta)$ we infer that q can be represented as $q = (x_3, f(x_3, r_1))$ with $x_3 \in B'(x_0, 2\delta)$. Hence, we obtain

$$\begin{aligned} |f(x, r_2) - f(x, r_1)| &\leq |f(x, r_2) - f(x_3, r_1)| + |f(x_3, r_1) - f(x, r_1)| \\ &\leq |r_2 - r_1| + \frac{2a}{\delta}|x_3 - x| \leq \left(\frac{2a}{\delta} + 1\right)|r_2 - r_1|. \end{aligned}$$

On the other hand, if $|r_2 - r_1| > \delta$, then

$$|f(x, r_2) - f(x, r_1)| \leq 2a \leq \frac{2a}{\delta}|r_2 - r_1|.$$

Thus,

$$|f(x, r_2) - f(x, r_1)| \leq \left(\frac{2a}{\delta} + 1\right)|r_2 - r_1|, \tag{2.5}$$

for any $x \in B'(x_0, \delta)$ and $r_1, r_2 \in [0, c]$. Now (2.4), (2.5) and the triangle inequality imply the assertion of Claim 1.

We set $\hat{f}(x) := f(x, r_0)$ and $\hat{g}(x) := g(x, r_0)$. Since $\hat{f}$ and $\hat{g}$ are convex functions, the subdifferentials $\partial\hat{f}(x_0)$ and $\partial\hat{g}(x_0)$ are defined as in convex geometry; confer [25, p. 30] or [4, p. 36 and Proposition 2.2.7]. By (2.3) and [4, p. 61, Corollary] (see also [25, Theorem 1.5.11]), we obtain that

$$\partial\hat{f}(x_0) \cap \partial\hat{g}(x_0) = \emptyset.$$

Moreover, $\partial\hat{f}(x_0)$ and $\partial\hat{g}(x_0)$ are non-empty compact convex subsets of $\mathbb{R}^{n-1}$; confer [4, Proposition 2.1.2]. Hence, we can find $u_0 \in \mathbb{R}^{n-1}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\langle v, u_0 \rangle \geq \gamma_1 > \gamma_2 \geq \langle w, u_0 \rangle \tag{2.6}$$

for all $v \in \partial\hat{f}(x_0)$ and $w \in \partial\hat{g}(x_0)$.

Next we extend the definition of the functions f and g by setting $f(x, -r) := f(x, r)$ and $g(x, -r) := g(x, r)$ for $x \in B'(x_0, \delta)$ and $r \geq 0$. Then $f(\cdot, r)$ is convex for all $r \in \mathbb{R}$ and, in addition, $f \in C^{0,1}(B'(x_0, \delta) \times [-c, c])$ for all $c > 0$. Then we define $\Gamma := f - g$.

Let $x \in B'(x_0, \delta)$ and $r \geq 0$. Then $(x,t) \in \partial K_r \cap \partial L_r$ for some $t < t_1$ if and only if $\Gamma(x,r) = 0$, and if this is the case, then $t = f(x,r) = g(x,r)$. The functions f and g are Lipschitz, and thus Γ is Lipschitz, too. Therefore, the generalized subdifferential (gradient) $\partial\Gamma(x_0, r_0)$ of Γ at $(x_0, r_0) \in \mathbb{R}^n$ can be defined as in [4, p. 27]. Here, of course, we identify $\mathbb{R}^n$ with its dual space. Also note that Γ is defined in a full neighbourhood of (x_0, r_0), since we extended the geometric definition of f and g.

Claim 2. Let $(\xi, \eta) \in \partial\Gamma(x_0, r_0)$. Then $\langle \xi, u_0 \rangle \geq \gamma := \gamma_1 - \gamma_2 > 0$; in particular, $(\partial\Gamma(x_0, r_0)) \cap (\{0\} \times \mathbb{R}) = \emptyset$.

Proof of Claim 2. Since f, g and Γ are Lipschitz functions, they are differentiable on $(B'(x_0, \delta) \times \mathbb{R}) - A$, where $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^n(A) = 0$. Theorem 2.5.1 in [4] then shows that

$$\partial\Gamma(x_0, r_0) = \operatorname{conv}\left\{\lim_{i\to\infty} \nabla\Gamma(x_i, r_i) \mid (x_i, r_i) \to (x_0, r_0), (x_i, r_i) \notin A\right\}.$$

On the right-hand side, we consider any sequence $((x_i, r_i))_{i\in\mathbb{N}} \subseteq (B'(x_0, \delta) \times \mathbb{R}) - A$, converging to (x_0, r_0), and such that the sequence $(\nabla\Gamma(x_i, r_i))_{i\in\mathbb{N}}$ converges. Then we take the convex hull of all such limit points.

Therefore, it is sufficient to prove that $\langle \xi, u_0 \rangle \geq \gamma$ is satisfied for all

$$(\xi, \eta) \in \left\{\lim_{i\to\infty} \nabla\Gamma(x_i, r_i) \mid (x_i, r_i) \to (x_0, r_0), (x_i, r_i) \notin A\right\}.$$

Let

$$(\xi, \eta) = \lim_{i\to\infty} \nabla\Gamma(x_i, r_i) = \lim_{i\to\infty} (\nabla f(x_i, r_i) - \nabla g(x_i, r_i))$$

for some $(x_i, r_i) \to (x_0, r_0)$ with $(x_i, r_i) \notin A$. We write

$$\nabla f(x_i, r_i) = (\nabla_X f(x_i, r_i), \partial_r f(x_i, r_i)),$$

and similarly for g. Taking subsequences, we may assume that

$$\nabla_X f(x_i, r_i) \to \xi_f \qquad \text{and} \qquad \nabla_X g(x_i, r_i) \to \xi_g$$

for $i \to \infty$. If $h \in \mathbb{R}^{n-1}$, $|h| < \delta$ and i is sufficiently large, then

$$f(x_i + h, r_i) - f(x_i, r_i) \geq \langle \nabla_X f(x_i, r_i), h \rangle$$

and

$$g(x_i + h, r_i) - g(x_i, r_i) \geq \langle \nabla_X g(x_i, r_i), h \rangle,$$

due to the convexity of $f(\cdot, r_i)$ and $g(\cdot, r_i)$. Passing to the limit, we obtain

$$f(x_0 + h, r_0) - f(x_0, r_0) \geq \langle \xi_f, h \rangle$$

and

$$g(x_0 + h, r_0) - g(x_0, r_0) \geq \langle \xi_g, h \rangle.$$

But since $\hat{f}(\cdot) = f(\cdot, r_0)$ and $\hat{g}(\cdot) = g(\cdot, r_0)$ are convex, this demonstrates that $\xi_f \in \partial\hat{f}(x_0)$ and $\xi_g \in \partial\hat{g}(x_0)$. Hence

$$\xi = \lim_{i\to\infty} (\nabla_X f(x_i, r_i) - \nabla_X f(x_i, r_i)) = \xi_f - \xi_g \in \partial\hat{f}(x_0) - \partial\hat{g}(x_0).$$

Now (2.6) yields that $\langle \xi, u_0\rangle \geq \gamma > 0$. This completes the proof of Claim 2.

After a rotation, we may assume that $u_0 = e_1$. Then Claim 2 yields that $\pi_1(\xi, \eta) \geq \gamma > 0$ for all $(\xi, \eta) \in \partial\Gamma(x_0, r_0)$, that is, $\pi_1\partial\Gamma(x_0, r_0)$ has maximal rank in the sense of [4, pp. 253/256]. We also refer to [4] for a definition of the projection operator π_1. Write $x_0 = (x_{01}, y_0)$ with $x_{01} \in \mathbb{R}$ and $y_0 \in \mathbb{R}^{n-2}$. According to Corollary 7.1.3 in [4], there are neighbourhoods $U(y_0, r_0) \subseteq \mathbb{R}^{n-2} \times \mathbb{R}$, $U(x_{01}) \subseteq \mathbb{R}$ and a Lipschitz map

$$\zeta : U(y_0, r_0) \to U(x_{01}),$$

satisfying $\zeta(y_0, r_0) = x_{01}$, $(\zeta(y, r), y) \subseteq B'(x_0, \delta)$ for all $(y, r) \in U(y_0, r_0)$, and such that $\Gamma(\zeta(y, r), y, r) = 0$ for $(y, r) \in U(y_0, r_0)$. Moreover, $\zeta(y, r)$ is the only solution of $\Gamma(\cdot, y, r) = 0$ on $U(x_{01})$ as long as $(y, r) \in U(y_0, r_0)$. Let $U(r_0) \subseteq [0, 1]$ be a neighbourhood of r_0 in $[0, 1]$ and let $U(y_0) \subseteq \mathbb{R}^{n-2}$ be a neighbourhood of y_0 in $\mathbb{R}^{n-2}$ such that $U(y_0) \times U(r_0) \subseteq U(y_0, r_0)$. If $r_0 \in (0, 1)$, then we can choose $U(r_0) = (r_1, r_2) \subseteq [0, 1]$ and set $U := K^\circ_{r_2} - K_{r_1}$. If $r_0 = 0$, then we choose $U(r_0) = [0, r_2) \subseteq [0, 1]$ and set $U := K^\circ_{r_2}$, and if $r_0 = 1$, then we choose $U(r_0) = (r_1, 1] \subseteq [0, 1]$ and set $U := \mathbb{R}^n - K_{r_1}$. Then we obtain

$$\begin{aligned} &M \cap U \cap ((U(x_{01}) \times U(y_0)) + (-\infty, 0)e_n) \\ &\quad = \{(x, t) \in \partial K_r \cap \partial L_r | (x, r) \in U(x_{01}) \times U(y_0) \times U(r_0)\} \\ &\quad = \{(\zeta(y, r), y, f(\zeta(y, r), y, r)) | (y, r) \in U(y_0) \times U(r_0)\} \\ &\quad = \text{im } h, \end{aligned}$$

where $h : U(y_0) \times U(r_0) \to \mathbb{R}^n$ is defined by

$$h(y, r) := (\zeta(y, r), y, f(\zeta(y, r), y, r)).$$

Clearly, h is Lipschitz and the map $q : \text{im } h \to U(y_0) \times U(r_0)$, defined by

$$q(x_1, y, t) := (y, d(K, (x_1, y, t))),$$

is Lipschitz, too, and the inverse of h. This proves that M is an $(n-1)$-dimensional Lipschitz manifold with boundary $\partial M = S_0 \cup S_1$.

Now let $z_0 \in S_r$ for some $r \geq 0$. Certainly, it suffices to assume that $r \in [0, 1]$. Proceeding as before, we obtain that

$$\begin{aligned} &S_r \cap U \cap ((U(x_{01}) \times U(y_0)) + (-\infty, 0)e_n) \\ &\quad = \{(x, t) \in \partial K_r \cap \partial L_r \mid x \in U(x_{01}) \times U(y_0)\} \\ &\quad = \text{im } h(\cdot, r), \end{aligned}$$

where $h(\cdot,r) : U(y_0) \to \mathbb{R}^n$ is Lipschitz with inverse $\tilde{q} : \text{im } h(\cdot,r) \to U(y_0)$, $(x_1,y,t) \mapsto y$. Hence, S_r is an $(n-2)$-dimensional Lipschitz manifold without boundary. $\square$

Proposition 2.3. *Under the assumptions* (2.1), (1.9) *and* (1.10), *the set* N, *defined in* (1.14), *is an* n*-dimensional compact Lipschitz manifold with boundary*

$$\partial N = M \cup H_0 \cup H_1 = \left(\bigcup_{0 \le r \le 1} S_r \right) \cup H_0 \cup H_1,$$

and H_r, *for any* $r \geq 0$, *is an* $(n-1)$*-dimensional compact Lipschitz manifold with boundary*

$$\partial H_r = S_r.$$

Proof. First, we show that N is an n-dimensional Lipschitz manifold with boundary $\partial N = M \cup H_0 \cup H_1$. We continue using the notation and results from the proof of Proposition 2.2. Let $z_0 \in N$. We distinguish several cases.

Case 1: $z_0 \notin M \cup H_0 \cup H_1$. Then $z_0 \in H_r$ for some $r \in (0,1)$. Further, $z_0 \in \partial K_r \cap L_r^\circ$, since $z_0 \notin M$. Let an open neighbourhood V of z_0 and $s \in (0,r)$ be chosen in such a way that $V \subseteq L_s$ and $V \subseteq K_1^\circ - K_s$. Hence, $z \in V$ implies that $z \in \partial K_t$ for some $t \in (s,1)$ and $z \in L_s \subseteq L_t$. Thus, $z \in \partial K_t \cap L_t = H_t \subseteq N$. This shows that $V \subseteq N$.

Case 2: $z_0 \in H_0 - M$. Then $z_0 \in \partial K \cap L^\circ$. There exist positive numbers $\delta > 0$ and $r_0 > 0$ such that

$$W := (B(p_0,\delta)^\circ + (-\infty,0)e_n) \cap K_{r_0}^\circ \subseteq L^\circ,$$

and hence

$$N \cap W = \{(y, f(y,r)) \mid y \in B'(x_0,\delta)^\circ, r \in [0,r_0)\}.$$

Case 3: $z_0 \in H_1 - M$. This case is analogous to Case 2.

Case 4: $z_0 \in M \cup H_0 \cup H_1$ and $z_0 \notin (H_0 - M) \cup (H_1 - M)$. This is equivalent to $z_0 \in M$. Thus, $z_0 \in S_{r_0}$ for some $r_0 \in [0,1]$. We can write $z_0 = (x_{01}, y_0, t_0)$ and choose neighbourhoods $U(x_{01})$, $U(y_0)$ and $U(r_0)$ as in the proof of Proposition 2.2. Certainly, we can assume that these neighbourhoods are open and convex sets. Consider the function

$$\Gamma(\cdot,y,r) = f(\cdot,y,r) - g(\cdot,y,r), \qquad (y,r) \in U(y_0) \times U(r_0),$$

defined on $U(x_{01})$. Recall from the proof of Proposition 2.2 that $\zeta(y,r)$ is the only solution of $\Gamma(\cdot,y,r) = 0$ on $U(x_{01})$ as long as $(y,r) \in U(y_0) \times U(r_0)$. Hence, $\Gamma(\cdot,y,r)$ is either strictly positive or strictly negative on each of the sets

$$U^+(y,r) := \{s \in U(x_{01}) \mid s > \zeta(y,r)\}$$

and

$$U^-(y,r) := \{s \in U(x_{01}) \mid s < \zeta(y,r)\}.$$

Claim 1. If $\Gamma(\cdot, y_0, r_0) > 0$ on $U^+(y_0, r_0)$, then $\Gamma(\cdot, y, r) > 0$ on $U^+(y, r)$ for any $(y, r) \in U(y_0) \times U(r_0)$.

Proof of Claim 1. Assume that there is some $(y_1, r_1) \in U(y_0) \times U(r_0)$ such that $\Gamma(\cdot, y_1, r_1) < 0$ on $U^+(y_1, r_1)$. Let $\epsilon > 0$ be so small that

$$\zeta((1-s)y_0 + sy_1, (1-s)r_0 + sr_1) + \epsilon \in U(x_{01})$$

for all $s \in [0, 1]$. Define

$$\omega(s) := (\zeta((1-s)y_0 + sy_1, (1-s)r_0 + sr_1) + \epsilon, (1-s)y_0 + sy_1, (1-s)r_0 + sr_1)\,,$$

where $s \in [0, 1]$. Then $\omega(s) \in U(x_{01}) \times U(y_0) \times U(r_0)$, for $s \in [0, 1]$, $\Gamma(\omega(0)) > 0$ and $\Gamma(\omega(1)) < 0$. Hence there is some $s^* \in [0, 1]$ for which $\Gamma(\omega(s^*)) = 0$. But this is a contradiction, since $\epsilon > 0$, which completes the proof of Claim 1.

Of course, similar statements hold under the assumption $\Gamma(\cdot, y_0, r_0) < 0$ or with $U^-(y_0, r_0)$ instead of $U^+(y_0, r_0)$.

Claim 2. Precisely one of the following two conditions is fulfilled.

(i) $\Gamma(\cdot, y, r) > 0$ on $U^+(y, r)$ and $\Gamma(\cdot, y, r) < 0$ on $U^-(y, r)$, for all $(y, r) \in U(y_0) \times U(r_0)$.

(ii) $\Gamma(\cdot, y, r) < 0$ on $U^+(y, r)$ and $\Gamma(\cdot, y, r) > 0$ on $U^-(y, r)$, for all $(y, r) \in U(y_0) \times U(r_0)$.

Proof of Claim 2. Assume that neither (i) nor (ii) is satisfied. Then according to Claim 1 we can infer that either $\Gamma(s, y, r) \geq 0$ for all $(s, y, r) \in U(x_{01}) \times U(y_0) \times U(r_0)$ or $\Gamma(s, y, r) \leq 0$ for all $(s, y, r) \in U(x_{01}) \times U(y_0) \times U(r_0)$. But then Γ has a local minimum or a local maximum at $(x_{01}, y_0, r_0) = (x_0, r_0)$. From Proposition 2.3.2 in [4] we then obtain that $0 \in \partial\Gamma(x_0, r_0)$, which contradicts Claim 2 in the proof of Proposition 2.2. This proves Claim 2.

Assume, for example, that (i) is satisfied. Then

$$\{(s, y, f(s, y, r)) \mid (s, y, r) \in U(x_{01}) \times U(y_0) \times U(r_0), s \geq \zeta(y, r)\} \subseteq \partial K_r \cap L_r$$

and

$$\{(s, y, f(s, y, r)) \mid (s, y, r) \in U(x_{01}) \times U(y_0) \times U(r_0), s < \zeta(y, r)\} \cap (\partial K_r \cap L_r) = \emptyset.$$

Defining the open set U as in the proof of Proposition 2.2, we now see that

$$\begin{aligned} &N \cap U \cap ((U(x_{01}) \times U(y_0)) + (-\infty, 0)e_n) \\ &\quad = \{(x, t) \in \partial K_r \cap L_r \mid (x, r) \in U(x_{01}) \times U(y_0) \times U(r_0)\} \\ &\quad = \{(s, y, f(s, y, r)) \mid (s, y, r) \in U(x_{01}) \times U(y_0) \times U(r_0), s \geq \zeta(y, r)\}. \end{aligned}$$

The case where (ii) is satisfied can be treated similarly.

If we collect the results that have been deduced in the preceding four cases, then we see that N enjoys the asserted properties.

Next we prove that H_r, $r \in [0,1]$, is an $(n-1)$-dimensional Lipschitz manifold with boundary $\partial H_r = S_r$. Let r be fixed and $z_0 \in H_r$.

Case 1: $z_0 \in H_r - S_r$. Then $z_0 \in \partial K_r \cap L_r^\circ$. Let $\delta > 0$ be sufficiently small so that

$$W := (B(p_0,\delta)^\circ + (-\infty,0)e_n) \cap \partial K_r \subseteq L_r^\circ.$$

Then

$$H_r \cap W = \{(x, f(x,r) \mid x \in B'(x_0,\delta)^\circ\}.$$

Case 2: $z_0 \in S_r$. Using the same arguments as in Case 4 above and under the assumption that, for instance, condition (i) of Claim 2 above is satisfied, we obtain

$$\begin{aligned} &H_r \cap U \cap ((U(x_{01}) \times U(y_0)) + (-\infty,0)e_n) \\ &\quad = \{(x,t) \in \partial K_r \cap L_r \mid x \in U(x_{01}) \times U(y_0)\} \\ &\quad = \{(s,y,f(s,y,r)) \mid (s,y) \in U(x_{01}) \times U(y_0), s \geq \zeta(y,r)\}. \end{aligned}$$

From the representations derived in these two cases the assertion follows immediately. □

2.2. A Morse type lemma

In this subsection, we apply elementary techniques from Morse theory to show that

$$S_r \cong S_s \quad \text{for all } r,s > 0. \tag{2.7}$$

We write $d_K(x) := d(K,x)$ for the distance of $x \in \mathbb{R}^n$ to K. Clearly, $d_K(\cdot)$ is a Lipschitz function. In $\mathbb{R}^n - K$, it is known that $d_K \in \mathrm{C}^1$ and

$$\nu_K(x) := \nabla d_K(x) = \frac{x - p_K(x)}{d_K(x)}$$

is the outer unit normal at $x \in \partial K_r$ of the parallel body K_r, $r = d_K(x)$, where $p_K(x)$ denotes the unique next point of K to x. Likewise, we define d_L for L. We see that

$$d_K \in \mathrm{C}^{1,1}_{loc}([d_K > 0]) \cap \mathrm{C}^{0,1}(\mathbb{R}^n), \qquad d_L \in \mathrm{C}^{1,1}_{loc}([d_L > 0]) \cap \mathrm{C}^{0,1}(\mathbb{R}^n). \tag{2.8}$$

Moreover, we observe that $\partial K_r = [d_K = r]$ and $\partial L_r = [d_L = r]$ for $r > 0$.
As in Morse theory, we consider a flow which is induced by

$$F(x) := \frac{\nu_K + \nu_L}{1 + \langle \nu_K, \nu_L \rangle}(x) \quad \text{for } x \in V, \tag{2.9}$$

where $V := \mathbb{R}^n - (K \cup L) = [d_K > 0, d_L > 0]$. The flow F is well-defined on V, that is, $1 + \langle \nu_K, \nu_L \rangle > 0$ on V. Indeed, if there is $x \in V$ which satisfies $d_K(x) = r > 0$,

$d_L(x) = s > 0$ and $1 + \langle \nu_K(x), \nu_L(x) \rangle = 0$, then we infer from $|\nu_K(x)| = |\nu_L(x)| = 1$ that

$$\nu_K(x) = -\nu_L(x).$$

As $\nu_K(x) \in N(K_r, x)$ and $\nu_L(x) \in N(L_s, x)$ for $r, s > 0$, this contradicts (2.2).

Obviously, F satisfies

$$\langle F, \nu_K \rangle = \langle F, \nu_L \rangle = 1.$$

Observing the continuity properties of $\nu_K = \nabla d_K,\ \nu_L = \nabla d_L$ given in (2.8), we obtain

$$F \in \mathrm{C}^{0,1}_{loc}(V). \tag{2.10}$$

Hence there is a unique local solution of the ordinary differential equation

$$x'(t) = F(x(t)), \quad x(r) = x_0, \tag{2.11}$$

for $x_0 \in S_r$, on some maximal interval I_{x_0} containing r. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} d_K(x(t)) = \langle \nu_K(x(t)), F(x(t)) \rangle = 1,$$

and likewise for L. We obtain $d_K(x(t)) = d_L(x(t)) = t$, hence

$$x(t) \in S_t \quad \text{for } t \in I_{x_0}.$$

Since $x(t)$ approaches the boundary of V when t approaches an endpoint of I_{x_0}, we conclude that

$$I_{x_0} = (0, \infty).$$

Taking any $x_0 \in S_1$, we obtain a continuous map

$$\Phi : S_1 \times (0, \infty) \to \bigcup_{r>0} S_r \tag{2.12}$$

such that $\Phi(x_0, t) = x(t) \in S_t$. Since the solution of (2.11) is unique, we see that Φ is bijective, and the theory of ordinary differential equations implies that Φ is even a homeomorphism. We put $\varphi_t(x) := \Phi(x, t)$ and see that φ_t induces a homeomorphism

$$\varphi_t : S_1 \to S_t \quad \text{for } t > 0. \tag{2.13}$$

Next we want to extend the map Φ to $t = 0$ in order to get a continuous, surjective map $\varphi : S_1 \to S_0$. To this end, we need a uniform Lipschitz bound on the solutions of (2.11) for small $t > 0$, that is a bound on F. The next lemma implies that

$$|F| \le C \quad \text{on } [0 < d_K, d_L \le 1], \tag{2.14}$$

for some $0 < C < \infty$.

Lemma 2.4. *There is a constant $c_0 > 0$ such that*

$$1 + \langle \nu_K(x), \nu_L(x) \rangle \ge c_0 \quad \text{for } x \in [0 < d_K, d_L \le 1]. \tag{2.15}$$

P r o o f. If there is no such c_0, then there are points $x_j \in \mathbb{R}^n$, $j \in \mathbb{N}$, such that

$$0 < d_K(x_j), d_L(x_j) \leq 1$$

and

$$1 + \langle \nu_K(x_j), \nu_L(x_j) \rangle \to 0 \qquad \text{for } j \to \infty.$$

As K_1, L_1 are compact and $|\nu_K| = |\nu_L| = 1$, we may assume, by passing to a subsequence, that $x_j \to x_0$, $d_K(x_0) = r \in [0,1]$, $d_L(x_0) = s \in [0,1]$, $\nu_K(x_j) \to \xi_K$, and $\nu_L(x_j) \to \xi_L$, as $j \to \infty$. We infer that $x_0 \in \partial K_r \cap \partial L_s$, $|\xi_K| = |\xi_L| = 1$, $1 + \langle \xi_K, \xi_L \rangle = 0$, and hence

$$\xi_K = -\xi_L.$$

As ν_K, ν_L are the outer normals at parallel bodies of the convex bodies K, L, respectively, we conclude that

$$\xi_K \in N(K_r, x_0) \quad \text{and} \quad \xi_L \in N(L_s, x_0).$$

As $\xi_K = -\xi_L$, this contradicts (2.2). □

Relation (2.14) implies that

$$|\Phi(x,t) - \Phi(x,s)| \leq C|t-s|$$

for $x \in S_1$ and $0 < s, t \leq 1$. We define

$$\Phi(x,0) := \varphi(x) := \lim_{t \downarrow 0} \Phi(x,t),$$

as the limit exists. Since $d_K(\Phi(x,t)) = d_L(\Phi(x,t)) = t$, we obtain $\Phi(x,0) = \varphi(x) \in S_0$. Some properties of Φ and φ, which we need subsequently, are stated in the next lemma.

Lemma 2.5. *The maps*

$$\Phi : S_1 \times [0,1] \to M = \bigcup_{0 \leq r \leq 1} S_r \qquad \text{and} \qquad \varphi : S_1 \to S_0$$

are continuous and surjective.

P r o o f. For $x, y \in S_1$ and $0 < t \leq 1$, we obtain

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq |\varphi(x) - \Phi(x,t)| + |\varphi_t(x) - \varphi_t(y)| + |\Phi(y,t) - \varphi(y)| \\ &\leq 2C|t| + \omega_t(|x-y|), \end{aligned}$$

where ω_t is a modulus of continuity for the continuous function φ_t. As $|t| > 0$ can be chosen arbitrarily small, we infer that φ is continuous.

Next we consider $(x_j, t_j) \to (x_0, 0)$ for $x_j, x_0 \in S_1$. We obtain

$$\begin{aligned} |\Phi(x_j,t_j) - \Phi(x_0,0)| &\leq |\Phi(x_j,t_j) - \Phi(x_j,0)| + |\varphi(x_j) - \varphi(x_0)| \\ &\leq C|t_j| + \omega(|x_j - x_0|), \end{aligned}$$

where ω is a modulus of continuity for the continuous function φ. As $t_j \to 0$ and $x_j \to x_0$, we see that $\Phi(x_j, t_j) \to \Phi(x_0, 0)$, and thus Φ is continuous.

To prove surjectivity, we choose $x_0 \in S_0$. By Proposition 2.2(a), M is a topological manifold with boundary $\partial M = S_0 \cup S_1$. Hence, there exist

$$x_j \in M - \partial M = \bigcup_{0<r<1} S_r, \quad j \in \mathbb{N},$$

such that

$$x_j \to x_0 \qquad \text{as } j \to \infty.$$

As $\Phi : S_1 \times (0, \infty) \to \bigcup_{r>0} S_r$ is surjective, for each $j \in \mathbb{N}$ there are $\tilde{x}_j \in S_1$ and $r_j > 0$ such that

$$\Phi(\tilde{x}_j, r_j) = x_j.$$

We infer that $r_j \to 0$ and, as S_1 is compact, passing to a subsequence (if necessary) we may assume that $\tilde{x}_j \to \tilde{x} \in S_1$. As Φ is continuous according to the first part of the proof, we obtain

$$\varphi(\tilde{x}) = \Phi(\tilde{x}, 0) \leftarrow \Phi(\tilde{x}_j, t_j) = x_j \to x_0 \qquad \text{as } j \to \infty,$$

and therefore Φ and φ are surjective. □

As continuous, surjective maps from compact sets are identification or quotient maps (see [6, VI.1.4 and XI.2.1]), the lemma yields that Φ and φ are quotient maps. Since the set $S_1 \times [0,1] \times [0,1]$ is compact, we obtain that

$$\Phi \times id : S_1 \times [0,1] \times [0,1] \to M \times [0,1]$$

is a quotient map, too. That $\Phi \times id$ is a quotient map for a quotient map Φ follows by a theorem of Whitehead (see [6, XII.4.1]), even without using that $S_1 \times [0,1] \times [0,1]$ is compact.

We use these quotient maps in the next proposition to construct a homotopy which will show that S_0 is a strong deformation retract of M.

Proposition 2.6. *The set S_0 is a strong deformation retract of M, hence these sets are of the same homotopy type*

$$S_0 \simeq M.$$

Proof. Clearly, $S_1 \times \{0\}$ is a strong deformation retract of $S_1 \times [0,1]$ via the homotopy $H : S_1 \times [0,1] \times [0,1] \to S_1 \times [0,1]$ defined by

$$H(x, t, s) := (x, t(1-s)).$$

We consider the commutative diagram

$$\begin{array}{ccc} S_1 \times [0,1] \times [0,1] & \xrightarrow{\quad H \quad} & S_1 \times [0,1] \\ \Big\downarrow \Phi \times id & & \Big\downarrow \Phi \\ M \times [0,1] & \xrightarrow{\quad \tilde{H} \quad} & M \end{array}$$

As H preserves the relations induced by $\Phi \times id$ and Φ, that is

$$[(\Phi \times id)(x,t,s) = (\Phi \times id)(x',t',s')] \Rightarrow [(\Phi \circ H)(x,t,s) = (\Phi \circ H)(x',t',s')],$$

passing to the quotient (see [6, I.7.7]), we obtain a map $\tilde{H} : M \times [0,1] \to M$ such that the diagram commutes. Further, $\tilde{H}$ is continuous by [6, VI.4.3], as $\Phi \times id$ is a quotient map.

The map $\tilde{H}$ is the desired strong deformation retraction for S_0. Indeed,

$$\begin{array}{lll} \tilde{H}(x,0) & = x & \text{for } x \in M, \\ \tilde{H}(y,s) & = y & \text{for } y \in S_0,\ s \in [0,1], \\ \tilde{H}(x,1) & \in S_0 & \text{for } x \in M. \end{array}$$

These assertions are easily verified from the definition of H. □

Since it has already been proved that Φ is a homeomorphism, the following lemma is much easier obtained than Proposition 2.6.

Lemma 2.7. *The set $S_{\frac{1}{2}}$ is a strong deformation retract of $M - \partial M$, hence these sets are of the same homotopy type*

$$S_{\frac{1}{2}} \simeq M - \partial M.$$

Proof. Clearly, $S_1 \times \{\frac{1}{2}\}$ is a strong deformation retract of $S_1 \times (0,1)$. As

$$\Phi : \left(S_1 \times (0,1), S_1 \times \{\tfrac{1}{2}\}\right) \to \left(\bigcup_{0<r<1} S_r, S_{\frac{1}{2}} \right)$$

induces a homeomorphism (see (2.12)), we obtain that $S_{\frac{1}{2}}$ is a strong deformation retract of $\bigcup_{0<r<1} S_r$. According to Proposition 2.2(a), we know that $\bigcup_{0<r<1} S_r = M - \partial M$, which concludes the proof. □

From (2.13) we now obtain

Proposition 2.8. *The sets S_r and $M - \partial M$ are of the same homotopy type, that is*

$$S_r \simeq M - \partial M \quad \textit{for } r > 0.$$

2.3. Collaring

Theorem 1.3 will follow from Propositions 2.6 and 2.8 when we know that

$$i : M - \partial M \hookrightarrow M \tag{2.16}$$

is a homotopy equivalence, that is $M - \partial M$ is a weak deformation retract of M. In [30, p. 297], (2.16) is mentioned as an easy consequence of the following theorem due to Brown [3, Theorem 2].

Theorem (Brown). *The boundary of a topological manifold with boundary is collared.*

A subset $B \subseteq M$ is called collared in M if there is a homeomorphism

$$h : (B \times [0,1), B \times \{0\}) \to (U, B)$$

onto some open neighbourhood U of B; see [3, II]. It is immediate that the boundary of a manifold is locally collared, and Brown's Theorem states that the boundary has a global collaring.

For the sake of completeness, we give the arguments for proving (2.16) with the use of Brown's Theorem.

Corollary 2.9. *Let Y be a compact topological manifold with boundary ∂Y. Then $Y - \partial Y$ is a weak deformation retract of Y.*

Proof. According to Brown's Theorem, we may identify an open neighbourhood $U \subseteq Y$ of ∂Y with $\partial Y \times [0,1)$, that is we assume that

$$U = \partial Y \times [0,1) \subseteq Y$$

is an open neighbourhood of $\partial Y = \partial Y \times \{0\}$. We put $Z := Y - (\partial Y \times [0, \frac{1}{2}))$ which is closed as $\partial Y \times [0, \frac{1}{2})$ is open, and $Z' := Y - (\partial Y \times [0, \frac{1}{2}])$ which is open as ∂Y and $\partial Y \times [0, \frac{1}{2}]$ are compact, and hence are closed in Y.

We claim that Z is a strong deformation retract of Y. To this end, we define a homotopy

$$H : Y \times [0,1] \to Y$$

by putting

$$H(y,s) := \begin{cases} y & \text{if } y \in Z', \\ (\tilde{y}, t - s\min\{0, t - \frac{1}{2}\}) & \text{if } y = (\tilde{y}, t) \in U = \partial Y \times [0,1). \end{cases}$$

The map H is well-defined and continuous, as $Y = Z' \cup U$ and Z' and U are open. Moreover,

$$\begin{array}{lll} H(y,0) & = y & \text{for } y \in Y, \\ H(z,s) & = z & \text{for } z \in Z,\ s \in [0,1], \\ H(y,1) & \in Z & \text{for } y \in Y, \end{array}$$

which yields that Z is a strong deformation retract of Y.

We observe that $H : (Y - \partial Y) \times [0,1] \to Y - \partial Y$, and $Z \subseteq Y - \partial Y$ is a strong deformation retract of $Y - \partial Y$. Therefore the inclusion maps on the left side in the commutative diagram

$$\begin{tikzcd} & Y - \partial Y \arrow[d, "i"] \\ Z \arrow[ur] \arrow[r] & Y \end{tikzcd}$$

are homotopy equivalences, hence so is i, and thus $Y - \partial Y$ is a weak deformation retract of Y. $\square$

As already mentioned, Theorem 1.3 is a consequence of (2.16) and the Propositions 2.6 and 2.8.

2.4. Intersection of boundary and body

In this subsection, we supplement the details for the proof of Theorem 1.4. We use the notation of Section 2.2.

First, we modify the flow defined in (2.9) to prove that

$$H_r \cong H_s \quad \text{for all } r, s > 0. \tag{2.17}$$

The flow F was defined in $V = [d_K > 0, d_L > 0]$. It transforms S_r into S_s. Now we are looking for a flow $\tilde{F}$ which transforms H_r into H_s. We recall that $S_r = [d_K = d_L = r]$ and $H_r = [d_K = r, d_L \leq r]$ for $r > 0$. As $\partial H_r = S_r$, according to Proposition 2.3, we take $\tilde{F} = F$ on $[d_K = d_L > 0]$ and extend it on $\tilde{V} := [d_K > 0]$. More precisely, we put

$$\tilde{F}(x) := \eta(d_K(x), d_L(x)) \frac{\nu_K + \nu_L}{1 + \langle \nu_K, \nu_L \rangle}(x) + (1 - \eta(d_K(x), d_L(x)))\nu_K(x), \tag{2.18}$$

for $x \in \tilde{V}$, where $\eta : (0,\infty) \times \mathrm{I\!R} \to [0,1]$, $\eta \in \mathrm{C}^{0,1}_{loc}((0,\infty) \times \mathrm{I\!R})$, is such that

$$\eta(r,r) = 1 \quad \text{for } r > 0,$$

and

$$\eta(r,s) = 0 \quad \text{for } r > 0, s < \frac{r}{2}.$$

For example, we may choose

$$\eta(r,s) := \min\left\{ r^{-1} \max\{2s - r, 0\}, 1 \right\} \quad \text{for } r > 0.$$

Observing (2.8), (2.10), and the properties of η, we see that

$$\tilde{F} \in \mathrm{C}^{0,1}_{loc}(\tilde{V}).$$

Moreover, we calculate that

$$\langle \tilde{F}, \nu_K \rangle = 1 \quad \text{on } \tilde{V}$$

and

$$\langle \tilde{F}, \nu_L \rangle = 1 \quad \text{on } \bigcup_{r>0} S_r.$$

As in Section 2.2, there is a unique local solution of the ordinary differential equation

$$x'(t) = \tilde{F}(x(t)), \quad x(r) = x_0, \tag{2.19}$$

for $x_0 \in \partial K_r \subseteq \tilde{V}$, on some maximal interval J_{x_0} containing $r > 0$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} d_K(x(t)) = \langle \nu_K(x(t)), \tilde{F}(x(t)) \rangle = 1,$$

hence

$$x(t) \in \partial K_t \quad \text{for } t \in J_{x_0}.$$

Furthermore, $J_{x_0} = (0, \infty)$ since $x(t)$ approaches the boundary of $\tilde{V} = [d_K > 0]$ when t approaches an endpoint of J_{x_0}. Since the solution of the ordinary differential equation (2.19) is unique, we see that the flow $\tilde{F}$ establishes a homeomorphism

$$\Psi_{r,s} : \partial K_r \cong \partial K_s \quad \text{for } r, s > 0$$

which maps $x(r) = x_0$ to $x(s)$.

Next, when $x_0 \in S_r \subseteq \partial K_r$, we consider the solution x of (2.11) and observe, as $x(t) \in S_t$ and $F = \tilde{F}$ on $\bigcup_{r>0} S_r$, that x is the solution of (2.19) as well. Therefore $\Psi_{r,s}(S_r) = S_s$. This yields

$$\Psi_{r,s}(H_r) = H_s \quad \text{for } r, s > 0. \tag{2.20}$$

Indeed, $H_r = \partial K_r \cap [d_L \le r]$. Let $x_0 \in H_r$. If $d_L(x_0) = r$, then $x_0 \in S_r$ and $x(t) \in S_t \subseteq H_t$ for all $t > 0$. If $d_L(x_0) < r$, then $x_0 \notin S_r$ and $x(t) \notin S_t$, as $\Psi_{r,t}(S_r) = S_t$ and $\Psi_{r,t} : \partial K_r \cong \partial K_t$ is bijective. Therefore $d_L(x(t)) \neq t$ for all $t > 0$. As $d_L(x(r)) < r$, we see that $d_L(x(t)) < t$ for all $t > 0$, and $x(t) \in H_t$. Hence $\Psi_{r,s}(H_r) \subseteq H_s$, and equality is concluded by symmetry in r, s.

Clearly, (2.20) implies (2.17).

Taking any $x_0 \in H_1$, we obtain a continuous map

$$\Psi : (H_1, S_1) \times (0, \infty) \to \left(\bigcup_{r>0} H_r, \bigcup_{r>0} S_r \right) \tag{2.21}$$

such that $\Psi(x_0, t) = x(t) \in H_t$. As the solutions of (2.19) are unique, Ψ is a homeomorphism. We put $\psi_t(x) := \Psi(x, t)$, and thus ψ_t induces a homeomorphism

$$\psi_t : (H_1, S_1) \to (H_t, S_t) \quad \text{for } t > 0. \tag{2.22}$$

We want to extend Ψ to $t = 0$. Lemma 2.4 shows that

$$|\tilde{F}| \leq C \quad \text{on } [0 < d_K \leq 1, d_L \leq 1],$$

and Ψ admits a uniform Lipschitz constant in the variable t. As in Lemma 2.5, we can extend Ψ and ψ_t to obtain continuous and surjective maps, hence quotient maps,

$$\Psi : H_1 \times [0,1] \to N = \bigcup_{0 \leq r \leq 1} H_r \quad \text{and} \quad \psi : H_1 \to H_0.$$

The surjectivity is obtained by using that $H_0 \subseteq \partial N \subseteq \overline{N - \partial N}$; see Proposition 2.3.

Next, using these quotient maps, we conclude as in Lemma 2.6 that H_0 is a strong deformation retract of N, in particular

$$H_0 \simeq N. \tag{2.23}$$

We recall that $\partial H_{\frac{1}{2}} = S_{\frac{1}{2}}$ and $\partial N = (\bigcup_{0<r<1} S_r) \cup H_0 \cup H_1$, see Proposition 2.3. Knowing that $\Psi : (H_1, S_1) \times (0,1) \to (\bigcup_{0<r<1} H_r, \bigcup_{0<r<1} S_r)$ is a homeomorphism, we see as in Lemma 2.7 that $H_{\frac{1}{2}} - \partial H_{\frac{1}{2}}$ is a strong deformation retract of $N - \partial N$. In particular,

$$H_{\frac{1}{2}} - \partial H_{\frac{1}{2}} \simeq N - \partial N. \tag{2.24}$$

Finally, the proof of Theorem 1.4 is concluded again by Brown's Theorem and its Corollary 2.9, which yields that

$$H_{\frac{1}{2}} \simeq H_{\frac{1}{2}} - \partial H_{\frac{1}{2}} \quad \text{and} \quad N \simeq N - \partial N. \tag{2.25}$$

Then (2.17), (2.23), (2.24) and (2.25) together imply

$$H_r \simeq H_0 \quad \text{for } r \geq 0,$$

which is the assertion of Theorem 1.4.

3. The translative integral formulas

In this final section, we shall prove Theorems 1.1 and 1.2. The basic idea of our approach has already been outlined in the Introduction.

The following crucial lemma is implicitly contained in a paper of Ewald, Larman and Rogers [7]. It is also used by Heinrich and Molchanov [16]. For the reader's convenience we include the complete argument. Subsequently, we write $\mathcal{H}^r$, $r \geq 0$, for the r-dimensional Hausdorff measure in a Euclidean space. It will always be clear from the context which space is meant.

Lemma 3.1. *Let $K, L \subset \mathbb{R}^n$ be convex bodies. Then the set $T(K, L)$ of all $t \in \mathbb{R}^n$ such that K and $L + t$ do not intersect almost transversally has $\mathcal{H}^n$ measure zero.*

Proof. Certainly, $T(K,L) = T_1(K,L) \cup T_2(K,L)$, where $T_1(K,L)$ denotes the set of all $t \in \mathbb{R}^n$ for which there is some $x \in \partial K \cap \partial(L+t)$ such that

$$N(K,x) \cap N(L+t,x) \neq \{0\}$$

and $T_2(K,L)$ is the set of all $t \in \mathbb{R}^n$ for which there is some $x \in \partial K \cap \partial(L+t)$ such that

$$N(K,x) \cap (-N(L+t,x)) \neq \{0\}.$$

It is easy to check that $T_2(K,L) = \partial(K + (-L))$.

For a convex body $M \subset \mathbb{R}^n$ and $u \in S^{n-1}$, let $F(M,u)$ denote the support set of M with exterior normal vector u, and let $h(M,\cdot) : \mathbb{R}^n \to \mathbb{R}$ denote the support function of M; see [25] for explicit definitions. Choose any $t \in T_1(K,L)$. Then there exists a point $x \in \partial K \cap \partial(L+t)$ and a unit vector $u \in N(K,x) \cap N(L+t,x)$, and thus

$$\langle x, u\rangle = h(K,u) \quad \text{and} \quad \langle x-t, u\rangle = h(L,u).$$

Then $t = x - (x-t) \in \partial K + (-\partial L)$ and $t \in F(K,u) + (-F(L,u))$. This shows that

$$T_1(K,L) \subseteq \bigcup_{u \in S^{n-1}} [F(K,u) + (-F(L,u))] =: \Lambda(L,K).$$

Let $\epsilon > 0$ be arbitrarily chosen. Then Theorem 2.3.2 in [25] implies that there are caps $C_1, \dots, C_m$ whose union covers $\partial(K+L)$ and for which

$$\sum_{i=1}^m \mathcal{H}^n(C_i) \leq c_1 \epsilon,$$

where c_1 is a constant independent of ϵ. Further, Lemma 2.3.9 in [25] yields that

$$\Lambda(L,K) \subseteq \bigcup_{i=1}^m [C_i + (-C_i) + a_i],$$

where $a_1, \dots, a_m$ are suitable translation vectors. This implies

$$\mathcal{H}^n(\Lambda(L,K)) \leq \sum_{i=1}^m \mathcal{H}^n(C_i + (-C_i)) \leq \sum_{i=1}^m (n+1)^n \mathcal{H}^n(C_i) \leq c_2 \epsilon,$$

where c_2 is a constant independent of ϵ. This concludes the proof since $\epsilon > 0$ was arbitrary. □

Using Lemma 3.1 and the continuity properties of mixed volumes, we shall see that Theorems 1.1 and 1.2 follow from the next lemma.

Lemma 3.2. *Let $K, L \subset \mathbb{R}^n$ be convex bodies, and let $r > 0$. Define $X := K_r$ and $Y := L_r$. Then Theorems 1.1 and 1.2 hold for the parallel bodies X and Y.*

Proof. First we note that ∂X and ∂Y are sets of positive reach in the sense of Federer [8]. In fact, these sets are $C^{1,1}$ Lipschitz submanifolds of $\mathbb{R}^n$. Of course, the same is true for X and Y. For general sets of positive reach, FEDERER [8] introduced curvature measures as Borel measures on $\mathbb{R}^n$. Subsequently, M. ZÄHLE [36] extended this concept to (generalized) curvature measures, as Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$, and simplified the original approach. For a set $R \subset \mathbb{R}^n$ of positive reach, these measures are denoted by $C_i(R, \cdot)$, $i = 0, \ldots, n-1$. They can be obtained as coefficient measures of a local Steiner formula. We refer to [36] and to [19] for a more detailed description of these measures; see also [36] and [19] for the particular normalization of these measures which is adopted in the present paper. Next we note that if $X \cap (Y + t) \neq \emptyset$ and $t \notin T_2(X, Y)$, then $\partial X \cap (\partial Y + t)$ is a set of positive reach and

$$\chi(\partial X \cap (\partial Y + t)) = C_0(\partial X \cap (\partial Y + t), \mathbb{R}^n \times \mathbb{R}^n). \tag{3.1}$$

This is implied by Theorems 4.10 and 5.19 in [8]. Hence it follows from results provided in [8] that $t \mapsto \chi(\partial X \cap (\partial Y + t))$ is Borel measurable if we possibly redefine this map on the Borel set $T_2(X, Y)$ of measure zero where both sides of equation (3.1) might not be defined.

Essentially, Lemma 3.2 will be proved by applying a translative integral formula, due to RATAJ and M. ZÄHLE [19], in the present special context. Since we shall need some specific information about certain mixed curvature measures which are involved in the statement of this formula, we introduce some additional notation.

Let $\nu_X(x)$, $x \in \partial X$, denote the uniquely determined exterior unit normal vector of X at x. Let $N(X)$ and $N(\partial X)$ denote the generalized unit normal bundles of X and ∂X, respectively. A general definition is given, e.g., in [36]. In the present particular setting,

$$N(X) = \{(x, \nu_X(x)) \in \mathbb{R}^n \times S^{n-1} : x \in \partial X\}$$

and

$$N(\partial X) = N(X) \cup \{(x, -\nu_X(x)) \in \mathbb{R}^n \times S^{n-1} : x \in \partial X\}.$$

Since $X = K_r$ and $r > 0$, the map ν_X is Lipschitz and hence differentiable $\mathcal{H}^{n-1}$-almost everywhere on ∂X. Thus, by specializing the general definitions in [36], the generalized curvature functions

$$\kappa_1(X, x, m), \ldots, \kappa_{n-1}(X, x, m) \in [0, \infty)$$

can be obtained, for $\mathcal{H}^{n-1}$-almost all $x \in \partial X$ and $m = \nu_X(x)$, as the eigenvalues of the symmetric linear map $D\nu_X(x) : \nu_X(x)^\perp \to \nu_X(x)^\perp$. Moreover, for $\mathcal{H}^{n-1}$-almost all $(x, m) \in N(X)$ and $i = 1, \ldots, n-1$,

$$\kappa_i(\partial X, x, m) = \kappa_i(X, x, m) \in [0, \infty)$$

and

$$\kappa_i(\partial X, x, -m) = -\kappa_i(X, x, m) \in (-\infty, 0].$$

A common eigenvector corresponding to $\kappa_i(\partial X, x, m)$, $\kappa_i(\partial X, x, -m)$ and $\kappa_i(X, x, m)$ is denoted by $a_i(\partial X, x, \pm m) = a_i(X, x, m)$, for $i = 1, \ldots, n-1$ and $\mathcal{H}^{n-1}$-almost all $(x, m) \in N(X)$. In particular, these eigenvectors are chosen in such a way that they represent an orthonormal basis of $m^\perp$. Similar results and notations are used for Y. All these definitions are consistent with those given in [36] for arbitrary sets of positive reach. Finally, for vectors $c_1, \dots, c_n \in \mathbb{R}^n$ the volume of the parallelepiped spanned by these vectors is denoted by $\|c_1 \wedge \dots \wedge c_n\|$, since this volume equals the norm of the n-vector $c_1 \wedge \dots \wedge c_n$.

After these preparations we can describe the mixed curvature measures

$$\Psi_{l,n-l}(R_1, R_2; \cdot), \quad l \in \{0, \dots, n\},$$

where $R_1, R_2 \in \{X, Y, \partial X, \partial Y\}$. These measures are Borel measures on subsets of $(\mathbb{R}^n)^3$. In [19], these measures are defined by an integration over the joint unit normal bundle of R_1 and R_2 (that is, by means of a joint unit normal current of R_1 and R_2), for general sets R_1, R_2 of positive reach. We shall merely need the projections of the mixed curvature measures onto the first two components, and hence we set

$$\overline{\Psi}_{l,n-l}(R_1, R_2; A) := \Psi_{l,n-l}(R_1, R_2; A \times \mathbb{R}^n),$$

for an arbitrary Borel set $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$. By Theorem 2 in [19], for $l = 1, \dots, n-1$ the measures $\overline{\Psi}_{l,n-l}(R_1, R_2; \cdot)$ can be represented as

$$\begin{aligned} \overline{\Psi}_{l,n-l}(R_1, R_2; A) \quad = \quad & \int_{N(R_1) \times N(R_2)} \mathbf{1}_A(x, y) F_{l,n-l}(\theta(m, n)) \\ & \left(\prod_{i=1}^{n-1} \sqrt{1 + \kappa_i(R_1, x, m)^2} \sqrt{1 + \kappa_i(R_2, y, n)^2} \right)^{-1} \\ & \sum_{|I|=l} \sum_{|J|=n-l} \prod_{i \in I^c} \kappa_i(R_1, x, m) \prod_{j \in J^c} \kappa_j(R_2, y, n) \\ & \left\| \bigwedge_{i \in I} a_i(R_1, x, m) \wedge \bigwedge_{j \in J} a_j(R_2, y, n) \right\|^2 \mathcal{H}^{2n-2}(d(x, m, y, n)), \end{aligned}$$

where $\theta(m, n) \in (0, \pi)$ denotes the angle between two vectors $m, n \in S^{n-1}$ with $m \neq \pm n$ and

$$F_{l,n-l}(\theta) = (\mathcal{H}^{n-1}(S^{n-1}))^{-1} \frac{\theta}{\sin\theta} \int_0^1 \left(\frac{\sin(t\theta)}{\sin\theta} \right)^{l-1} \left(\frac{\sin((1-t)\theta)}{\sin\theta} \right)^{n-1-l} dt,$$

$\theta \in (0, \pi)$. It should be emphasized that for $m = \pm n$ and $|I| = n - |J|$ we obtain

$$\left\| \bigwedge_{i \in I} a_i(R_1, x, m) \wedge \bigwedge_{j \in J} a_j(R_2, y, n) \right\| = 0,$$

since then all vectors lie in the orthogonal complement of m. So if $m = \pm n$, then the integrand is defined to be zero. We also mention that

$$\overline{\Psi}_{0,n}(R_1, R_2; A \times B) = C_0(R_1, A \times \mathbb{R}^n) \mathcal{H}^n(R_2 \cap B) \tag{3.2}$$

and

$$\overline{\Psi}_{n,0}(R_1, R_2; A \times B) = \mathcal{H}^n(R_1 \cap A) C_0(R_2, B \times \mathbb{R}^n), \tag{3.3}$$

for Borel sets $A, B \subseteq \mathbb{R}^n$. By Lemma 3.1, condition (6) of Theorem 1 in [19] is satisfied for the sets ∂X and ∂Y. Thus we can apply this theorem to the sets ∂X and ∂Y in $\mathbb{R}^n$ (and for $k = 0$ in the notation of [19]). Let $h : \mathbb{R}^n \to [0, \infty)$ be an arbitrary Borel measurable function. Furthermore, let $B \subset (\mathbb{R}^n)^3$ denote a bounded Borel set which contains $(X + (-Y)) \times X \times S^{n-1}$ in its interior. If we use the fact that the curvature measure $C_0(\partial X \cap (\partial Y + t), \cdot)$ is defined at least for $\mathcal{H}^n$-almost all $t \in \mathbb{R}^n$ such that $\partial X \cap (\partial Y + t) \neq \emptyset$ and concentrated on $\partial X \times S^{n-1}$, $\partial X \cap (\partial Y + t) = \emptyset$ if $t \notin X + (-Y)$, $\Psi_{l,n-l}(\partial X, \partial Y; \cdot)$ is concentrated on $\partial X \times \partial Y \times S^{n-1}$ for $l = 0, \ldots, n$, and equation (3.1), then an application of Theorem 1 in [19] to the function $\mathbf{1}_B h$ yields

$$\begin{aligned}\int_{\mathbb{R}^n} & h(t)\chi(\partial X \cap (\partial Y + t))\mathcal{H}^n(dt) \\ &= \sum_{l=0}^{n} \int_{(\mathbb{R}^n)^2} h(x - y)\overline{\Psi}_{l,n-l}(\partial X, \partial Y; d(x, y)).\end{aligned} \tag{3.4}$$

Since $\mathcal{H}^n(\partial X) = \mathcal{H}^n(\partial Y) = 0$, equations (3.2) and (3.3) imply that the summands corresponding to $l = 0$ and $l = n$ vanish.

Next we consider the following decomposition of the essential domain of integration for each summand $I(h, l)$, $l = 1, \ldots, n - 1$, on the right-hand side of equation (3.4):

$$\begin{aligned} N(\partial X) \times N(\partial Y) \quad = \quad & (N(X) \times N(Y)) \cup (\{(x, -m) : (x, m) \in N(X)\} \times N(Y)) \\ & \cup (N(X) \times \{(y, -n) : (y, n) \in N(Y)\}) \\ & \cup (\{(x, -m) : (x, m) \in N(X)\} \times \{(y, -n) : (y, n) \in N(Y)\}). \end{aligned}$$

This decomposition leads to a corresponding decomposition of $I(h, l)$ according to

$$I(h, l) = I_1(h, l) + \ldots + I_4(h, l), \quad l = 1, \ldots, n - 1. \tag{3.5}$$

Thus, for $l = 1, \ldots, n - 1$ we obtain

$$I_1(h, l) = \int_{(\mathbb{R}^n)^2} h(x - y)\overline{\Psi}_{l,n-l}(X, Y; d(x, y)).$$

In order to treat the second summand $I_2(h, l)$, we first apply the isometric map $(x, m, y, n) \mapsto (-x, m, y, n)$ and then use the relations

$$\kappa_i(\partial X, -x, m) = -\kappa_i(-\partial X, x, m) = -\kappa_i(-X, x, m)$$

and

$$a_i(\partial X, -x, m) = \pm a_i(-\partial X, x, m) = \pm a_i(-X, x, m),$$

for $i = 1, \ldots, n - 1$ and $\mathcal{H}^{n-1}$-almost all $(x, m) \in N(-X)$. Thus we obtain

$$I_2(h, l) = (-1)^{n-1-l} \int_{(\mathbb{R}^n)^2} h(-x - y)\overline{\Psi}_{l,n-l}(-X, Y; d(x, y)).$$

Similarly, we find

$$I_3(h, l) = (-1)^{l-1} \int_{(\mathbb{R}^n)^2} h(x + y)\overline{\Psi}_{l,n-l}(X, -Y; d(x, y))$$

and

$$I_4(h,l) = (-1)^n \int_{(\mathbb{R}^n)^2} h(-x+y)\overline{\Psi}_{l,n-l}(-X,-Y;d(x,y)).$$

Obviously, the measures $\overline{\Psi}_{l,n-l}(\pm X, \pm Y; \cdot)$ are non-negative. Therefore we can deduce the integrability of the map $t \mapsto \chi(\partial X \cap (\partial Y + t))$ as soon as $|I_j(h,l)| < \infty$ has been established for $j = 1, \ldots, 4$ and $h \equiv 1$. Set $h_0 \equiv 1$. Then it follows from equation (4) in [19] that

$$\begin{aligned} I_1(h_0,l) &= \binom{n}{l} V(X[l], -Y[n-l]), \\ I_2(h_0,l) &= (-1)^{n-1-l}\binom{n}{l} V(-X[l], -Y[n-l]), \\ I_3(h_0,l) &= (-1)^{l-1}\binom{n}{l} V(X[l], Y[n-l]) \end{aligned}$$

and

$$I_4(h_0,l) = (-1)^n \binom{n}{l} V(-X[l], Y[n-l]).$$

Since mixed volumes are invariant with respect to simultaneous applications of volume preserving affine maps, we are finally led to

$$\begin{aligned} & I_1(h_0,l) + \ldots + I_4(h_0,l) \\ & = \binom{n}{l}(1+(-1)^n)\left\{V(X[l], -Y[n-l]) + (-1)^{l-1}V(X[l], Y[n-l])\right\}. \end{aligned} \tag{3.6}$$

This proves the assertion of integrability. Moreover, combining (3.4), (3.5) and (3.6), we also conclude the proof of the translative integral formula for the intersections of boundaries of parallel sets.

The argument for the intersection of boundary and body is analogous, actually it is simpler since we merely have to apply the decomposition

$$N(\partial X) \times N(Y) = (N(X) \times N(Y)) \cup (\{(x,-m) : (x,m) \in N(X)\} \times N(Y))$$

to the right-hand side of

$$\begin{aligned} \int_{\mathbb{R}^n} h(t)\chi(\partial X \cap (Y+t))\mathcal{H}^n(dt) &= \sum_{l=1}^{n-1} \int_{(\mathbb{R}^n)^2} h(x-y)\overline{\Psi}_{l,n-l}(\partial X, Y; d(x,y)) \\ &\quad + \int_Y \int_{\mathbb{R}^n} h(x-y) C_0(\partial X, dx \times \mathbb{R}^n)\mathcal{H}^n(dy) \end{aligned}$$

The last summand on the right-hand side can be treated as the remaining summands if we use the representation

$$\begin{aligned} & \mathcal{H}^{n-1}(S^{n-1}) C_0(\partial X, A \times \mathbb{R}^n) \\ & = \int_{N(\partial X)} \mathbf{1}_A(x) \prod_{i=1}^{n-1} \frac{\kappa_i(\partial X, x, m)}{\sqrt{1+\kappa_i(\partial X, x, m)^2}} \mathcal{H}^{n-1}(d(x,m)), \end{aligned}$$

for Borel sets $A \subseteq \mathbb{R}^n$; confer [36] or [19]. The proof can then be completed as before by either using that (for this particular normalization of the curvature measures)

$$C_0(\partial X, (\mathbb{R}^n)^2) = \chi(S^{n-1}) = 1 - (-1)^n$$

or by recalling the well-known fact that $C_0(X, (\mathbb{R}^n)^2) = 1$. □

It remains to establish the general case of Theorems 1.1 and 1.2. This is accomplished by the following

Proof. Fix two convex bodies $K, L \subset \mathbb{R}^n$. First, we consider the case of the intersection of boundaries. Choose a decreasing sequence $r_i \in (0,1]$ satisfying $r_i \to 0$ for $i \to \infty$ and $r_1 = 1$. By $\mathbf{T}^*$ we denote the set of all $t \in \mathbb{R}^n$ for which there is some $r \in \{r_i \mid i \in \mathbb{N}\} \cup \{0\}$ such that ∂K_r and $\partial L_r + t$ do not intersect almost transversally. Lemma 3.1 implies that $\mathcal{H}^n(\mathbf{T}^*) = 0$. For $r \geq 0$ we set

$$\upsilon_r(t) := \chi(\partial K_r \cap (\partial L_r + t)).$$

Let $t \in \mathbb{R}^n - \mathbf{T}^*$ be fixed for the moment. We distinguish several cases.

Case 1: $K^\circ \cap (L^\circ + t) \neq \emptyset$.

Case 1.1: $\partial K \cap (\partial L + t) \neq \emptyset$. Then ∂K and $\partial L + t$ must intersect almost transversally, since $t \notin \mathbf{T}^*$. Thus $\upsilon_0(t) = \upsilon_{r_i}(t)$, for all $i \in \mathbb{N}$, by Theorem 1.3.

Case 1.2: $\partial K \cap (\partial L + t) = \emptyset$. It is easy to check that then $K \subseteq L^\circ + t$ or $L + t \subseteq K^\circ$. Hence, $K_r \subseteq L_r^\circ + t$ for all $r \geq 0$ or $L_r + t \subseteq K_r^\circ$ for all $r \geq 0$, and therefore $\upsilon_0(t) = \upsilon_{r_i}(t) = 0$ for all $i \in \mathbb{N}$.

Case 2: $K^\circ \cap (L^\circ + t) = \emptyset$. Then $\partial K \cap (\partial L + t) = \emptyset$, since $t \notin \mathbf{T}^*$. Hence, $K \cap (L + t) = \emptyset$. Note that $K_{r_1} = (K_r)_{r_1 - r}$ and $L_{r_1} = (L_r)_{r_1 - r}$, for $r \in [0, r_1]$. Let $r \in \{r_i : i \in \mathbb{N}\}$. Using again the assumption that $t \notin \mathbf{T}^*$ and Theorem 1.3, we obtain

$$\upsilon_r(t) = \begin{cases} \upsilon_1(t) & \text{if } K_r \cap (L_r + t) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\lim_{i \to \infty} \upsilon_{r_i}(t) = 0 = \upsilon_0(t).$$

In any case, for $t \notin \mathbf{T}^*$ we obtain that

$$|\upsilon_r(t)| \leq |\upsilon_1(t)|, \qquad \text{for } r \in \{r_i : i \in \mathbb{N}\} \cup \{0\},$$

and

$$\lim_{i \to \infty} \upsilon_{r_i}(t) = \upsilon_0(t).$$

Lemma 3.2 shows that $\upsilon_1(\cdot)$ is integrable with respect to $\mathcal{H}^n$. By the dominated convergence theorem, we thus conclude that

$$\lim_{i \to \infty} \int_{\mathbb{R}^n} \chi(\partial K_{r_i} \cap (\partial L_{r_i} + t)) \mathcal{H}^n(dt) = \int_{\mathbb{R}^n} \chi(\partial K \cap (\partial L + t)) \mathcal{H}^n(dt).$$

An application of Lemma 3.2 then completes the proof in the case considered, since the mixed volumes are continuous in each component with respect to the Hausdorff metric.

The case of the intersection of boundaries and bodies can be treated similarly. Instead of Theorem 1.3 one now has to use Theorem 1.4. Moreover, Case 1.2 has to be split into two subcases due to the asymmetry of the situation. If $K_r \subseteq L_r^\circ + t$ for all $r \geq 0$, then

$$\chi(\partial K_r \cap (L_r + t)) = \chi(S^{n-1}) = \chi(\partial K_1 \cap (L_1 + t))$$

for all $r \geq 0$. If $L_r + t \subseteq K_r^\circ$ for all $r \geq 0$, then $\chi(\partial K_r \cap (L_r + t)) = \chi(\emptyset) = 0$ for all $r \geq 0$. □

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