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Absolute Continuity for Curvature Measures of Convex Sets, III

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This work is devoted to the investigation of the basic relationship between the geometric shape of a convex set and measure theoretic properties of the associated curvature and surface area measures. We study geometric consequences of and conditions for absolute continuity of curvature and surface area measures with respect to $(d-1)$-dimensional Hausdorff measure in Euclidean space $\mathbb{R}^d$. Our main results are two "transfer principles" which allow one to translate properties connected with the absolute continuity of the rth curvature measure of a convex body to dual properties related to the absolute continuity of the $(d-1-r)$th surface area measure of the polar body, and conversely. Applications are also considered.

Key Words: curvature measures; surface area measures; absolute continuity of measures; bounded density; polar body; curvatures on the unit normal bundle; integral geometry; stability results.

1. INTRODUCTION

The theory of *curvature* and *surface area measures* is of central importance in convexity (see [26, 27, 32, 34]). Both types of measures emerge in the study of general closed convex sets, since in many cases pointwise (almost everywhere) defined functions of principal curvatures or radii of curvature are an insufficient tool of investigation. Perhaps, the most natural way to arrive at the curvature and surface area measures, or at their common generalizations, the support measures, of general convex sets, is to consider a local version of the classical Steiner formula. Thus, these measures provide local extensions of the well-known Minkowski functionals (quermassintegrals, intrinsic volumes).

The crucial role of the support measures is, to some extent, explained by the fact that they can be characterized by a certain number of basic properties (cf. [13, 39]) similar to Hadwiger's famous characterization

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theorem for the intrinsic volumes. Characterization theorems for curvature and surface area measures, which are similar in spirit, have been established in [24, 25]. Recent applications of such (axiomatic) results and methods are described in [13, 28]. Furthermore, both types of measures are an indispensible tool for various investigations in such diverse fields as stochastic geometry [33], geometric tomography [12], the study of additive functions [1, 22, 29] or the theory of mixed volumes. In fact, curvature measures can be introduced for other classes of sets as well; important examples are unions of convex sets, certain unions of sets with positive reach, and special classes of tame sets (cf. [3, 11]). The connection to Hessian measures of semi-convex functions was explored in [5, 6]. For the present purpose, however, the assumption of convexity will be essential.

Our main objective is to study the basic relationship between geometric properties of convex sets and measure theoretic properties of the associated curvature and surface area measures. The measure theoretic property which is relevant here is the *absolute continuity* with respect to an appropriate Hausdorff measure. A systematic investigation of this subject was initiated in [18, 19], and then continued in [20]; we also refer to [19, 20] for a description of the historical context and motivation. There it is explained how the absolute continuity of curvature measures is related to the characterization of Euclidean balls and to the corresponding splitting and *stability results*. Moreover, it is shown how regularity results for convex sets can be deduced under the assumption of absolute continuity of some curvature or surface area measure. We now continue this line of research by studying absolutely continuous measures with *bounded densities* and by exhibiting the role which *polarity* plays in this context.

A further thorough study of absolutely continuous curvature and surface area measures with bounded densities will be carried out in a subsequent paper which will rely in an essential way on the present work. There, for instance, we shall establish regularity results and characterize absolute continuity with bounded density in terms of integral-geometric Crofton and projection formulae. Thus, we also continue recent works of Bangert [2] and Burago and Kalinin [4].

A brief description of the scope of the paper is appropriate. In Section 2, we introduce our notation and provide some background information, for later reference and as a motivation for our main results. Section 3 contains a detailed description of the main results which will be proved in the subsequent sections. In Section 4, we investigate the relationship between absolutely continuous curvature and surface area measures of polar pairs of convex bodies, and in Section 5 such a relationship is studied for absolutely continuous measures with bounded densities. In Section 6, two applications are considered. Here, for instance, we prove a stability result of optimal order of the uniqueness assertion for the Minkowski problem.

2. NOTATION AND BACKGROUND INFORMATION

The starting point for the present investigation was an explicit description of the Lebesgue decomposition for the curvature and surface area measures of convex sets in $\mathbb{R}^d$ with respect to the appropriate $(d-1)$-dimensional Hausdorff measures. As a preparation for a description of this result and its consequences, we introduce some terminology. However, we shall assume that the reader is already familiar with curvature and surface area measures as introduced in [26]. Subsequently, we shall sketch some results of the previous paper [20]. In particular, we shall try to emphasize the dual nature of the results obtained for curvature and surface area measures. This should serve as a motivation for our new results.

Let $\mathscr{C}^d$ be the set of all non-empty closed convex sets $K \subset \mathbb{R}^d$ with $K \neq \mathbb{R}^d$. Let $\mathscr{H}^s$, $s \geqslant 0$, denote the s-dimensional Hausdorff measure in a Euclidean space. The unit sphere of $\mathbb{R}^d$ with respect to the Euclidean norm $|\cdot|$ is denoted by S^{d-1}, the unit ball centred at the origin o is denoted by B^d. Furthermore, we write $B^d(x,r)$ instead of $x + rB^d$. The scalar product is denoted by $\langle \cdot, \cdot \rangle$. If $K \in \mathscr{C}^d$ and $x \in \operatorname{bd} K$ (the boundary of K), then the *normal cone* of K at x is denoted by $N(K,x)$; see [26] for notions of convex geometry which are not explicitly defined here. For our approach, the (generalized) *unit normal bundle* $\mathscr{N}(K)$ of a convex set $K \in \mathscr{C}^d$ plays an important role. It is defined as the set of all pairs $(x,u) \in \operatorname{bd} K \times S^{d-1}$ such that $u \in N(K,x)$. Walter (see [35] or [36]) has shown that this set is a strong $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^{2d}$. For $\mathscr{H}^{d-1}$ almost all $(x,u) \in \mathscr{N}(K)$, one can introduce non-negative (*generalized*) *curvatures* $k_i(x,u)$, $i \in \{1, \ldots, d-1\}$, to which we also refer to as curvatures on the unit normal bundle.

It is appropriate to describe the definition of these curvatures more explicitly, since the details will become relevant in the following. For that purpose, we write $p(K,\cdot)$ for the metric projection onto K, we set $d(K,y) := |y - p(K,y)|$ and define $u(K,y) := d(K,y)^{-1}(y - p(K,y))$ for $y \in \mathbb{R}^d \backslash K$. For any $\varepsilon > 0$, we set $K_\varepsilon := K + \varepsilon B^d$. Then, for all $\varepsilon > 0$, the map $(p(K,\cdot), u(K,\cdot))|_{\operatorname{bd} K_\varepsilon}$ provides a bi-Lipschitz homeomorphism between $\operatorname{bd} K_\varepsilon$ and $\mathscr{N}(K)$. Furthermore, let $\mathscr{D}_K$ denote the set of all $y \in \mathbb{R}^d \backslash K$ for which $p(K,\cdot)$ is differentiable at y. It is known that if $y \in \mathbb{R}^d \backslash K$, then $y \in \mathscr{D}_K$ if and only if $p(K,y) + (0,\infty)u(K,y) \subset \mathscr{D}_K$. For any $(x,u) \in \mathscr{N}(K)$ such that $x + (0,\infty)u \subset \mathscr{D}_K$, and thus for $\mathscr{H}^{d-1}$ almost all $(x,u) \in \mathscr{N}(K)$, the spherical image map $u(K,\cdot)|_{\operatorname{bd} K_\varepsilon}$ is differentiable at $x + \varepsilon u$ for all $\varepsilon > 0$ (see [35]). Therefore, curvatures $k_1(x + \varepsilon u), \ldots, k_{d-1}(x + \varepsilon u)$ are defined as the eigenvalues of the symmetric linear map $Du(K, x + \varepsilon u)|_{u^\perp}$, where $u^\perp$ denotes the orthogonal complement of u. The corresponding eigenvectors will be denoted by $u_1, \ldots, u_{d-1}$. It is easy to see that they can be chosen in such a way that they do not depend on ε and constitute an orthonormal basis of $u^\perp$.

Of course, they depend on (x, u), but we shall often omit the argument. Hence, especially for $\mathcal{H}^{d-1}$ almost all $(x, u) \in \mathcal{N}(K)$ and any $\varepsilon > 0$, we can define the generalized curvatures

$$k_i(x, u) := \begin{cases} \frac{k_i(x+\varepsilon u)}{1-\varepsilon k_i(x+\varepsilon u)} & \text{if } k_i(x + \varepsilon u) < \varepsilon^{-1}, \\ \infty & \text{if } k_i(x + \varepsilon u) = \varepsilon^{-1}, \end{cases}$$

where $i \in \{1, \ldots, d - 1\}$, independent of the particular choice of $\varepsilon > 0$ (see [38]). We shall always assume that the ordering of these curvatures is such that

$$0 \leqslant k_1(x, u) \leqslant \cdots \leqslant k_{d-1}(x, u) \leqslant \infty.$$

In addition, we set $k_0(x, u) := 0$ and $k_d(x, u) := \infty$ for all $(x, u) \in \mathcal{N}(K)$. The preceding notation does not make explicit the dependence of the various curvature functions on the convex set K. If necessary, however, we shall be more precise. Further details of this construction, in the general context of sets with positive reach, can be found in [18, 19, 38].

Let X be a locally compact Hausdorff space with a countable base. (We are interested in the cases $X = \mathbb{R}^d$ and $X = S^{d-1}$.) In the following, we refer to [9, Chap. 1] for the basic notation and results concerning measure theory. However, there is one minor difference. For us a Radon measure over X will be defined on the Borel subsets of X, whereas in [9] Radon measures are understood to be outer measures defined on all subsets of X. We write $\mathfrak{B}(Y)$ for the σ-algebra of Borel sets of an arbitrary topological space Y.

Now let μ and ν be two Radon measures over X. If $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathfrak{B}(X)$, then we say that μ is *absolutely continuous* with respect to ν, and we write $\mu \ll \nu$. By the Radon–Nikodym theorem, $\mu \ll \nu$ if and only if there is a non-negative Borel measurable function $f : X \to \mathbb{R}$ such that

$$\mu(A) = \int_A f(x)\nu\,(dx)$$

for all $A \in \mathfrak{B}(X)$. In particular, the *density function* f is locally integrable with respect to ν. Furthermore, we say that μ is *singular* with respect to ν if there is a Borel set $B \subset X$ such that $\mu(X \backslash B) = 0 = \nu(B)$, and in this case we write $\mu \perp \nu$. Certainly, this is a symmetric relationship. A version of the Lebesgue decomposition theorem says that for arbitrary Radon measures μ and ν there are two Radon measures μ^{a} and μ^{s} such that $\mu = \mu^{\mathrm{a}} + \mu^{\mathrm{s}}$, $\mu^{\mathrm{a}} \ll \nu$ and $\mu^{\mathrm{s}} \perp \nu$. Moreover, the absolutely continuous part μ^{a} and the singular part μ^{s} (of μ with respect to ν) are uniquely determined by these conditions. We shall also consider the restriction $(\mu \llcorner A)(\cdot) := \mu(A \cap \cdot)$ of a Radon measure μ to a set $A \in \mathfrak{B}(X)$, which is again a Radon measure.

These notions and results will now be applied to the curvature and surface area measures of a convex set $K \in \mathscr{C}^d$. As the curvature measures are Borel measures over $\mathbb{R}^d$ which are locally finite and concentrated on $\operatorname{bd} K$, the curvature measure $C_r(K,\cdot)$, for any $r \in \{0,\dots,d-1\}$, can be written as the sum of two measures, that is,

$$C_r(K,\cdot) = C_r^{\mathrm{a}}(K,\cdot) + C_r^{\mathrm{s}}(K,\cdot),$$

where $C_r^{\mathrm{a}}(K,\cdot)$ is absolutely continuous and $C_r^{\mathrm{s}}(K,\cdot)$ is singular with respect to the boundary measure $C_{d-1}(K,\cdot)$. Recall that if $K \in \mathscr{C}^d$, then $C_{d-1}(K,\cdot) = \mathscr{H}^{d-1} \llcorner \operatorname{bd} K$ if K has non-empty interior or $\dim K \leqslant d-2$. If $\dim K = d-1$, then $C_{d-1}(K,\cdot) = 2(\mathscr{H}^{d-1} \llcorner \operatorname{bd} K)$. Subsequently, we often say that the rth curvature measure of a convex set is absolutely continuous, by which we wish to express that this measure is absolutely continuous with respect to the boundary measure of the set.

The surface area measures $S_r(K,\cdot)$ of non-empty compact convex sets (convex bodies) $K \subset \mathbb{R}^d$ are finite Borel measures over S^{d-1}. Hence, if $K \subset \mathbb{R}^d$ is a convex body and $r \in \{0,\dots,d-1\}$, then we can write

$$S_r(K,\cdot) = S_r^{\mathrm{a}}(K,\cdot) + S_r^{\mathrm{s}}(K,\cdot),$$

where $S_r^{\mathrm{a}}(K,\cdot)$ is absolutely continuous and $S_r^{\mathrm{s}}(K,\cdot)$ is singular with respect to $S_0(K,\cdot)$. In this case, the surface area measure of order 0 is just the restriction of the $(d-1)$-dimensional Hausdorff measure to the Borel sets of the unit sphere, and thus it is independent of the convex body K.

In the remainder of this section, we recall various results and some notation from [19] and [20]. Let $\mathscr{K}^d$ denote the set of all convex bodies in $\mathbb{R}^d$. We write $\mathscr{C}_o^d$ ($\mathscr{K}_o^d$) for the set of all $K \in \mathscr{C}^d$ ($K \in \mathscr{K}^d$) for which $\operatorname{int} K \neq \emptyset$. In [19] an explicit description of the singular parts of the curvature and surface area measures of a convex set K was given in terms of the generalized curvature functions of the unit normal bundle of K. The corresponding theorems are the essential tools that allow one to establish geometric results. Moreover, they serve as a main ingredient in the proof of the regularity theorems contained in [19].

The absolutely continuous parts of the curvature and surface area measures of convex sets were recovered in [19] as well. To describe these, we write $k_1(K,x),\dots,k_{d-1}(K,x)$ for the principal curvatures of K at a *normal boundary point* $x \in \operatorname{bd} K$; thus these curvatures are defined for $\mathscr{H}^{d-1}$ almost all boundary points (see [26, Sect. 2.5]). Then the density function of $C_r^{\mathrm{a}}(K,\cdot)$ with respect to $C_{d-1}(K,\cdot)$ is given by

$$H_{d-1-r}(K,x) := \binom{d-1}{r}^{-1} \sum_{|I|=d-1-r} \prod_{i \in I} k_i(K,x),$$

where the summation extends over all sets $I \subset \{1, \ldots, d-1\}$ of cardinality $d-1-r$. For $r = d-1$ the product over the empty set has to be interpreted as one.

Similarly, the principal radii of curvature $r_1(K,u), \ldots, r_{d-1}(K,u)$ of K at $u \in S^{d-1}$ are defined for all $u \in S^{d-1}$ such that $h_K = h(K,\cdot)$ is the second-order differentiable at u as the eigenvalues of the restriction to $u^\perp$ of the second-order differential of the *support function* h_K of K at u. Then the density function of $S_r^{\mathrm{a}}(K,\cdot)$ with respect to $S_0(K,\cdot)$ is

$$D_r h(K,u) := \binom{d-1}{r}^{-1} \sum_{|I|=r} \prod_{i \in I} r_i(K,u).$$

In [20], useful conditions were derived which are necessary and sufficient for the absolute continuity of a particular curvature or surface area measure of a convex set. These characterization results, stated in Theorems 2.1 and 2.2, allow one to express the measure theoretic property of absolute continuity of a particular curvature or surface area measure of a convex set in terms of the generalized curvatures of this set. The following two theorems also play a key role in the present investigation.

THEOREM 2.1. *Let* $K \in \mathscr{C}^d$, $r \in \{0, \ldots, d-1\}$, *and* $\beta \in \mathfrak{B}(\mathbb{R}^d)$. *Then*

$$C_r(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

if and only if, for $\mathscr{H}^{d-1}$ *almost all* $(x,u) \in \mathscr{N}(K)$ *with* $x \in \beta$, *one of the conditions*

$$k_{d-1}(x,u) < \infty \quad \textit{or} \quad k_{r+1}(x,u) = 0 \quad \textit{or} \quad k_r(x,u) = \infty$$

is satisfied.

THEOREM 2.2. *Let* $K \in \mathscr{K}^d$, $r \in \{0, \ldots, d-1\}$, *and* $\omega \in \mathfrak{B}(S^{d-1})$. *Then*

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

if and only if, for $\mathscr{H}^{d-1}$ *almost all* $(x,u) \in \mathscr{N}(K)$ *with* $u \in \omega$, *one of the conditions*

$$k_1(x,u) > 0 \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty$$

is satisfied.

In the special but important case of the curvature measure $C_0(K,\cdot)$ of a convex body K, a characterization of absolute continuity can be stated which involves a spherical supporting property of K. Using a *Crofton intersection formula* and various integral-geometric transformations, this result can be extended to curvature measures of any order. The precise formulation involves the conveniently normalized motion invariant Haar measure μ_r on the homogeneous space $\mathbf{A}(d,r)$ of r-dimensional affine subspaces in $\mathbb{R}^d$; cf. [26] for further explanations. Finally, we write $U(E)$ for the unique linear subspace which is parallel to a given affine subspace E, and we denote by $\sigma_{U(E)}(K \cap E, \beta \cap E)$ the *spherical image* of $K \cap E$ at $\beta \cap E$ with respect to $U(E)$ if $E \cap \operatorname{int} K \neq \emptyset$ and $\beta \subset \mathbb{R}^d$ (see [26, Sect. 2.2]).

Theorem 2.3. *Let $K \in \mathscr{C}_o^d$, $\beta \in \mathfrak{B}(\mathbb{R}^d)$, and $r \in \{2, \ldots, d\}$. Then*

$$C_{d-r}(K,\cdot) \llcorner \beta \ll C_{d-1}(K,\cdot) \llcorner \beta$$

if and only if, for μ_r almost all $E \in \mathbf{A}(d,r)$ such that $E \cap \operatorname{int} K \neq \emptyset$, and in $\mathscr{H}^{r-1}$ almost all directions of the set $\sigma_{U(E)}(K \cap E, \beta \cap E) \subset U(E)$, the intersection $K \cap E$ is supported from inside by an r-dimensional ball contained in E.

In fact, Theorem 2.3 was stated in [20] for $K \in \mathscr{K}_o^d$, but for unbounded sets the assertion follows immediately, since the curvature measures are locally defined. The case $r = 2$ is of particular interest because then the assumption $C_{d-2}(K,\cdot) \ll C_{d-1}(K,\cdot)$ implies that almost all two-dimensional sections of K are smooth.

Analogous results for surface area measures have been established in [20] as well. One of the basic tools which one uses now are integral-geometric *projection formulae*. Such formulae involve the Grassmann space $\mathbf{G}(d,j)$ of j-dimensional linear subspaces of $\mathbb{R}^d$ and the normalized rotation invariant Haar measure ν_j over $\mathbf{G}(d,j)$. We write $K|V$ for the orthogonal projection of a convex body K onto $V \in \mathbf{G}(d,j)$. Finally, $\tau(K|U, \omega \cap U) \subset U$ denotes the *reverse spherical image* of $K|U$ at $\omega \cap U$ where $\omega \subset S^{d-1}$ (see [26, Sect. 2.2]). The result corresponding to Theorem 2.3 is the following.

Theorem 2.4. *Let $K \in \mathscr{K}^d$, $\omega \in \mathfrak{B}(S^{d-1})$, and $i \in \{1, \ldots, d-1\}$. Then*

$$S_i(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega$$

if and only if, for ν_{i+1} almost all $U \in \mathbf{G}(d, i+1)$, and at $\mathscr{H}^i$ almost all points of the set $\tau(K|U, \omega \cap U)$, the projection $K|U$ is supported from outside by an $(i+1)$-dimensional ball contained in U.

Hence, if $S_1(K,\cdot) \ll S_0(K,\cdot)$, then almost all projections of K onto two-dimensional subspaces are strictly convex.

Further characterization results and consequences are discussed in [20]. A close inspection of these results suggests some underlying duality principles, the corresponding results, which make this idea precise, are described in the following section and are referred to as transfer principles. It should be pointed out that these principles do not apply to the main theorems of [20] in an obvious way. However, other applications will be given in the present paper and in the subsequent work [21].

3. MAIN RESULTS

A review of results on the absolute continuity of curvature and surface area measures clearly indicates that there should be a general principle by which results for curvature and surface area measures are related. Indeed, the corresponding pairs of notions such as boundary point—normal vector, support set—normal cone, principal curvatures—radii of curvature, intersection by an affine plane—projection onto a linear subspace, are connected by polarity; compare [17, 26, p. 75]. Therefore, it is natural to conjecture that the characterizations of absolute continuity of curvature measures correspond in a precise sense to the characterizations of absolute continuity of surface area measures via polarity.

The formation of the polar body of a given convex body is a non-linear operation and it requires the non-canonical choice of a reference point (cf. [26, Sect. 1.6]). Subsequently, it will be convenient to fix the origin o as the reference point, but this does not restrict the generality of our statements. We shall see that often the choice of a reference point is immaterial for geometric consequences which appear in a translation invariant setting. Most of the results, which we intend to discuss, thus refer to the set $K \in \mathcal{K}^d_{oo}$ of all convex bodies $K \in \mathcal{K}^d$ for which $o \in \operatorname{int} K$. For a given convex body $K \in \mathcal{K}^d_{oo}$, we write K^* for the polar body of K and introduce the map

$$f : S^{d-1} \to \operatorname{bd} K^*, \qquad u \mapsto h(K, u)^{-1} u.$$

It provides the required correspondence between normal vectors of K and boundary points of K^*.

Now we are prepared to state our *first transfer principle*, which allows us to transfer properties connected with the absolute continuity of the rth curvature measure $C_r(K, \cdot)$ of a convex body K to dual properties connected with the absolute continuity of the $(d-1-r)$th surface area measure $S_{d-1-r}(K^*, \cdot)$ of the polar body K^*, and conversely.

THEOREM 3.1. *Let* $K \in \mathcal{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, *and* $r \in \{0, \ldots, d-1\}$. *Then*

$$S_r(K, \cdot) \llcorner \omega \ll S_0(K, \cdot) \llcorner \omega$$

if and only if

$$C_{d-1-r}(K^*, \cdot) \llcorner f(\omega) \ll C_{d-1}(K^*, \cdot) \llcorner f(\omega).$$

Clearly, by the bi-polar theorem the roles of K and K^* can be interchanged. The proof of this result uses Theorems 2.1 and 2.2 in an essential way. Therefore, Theorem 3.1 cannot be used to deduce these two theorems from each other. Apparently, a similar remark applies to the other main results contained in [20] (except for parts of Theorems 2.3 and 3.7 in [20]). However, we shall encounter other applications of Theorem 3.1 in the present work and in [21]. In particular, Theorem 3.1 is an important ingredient for the proof of our *second transfer principle*.

Let us put Theorem 3.1 into a broader context. In spherical space, the connection between curvature measures of a convex set and surface area measures of the polar set is much simpler and actually extends to support measures; see [13]. This is due to the fact that polarity on the sphere essentially is the duality of cones, which is much easier to treat from a technical point of view. A similar phenomenon can be observed when one tries to extend certain integral-geometric results from the sphere to Euclidean space; cf. the discussion in [14]. Still another kind of duality for Hessian measures of convex functions was discovered in [6]. In this context, the right notion of duality turned out to be the classical formation of the conjugate function. However, the theory developed in [6] does not seem to apply to the present situation.

Previously, we considered the case of absolutely continuous curvature or surface area measures. The next step and our primary concern here is to study absolutely continuous measures with bounded densities. We say that a particular curvature or surface area measure is absolutely continuous with bounded density (function) if it is absolutely continuous with respect to the $(d-1)$-dimensional Hausdorff measure and the density function is bounded from above by a constant. Clearly, if the density of a measure with respect to another measure is bounded, then the former need not be absolutely continuous with respect to the latter. Again it is natural to ask for conditions which characterize the absolute continuity with bounded density of a particular curvature or surface area measure. Moreover, one will be interested in finding geometric consequences of the assumption of absolute continuity for the structure of the set of singular points or the set of singular normal vectors of convex sets.

A first general result concerning bounded densities is given by our *second transfer principle*, which is stated as Theorem 3.2. It is implied by Theorem 3.1 and by some new estimates involving elementary symmetric functions of principle curvatures of K^* and elementary symmetric functions of radii of curvature of K at corresponding points; see Corollary 5.1. These estimates again are consequences of a more general connection between elementary symmetric functions of principle curvatures of K^* and suitably weighted elementary symmetric functions of radii of curvature of K. A very special instance of such a relationship was found in [17, Theorem 2.2], but the present approach is completely different.

THEOREM 3.2. *Let* $K \in \mathscr{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, *and* $r \in \{0, \ldots, d-1\}$. *Then there is a constant* c *such that*

$$S_r(K, \cdot) \llcorner \omega \leqslant c\, S_0(K, \cdot) \llcorner \omega$$

if and only if there is a constant c^* *such that*

$$C_{d-1-r}(K^*, \cdot) \llcorner f(\omega) \leqslant c^* C_{d-1}(K^*, \cdot) \llcorner f(\omega).$$

In order to demonstrate how Theorem 3.2, together with Corollary 5.1, can be applied to obtain new results, we combine these results with a theorem of Weil [37] concerning the surface area measures to obtain a new theorem about curvature measures. Part (a) of Theorem 3.3 shows how an integrability assumption on the Radon–Nikodym derivative $H_1(K, \cdot)$ of the mean curvature measure $C_{d-2}(K, \cdot)$ of a convex set K implies the absolute continuity of certain lower-dimensional curvature measures with precise information about the integrability of the corresponding densities. In a certain sense this result is optimal as an example shows. By constructing suitable examples one can also see that in general the absolute continuity of the rth curvature measure of a convex body K does not imply the absolute continuity of any other curvature measure of order s ($s \neq r$) of K.

Subsequently, for $q > 0$, an open set $\beta \subset \mathbb{R}^d$ and a measurable function $g\colon \operatorname{bd} K \to \mathbb{R}$, we write $g \in \mathrm{L}^q_{\mathrm{loc}}(\beta \cap \operatorname{bd} K)$ if $\int_\alpha |g(x)|^q \mathscr{H}^{d-1}(dx) < \infty$ for all compact sets $\alpha \subset \beta \cap \operatorname{bd} K$.

THEOREM 3.3. *Let* $K \in \mathscr{C}^d_o$, *and let* $\beta \subset \mathbb{R}^d$ *be open.*

(a) *Assume that*

$$C_{d-2}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta,$$

and further assume that $H_1(K, \cdot) \in \mathrm{L}^p_{\mathrm{loc}}(\beta \cap \operatorname{bd} K)$ *for some* $p \in [1, \infty)$. *Then*

$$C_{d-1-j}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

and $H_j(K, \cdot) \in \mathrm{L}^{[p/j]}_{\mathrm{loc}}(\beta \cap \operatorname{bd} K)$ *for* $j \in \{1, \ldots, [p]\}$.

(b) *Assume that*

$$C_{d-2}(K, \cdot) \llcorner \beta \leqslant \bar{c}\, C_{d-1}(K, \cdot) \llcorner \beta$$

for some constant $\bar{c} > 0$. *Then*

$$C_{d-1-j}(K, \cdot) \llcorner \beta \leqslant \bar{c}^j C_{d-1}(K, \cdot) \llcorner \beta$$

for $j \in \{1, \ldots, d-1\}$.

Another application of the investigation of absolutely continuous curvature and surface area measures concerns a stability result. In [20], it was explained how stability results for curvature measures can be studied in the context of absolutely continuous measures. Indeed, this point of view is also useful for obtaining a stability result of the uniqueness assertion for the Minkowski problem. The uniqueness assertion states that if K and L are convex bodies for which $S_{d-1}(K,\cdot) = S_{d-1}(L,\cdot)$ is satisfied, then K and L are translates of each other. For the case where L is a ball, Diskant obtained in [8] a corresponding stability result. This can be seen by observing that $S_{d-1}(B^d,\cdot) = S_0(K,\cdot)$. The following theorem improves Diskant's result.

THEOREM 3.4. *Let* $K \in \mathscr{K}^d$ *and* $0 \leqslant \varepsilon < \frac{1}{4}$. *Assume that*

$$(1-\varepsilon)S_0(K,\cdot) \leqslant S_{d-1}(K,\cdot) \leqslant (1+\varepsilon)S_0(K,\cdot).$$

Then K lies in a $\gamma\varepsilon$-neighbourhood of a unit ball, where the constant γ depends only on the dimension d.

In fact, Diskant proved under the same assumptions that the given convex body K lies in a $\gamma\varepsilon^{1/(d-1)}$-neighbourhood of a unit ball. Our result shows that the exponent $1/(d-1)$ can be replaced by 1, which is the right order.

4. ABSOLUTE CONTINUITY AND POLARITY

In the present section, we give a proof of the first transfer principle. The major problem here in treating polarity is that the map $K \mapsto K^*$ cannot be described by a tractable analytic expression. Therefore, the idea is to pass to the normal bundles and to study instead a certain map $T : \mathscr{N}(K) \to \mathscr{N}(K^*)$, which turns out to be much more convenient. The same map has been used for a different problem in [31].

For a convex body $K \in \mathscr{K}^d_{oo}$, we define the map T by

$$T : \mathscr{N}(K) \to \mathscr{N}(K^*), \qquad (x,u) \mapsto (\langle x,u\rangle^{-1}u, |x|^{-1}x).$$

First, we check that T is properly defined. Let $\rho(L,\cdot)$ denote the radial function of $L \in \mathscr{K}^d_{oo}$. Choose any $(x,u) \in \mathscr{N}(K)$. Then $\langle x,u\rangle = h(K,u) = \rho(K^*,u)^{-1}$, and hence $\langle x,u\rangle^{-1}u = \rho(K^*,u)u \in \operatorname{bd} K^*$. In addition, we have $\langle |x|^{-1}x, \langle x,u\rangle^{-1}u\rangle = h(K^*,|x|^{-1}x)$, since this is equivalent to $\rho(K,x) = 1$. Thus $|x|^{-1}x \in N(K^*,\langle x,u\rangle^{-1}u) \cap S^{d-1}$. It is also easy to see that the inverse of T is given by

$$T^* : \mathscr{N}(K^*) \to \mathscr{N}(K), \qquad (x^*,u^*) \mapsto (\langle x^*,u^*\rangle^{-1}u^*, |x^*|^{-1}x^*).$$

From this we can conclude that T is a bi-Lipschitz homeomorphism. In addition, T is differentiable, if considered as a map from a neighbourhood of $\mathcal{N}(K) \subset \mathbb{R}^{2d}$ into $\mathbb{R}^{2d}$.

In the following, as a rule we shall attach an asterisk to quantities which are associated with K^*. For example, we write $k_1^*(\cdot), \ldots, k_{d-1}^*(\cdot)$ for the (generalized) curvatures of K^* instead of $k_1(K^*, \cdot), \ldots, k_{d-1}(K^*, \cdot)$. Finally, we set $\mathrm{I}_{d-1} := \{1, \ldots, d-1\}$.

Now we are prepared for Proposition 4.1, which relates generalized curvatures of K to those of K^*. Basically, the equations which are asserted in this proposition result from counting one and the same quantity in two different ways.

PROPOSITION 4.1. *Let $K \in \mathcal{K}^d_{oo}$. Then, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$,*

$$\operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i(x,u) = \infty\} = \operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = 0\}$$

and

$$\operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i(x,u) = 0\} = \operatorname{card}\{i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = \infty\}.$$

Proof. In the proof, we consider a pair $(x,u) \in \mathcal{N}(K)$ such that $x + \varepsilon u \in \mathcal{D}_K$ for all $\varepsilon > 0$. This condition is satisfied for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$. For any such pair (x,u) an orthonormal basis of the $(d-1)$-dimensional linear subspace $\operatorname{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), (x,u)) \subset \mathbb{R}^d \times \mathbb{R}^d$ of $(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K), d-1)$ approximate tangent vectors of $\mathcal{N}(K)$ at (x,u) is given by

$$w_i := \left(\frac{1}{\sqrt{1 + k_i(x,u)^2}} u_i, \frac{k_i(x,u)}{\sqrt{1 + k_i(x,u)^2}} u_i \right), \quad i \in \{1, \ldots, d-1\},$$

where the vectors $u_1, \ldots, u_{d-1} \in S^{d-1}$ constitute a suitable orthonormal basis of $u^\perp$, and $k_1(x,u), \ldots, k_{d-1}(x,u) \in [0, \infty]$ are the generalized curvatures of the unit normal bundle $\mathcal{N}(K)$; see [10, Sect. 3.2.16] for the terminology of geometric measure theory. The generalized curvatures of $\mathcal{N}(K^*)$ at $T(x,u)$ are denoted by $k_1^*(T(x,u)), \ldots, k_{d-1}^*(T(x,u))$. Since T is bi-Lipschitz, we can assume that $(x^*, u^*) := T(x,u)$ is such that $x^* + \varepsilon u^* \in \mathcal{D}_{K^*}$ for all $\varepsilon > 0$.

Let (x,u) be chosen as described. We also write T for the canonical extension of T to a neighbourhood of $\mathcal{N}(K)$ in $\mathbb{R}^{2d}$. In order to determine the special basis $DT(x,u)(w_i)$, $i \in \{1, \ldots, d-1\}$, of $\operatorname{Tan}^{d-1}(\mathcal{H}^{d-1} \llcorner \mathcal{N}(K^*), T(x,u))$, we first determine the values $DT(x,u)(v,o)$ and $DT(x,u)(o,\bar{v})$ with

$v, \bar{v} \in \mathbb{R}^d$ for the extended map T. By elementary calculus,

$$DT(x,u)(v,o) = \left(-\frac{\langle v,u \rangle}{\langle x,u \rangle^2} u,\ \frac{1}{|x|}\left[v - \left\langle \frac{x}{|x|}, v \right\rangle \frac{x}{|x|} \right] \right)$$

and

$$DT(x,u)(o,\bar{v}) = \left(\frac{1}{\langle x,u \rangle}\left[\bar{v} - \frac{\langle x,\bar{v} \rangle}{\langle x,u \rangle} u \right], o \right).$$

Since $\langle u_i, u \rangle = 0$, we obtain for $i \in \{1, \ldots, d-1\}$ that

$$DT(x,u)(w_i) = \left(\frac{k_i}{\sqrt{1+k_i^2}} \frac{1}{\langle x,u \rangle}\left[u_i - \frac{\langle x,u_i \rangle}{\langle x,u \rangle} u \right], \right.$$
$$\left. \frac{1}{\sqrt{1+k_i^2}} \frac{1}{|x|}\left[u_i - \left\langle \frac{x}{|x|}, u_i \right\rangle \frac{x}{|x|} \right] \right),$$

where the argument (x,u) of k_i has been omitted. If we attach an asterisk to the corresponding expressions for K^*, another basis of the tangent space $\mathrm{Tan}^{d-1}(\mathscr{H}^{d-1} \llcorner \mathscr{N}(K^*), T(x,u))$ is given by

$$w_i^* = \left(\frac{1}{\sqrt{1+(k_i^*)^2}} u_i^*, \frac{k_i^*}{\sqrt{1+(k_i^*)^2}} u_i^* \right), \qquad i \in \{1, \ldots, d-1\},$$

where the argument $T(x,u)$ of k_i^* has been omitted, and $(u_1^*, \ldots, u_{d-1}^*)$ is a suitable orthonormal basis of $x^\perp$. From this representation it is easy to see that the integer

$$\mathrm{card}\,\{i \in \mathrm{I}_{d-1} : k_i^*(T(x,u)) = \infty\}$$

equals the dimension of the kernel of the linear map π_1 which is given by

$$\pi_1 :\ \mathrm{lin}\{w_1^*, \ldots, w_{d-1}^*\} \to x^\perp, \quad (y,z) \mapsto y.$$

Since the vectors

$$u_i - \frac{\langle x,u_i \rangle}{\langle x,u \rangle} u, \qquad i \in \{1, \ldots, d-1\}$$

are linearly independent, and since

$$\frac{k_i}{\sqrt{1+k_i^2}} \in (0,\infty) \qquad \text{if } k_i \in (0,\infty],$$

the dimension of the kernel of π_1 is also equal to

$$\mathrm{card}\,\{i \in \mathrm{I}_{d-1} : k_i(x,u) = 0\}.$$

To see this, recall that

$$\mathrm{lin}\{w_1^*, \ldots, w_{d-1}^*\} = \mathrm{lin}\{DT(x,u)(w_1), \ldots, DT(x,u)(w_{d-1})\}.$$

Now the remaining statement of the lemma follows since $K^{**} = K$. ∎

By combining the preceding proposition with results from [20], we can now establish the first transfer principle which was announced as Theorem 3.1 in Section 3.

Proof of Theorem 3.1. We continue to use the notation introduced in the proof of Proposition 4.1 and in the preceding remarks. Let as assume that

$$S_r(K,\cdot) \llcorner \omega \ll S_0(K,\cdot) \llcorner \omega.$$

Hence, by Theorem 2.2, for $\mathcal{H}^{d-1}$ almost all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$,

$$k_1(x,u) > 0 \quad \text{or} \quad k_{r+1}(x,u) = 0 \quad \text{or} \quad k_r(x,u) = \infty. \tag{1}$$

Denote by $\mathcal{N}_1 \subset \mathcal{N}(K)$ the set of all $(x,u) \in \mathcal{N}(K)$ such that $u \in \omega$ and (1) is violated. Then $\mathcal{H}^{d-1}(\mathcal{N}_1) = \mathcal{H}^{d-1}(T(\mathcal{N}_1)) = 0$. Let $\mathcal{N}_2$ be the set of all $(x,u) \in \mathcal{N}(K)$ such that at least one of the two relations of Proposition 4.1 is not satisfied. Again $\mathcal{H}^{d-1}(\mathcal{N}_2) = \mathcal{H}^{d-1}(T(\mathcal{N}_2)) = 0$, since T is bi-Lipschitz.

Recall the definition of the map

$$f : S^{d-1} \to \mathrm{bd}\, K^*, \qquad u \mapsto h(K,u)^{-1}u,$$

choose $(x^*, u^*) \in \mathcal{N}(K^*) \setminus T(\mathcal{N}_1 \cup \mathcal{N}_2)$ such that $x^* \in f(\omega)$, and set $(x,u) := T^{-1}(x^*, u^*)$. Then $(x,u) \in \mathcal{N}(K) \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ and $u \in \omega$ because f is bijective and $f(u) \in f(\omega)$.

By relation (1) and using Proposition 4.1 thrice, we conclude that

$$k_{d-1}^*(x^*, u^*) < \infty \quad \text{or} \quad k_{d-r}^*(x^*, u^*) = 0 \quad \text{or} \quad k_{d-1-r}^*(x^*, u^*) = \infty.$$

Since $\mathscr{H}^{d-1}(T(\mathscr{N}_1 \cup \mathscr{N}_2)) = 0$, an application of Theorem 2.1 to the polar body K^* now yields that

$$C_{d-1-r}(K^*, \cdot) \llcorner f(\omega) \ll C_{d-1}(K^*, \cdot) \llcorner f(\omega).$$

The reverse implication is proved in a similar manner. ∎

5. BOUNDED DENSITIES AND POLARITY

In order to prove the second transfer principle, which deals with the case of bounded densities, it will be necessary to have sharp inequalities between elementary symmetric functions of principle curvatures of K^* and elementary symmetric functions of radii of curvature of K at corresponding points. Such inequalities will be derived from the following more general theorem. Instead of an elementary symmetric function of radii of curvature, it involves a weighted sum of products of radii of curvature.

It is remarkable that although the assertion of Theorem 5.1 does not involve generalized curvatures on unit normal bundles, the proof essentially uses this concept. Furthermore, recall that for a convex body $K \in \mathscr{K}^d$ the *reverse spherical image map* $\tau(K, u) = \tau_K(u)$ is well defined for $\mathscr{H}^{d-1}$ almost all $u \in S^{d-1}$; see [26, pp. 77–78].

THEOREM 5.1. *Let* $K \in \mathscr{K}^d_{oo}$ *and* $l \in \{1, \ldots, d-1\}$. *Then, for* $\mathscr{H}^{d-1}$ *almost all* $u \in S^{d-1}$,

$$\binom{d-1}{l} H_l(K^*, h(K,u)^{-1}u) = \left\langle \frac{x}{|x|}, u \right\rangle^l \sum_{|I|=l} \left[1 - \sum_{i \in I} \left\langle \frac{x}{|x|}, u_i \right\rangle^2 \right] \prod_{i \in I} r_i(K, u),$$

if $(u_1, \ldots, u_{d-1})$ *is a suitable orthonormal basis of* $u^\perp$, $x := \tau_K(u)$, *and the summation extends over all subsets* $I \subset \{1, \ldots, d-1\}$ *of cardinality* l.

Proof. Again we use the notation of the proof of Proposition 4.1. From the proofs of Lemma 3.1 in [19], applied to K^*, and Lemma 3.4 in [19], applied to K, as well as from the fact that $u \mapsto h(K,u)^{-1}u$, $u \in S^{d-1}$, is a bi-Lipschitz homeomorphism from S^{d-1} onto $\operatorname{bd} K^*$, we infer that for $\mathscr{H}^{d-1}$ almost all $u \in S^{d-1}$ the following conditions are simultaneously satisfied:

1. The support function $h(K, \cdot)$ of K is second-order differentiable at u and $(\tau_K(u), u) \in \mathscr{N}(K)$ is such that $\tau_K(u) + \varepsilon u \in \mathscr{D}_K$ for all $\varepsilon > 0$.

2. The point $h(K,u)^{-1}u = \langle \tau_K(u), u\rangle^{-1}u$ of K^* is a normal boundary point, and hence $\langle x,u\rangle^{-1}u + \varepsilon|x|^{-1}x \in \mathscr{D}_{K^*}$ for all $\varepsilon > 0$, if $x := \tau_K(u)$.

Let us fix one such $u \in S^{d-1}$, and set $x := \tau_K(u)$ and $(x^*, u^*) := T(x,u)$. Then by the proof of Lemma 3.4 in [19], we especially get that

$$k_i := k_i(x,u) > 0, \qquad i \in \{1,\ldots,d-1\};$$

moreover, by Lemma 3.1 in [19],

$$k_i^* := k_i^*(x^*,u^*) < \infty, \qquad i \in \{1,\ldots,d-1\}.$$

Also note that again by Lemmas 3.1 and 3.4 in [19],

$$\binom{d-1}{l}^{-1} \sum_{|I|=l} \prod_{i\in I} k_i^* = H_l(K^*, x^*), \qquad x^* = h(K,u)^{-1}u \tag{2}$$

and

$$k_i(x,u)^{-1} = r_i(K,u), \qquad i \in \{1,\ldots,d-1\}. \tag{3}$$

Hence, the proof of Proposition 4.1 implies that

$$\left(u_i - \frac{\langle x, u_i\rangle}{\langle x,u\rangle}u,\ \frac{1}{k_i}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_i - \left\langle \frac{x}{|x|}, u_i\right\rangle \frac{x}{|x|}\right]\right), \qquad i \in \{1,\ldots,d-1\}$$

is a basis of $\mathrm{Tan}^{d-1}(\mathscr{H}^{d-1} \llcorner \mathscr{N}(K^*), (x^*,u^*))$. Observe that the case $k_i = \infty$ is not excluded. Define

$$a_i := u_i - \frac{\langle x,u_i\rangle}{\langle x,u\rangle}u \quad \text{and} \quad b_i := \frac{1}{k_i}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_i - \left\langle \frac{x}{|x|}, u_i\right\rangle \frac{x}{|x|}\right],$$

for $i \in \{1,\ldots,d-1\}$. Note that the vectors $a_1,\ldots,a_{d-1} \in x^\perp$ are linearly independent. The linear mapping $\varphi : x^\perp \to x^\perp$, defined by

$$\varphi(a_i) := b_i, \qquad i \in \{1,\ldots,d-1\}$$

can also be determined by prescribing that

$$\varphi\left(\frac{1}{\sqrt{1+(k_i^*)^2}}u_i^*\right) = \frac{k_i^*}{\sqrt{1+(k_i^*)^2}}u_i^*, \qquad i \in \{1,\ldots,d-1\}.$$

To check this one can use that $(a_1,\ldots,a_{d-1})$ and

$$\left((1+(k_1^*)^2)^{-1/2}u_1^*,\ldots,(1+(k_{d-1}^*)^2)^{-1/2}u_{d-1}^*\right)$$

are two bases of $x^\perp$ and that

$$\mathrm{lin}\,\{w_1^*,\ldots,w_{d-1}^*\} = \mathrm{lin}\,\{(a_1,b_1),\ldots,(a_{d-1},b_{d-1})\}.$$

Therefore, the linear mapping φ has the eigenvalues $k_1^*,\ldots,k_{d-1}^*$. These eigenvalues are the zeros of the characteristic polynomial

$$t \mapsto \det\,(B - t\,E_{d-1}), \qquad t \in \mathbb{R},$$

where E_{d-1} is the unit $(d-1)$-by-$(d-1)$ matrix, and the matrix $B = (\beta_{ij})$, $i,j \in \{1,\ldots,d-1\}$, is defined by the relations

$$b_j = \sum_{i=1}^{d-1} \beta_{ij} a_i, \qquad j \in \{1,\ldots,d-1\}. \tag{4}$$

Substituting the expressions for a_i and b_j into (4), we arrive at

$$\frac{1}{k_j}\left\langle \frac{x}{|x|}, u\right\rangle \left[u_j - \left\langle \frac{x}{|x|}, u_j\right\rangle \frac{x}{|x|}\right] = \sum_{i=1}^{d-1} \beta_{ij}\left[u_i - \frac{\langle x, u_i\rangle}{\langle x, u\rangle} u\right]. \tag{5}$$

Since $(u_1,\ldots,u_{d-1},u)$ is an orthonormal basis of $\mathbb{R}^d$, we have

$$\frac{x}{|x|} = \sum_{k=1}^{d-1}\left\langle \frac{x}{|x|}, u_k\right\rangle u_k + \left\langle \frac{x}{|x|}, u\right\rangle u. \tag{6}$$

If we use (6) for the unit vector $|x|^{-1}x$ within the bracket on the left-hand side of Eq. (5), a comparison of the coefficients of $u_1,\ldots,u_{d-1}$ then yields, for $i,j \in \{1,\ldots,d-1\}$, that

$$\beta_{ij} = \frac{\langle \frac{x}{|x|}, u\rangle}{k_j}\left\{\delta_{ij} - \left\langle \frac{x}{|x|}, u_i\right\rangle\left\langle \frac{x}{|x|}, u_j\right\rangle\right\}.$$

Here, as usual, δ_{ij} denotes the Kronecker symbol. Moreover, for an arbitrary subset $I \subset \{1,\ldots,d-1\}$ with $|I| = l$, we set

$$B_I := (\beta_{jk})_{j,k\in I};$$

thus the determinants of the matrices B_I are the principal minors of order l of the matrix B. Furthermore, we know from (2) that $\binom{d-1}{l} H_l(K^*, x^*)$ can be calculated as the sum of these principal minors.

Therefore, we obtain that

$$\begin{aligned}\binom{d-1}{l} & H_l(K^*, x^*) \\ &= \sum_{|I|=l} \det B_I \\ &= \sum_{|I|=l} \left\langle \frac{x}{|x|}, u \right\rangle^l \left(\prod_{i \in I} k_i \right)^{-1} \det\left(\left(\delta_{jk} - \left\langle \frac{x}{|x|}, u_j \right\rangle \left\langle \frac{x}{|x|}, u_k \right\rangle \right)_{j,k \in I} \right) \\ &= \sum_{|I|=l} \left\langle \frac{x}{|x|}, u \right\rangle^l \left(\prod_{i \in I} k_i \right)^{-1} \left[1 - \sum_{j \in I} \left\langle \frac{x}{|x|}, u_j \right\rangle^2 \right].\end{aligned}$$

An application of (3) then implies the theorem. ∎

The following Corollary 5.1, which is an immediate consequence of Theorem 5.1, can not only be used to characterize absolute continuity with bounded density in terms of polarity, but it also leads to a characterization of the case where the measures are purely singular; see Corollary 5.2.

COROLLARY 5.1. *Let $K \in \mathscr{K}^d_{oo}$ and $l \in \{0, \ldots, d-2\}$. Then, for $\mathscr{H}^{d-1}$ almost all $u \in S^{d-1}$,*

$$\left\langle \frac{x}{|x|}, u \right\rangle^{l+2} D_l h(K, u) \leqslant H_l(K^*, h(K,u)^{-1} u) \leqslant \left\langle \frac{x}{|x|}, u \right\rangle^l D_l h(K, u),$$

where $x := \tau_K(u)$. In addition, for $\mathscr{H}^{d-1}$ almost all $u \in S^{d-1}$,

$$H_{d-1}(K^*, h(K,u)^{-1} u) = \left\langle \frac{x}{|x|}, u \right\rangle^{d+1} D_{d-1} h(K, u).$$

Remark 5.1. The special case $l = d-1$ of the preceding theorem and its corollary has already been established in [17] by a completely different method of proof. However, it does not seem to be possible to extend the approach of [17] to cover the present situation.

COROLLARY 5.2. *Let $K \in \mathscr{K}^d_{oo}$, $\omega \in \mathfrak{B}(S^{d-1})$, and $r \in \{0, \ldots, d-1\}$. Then*

$$S_r(K, \cdot) \llcorner \omega = S_r^s(K, \cdot) \llcorner \omega$$

if and only if

$$C_{d-1-r}(K^*, \cdot) \llcorner f(\omega) = C^s_{d-1-r}(K^*, \cdot) \llcorner f(\omega).$$

After these preparations it is now easy to provide a proof for the second transfer principle by just combining what we have proved so far.

Proof of Theorem 3.2. Assume that there is a constant c such that

$$S_r(K,\cdot)\llcorner\omega \leqslant c\, S_0(K,\cdot)\llcorner\omega.$$

Then $S_r(K,\cdot)\llcorner\omega$ is absolutely continuous with respect to $S_0(K,\cdot)\llcorner\omega$, and $D_r h(K,u)\leqslant c$ is satisfied for $\mathscr{H}^{d-1}$ almost all $u\in\omega$. By Theorem 3.1, $C_{d-1-r}(K^*,\cdot)\llcorner f(\omega)$ is absolutely continuous with respect to $C_{d-1}(K^*,\cdot)\llcorner f(\omega)$ and, for $\mathscr{H}^{d-1}$ almost all $x^*\in f(\omega)$, the density is given by $H_r(K^*,x^*)$. Now Corollary 5.1 implies that $H_r(K^*,x^*)\leqslant c$ for $\mathscr{H}^{d-1}$ almost all $x^*\in f(\omega)$. This finally shows that

$$C_{d-1-r}(K^*,\cdot)\llcorner f(\omega)\leqslant c^*\, C_{d-1}(K^*,\cdot)\llcorner f(\omega)$$

is satisfied with $c^*=c$.

Similarly, the reverse implication follows from the inequality on the left-hand side of Corollary 5.1. In fact, let $r,R\in(0,\infty)$ be chosen in such a way that $B^d(o,r)\subset K\subset B^d(o,R)$; hence $\langle |x|^{-1}x,u\rangle\geqslant r/R$, for $\mathscr{H}^{d-1}$ almost all $u\in S^{d-1}$ and $x=\tau_K(u)$, and we can proceed as before. ∎

6. APPLICATIONS

The following theorem has been established by Weil [37]. Its proof is based on a sophisticated convolution procedure which is applied to the restriction of the support function of a given convex body to properly chosen hyperplanes. Such a procedure is necessary in order to be able to exert control over the radii of curvature of a suitably constructed sequence of approximating smooth convex bodies.

THEOREM 6.1 (Weil [31]). *Let $K\in\mathscr{K}^d_o$, and let ω be an open subset of S^{d-1}.*

(a) *Assume that*

$$S_1(K,\cdot)\llcorner\omega \ll S_0(K,\cdot)\llcorner\omega,$$

and further assume that $D_1h(K,\cdot)\in \mathrm{L}^p(\omega)$ for some $p\in[1,\infty)$. Then

$$S_j(K,\cdot)\llcorner\omega \ll S_0(K,\cdot)\llcorner\omega$$

and $D_jh(K,\cdot)\in \mathrm{L}^{[p/j]}(\omega)$ for $j\in\{1,\dots,[p]\}$.

(b) *Assume that*

$$S_1(K,\cdot)\llcorner\omega \leqslant c\, S_0(K,\cdot)\llcorner\omega$$

for some constant $c > 0$. *Then,*

$$S_j(K,\cdot)\llcorner\omega \leqslant c^j\, S_0(K,\cdot)\llcorner\omega$$

for $j \in \{1,\ldots,d-1\}$.

The corresponding new result for curvature measures is stated as Theorem 3.3 in Section 3. It will be implied by our first transfer principle, Corollary 5.1 and Theorem 6.1. Alternatively, one could try to deduce Theorem 3.3 by a more direct application of a convolution procedure to the distance function of the convex body K. If it were indeed possible to establish Theorem 3.3 by such an argument, independent of Theorem 6.1, then one could deduce Theorem 6.1 from Theorem 3.3 again by using Corollary 5.1 and the first transfer principle. However, it remains unresolved whether a more direct approach to Theorem 3.3 exists. One difficulty is that such a direct proof of Theorem 3.3 probably requires results analogous to Satz 1.1 and Satz 4.1 in [37]. Here the problem arises that the curvatures of a sequence of smooth convex bodies, which approximate a given convex body, are defined on different domains. Furthermore, the proof of Satz 4.1 in [37] exploits the connection of surface area measures to mixed volumes and such a relationship is not available for curvature measures.

After this brief discussion, we turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. It is sufficient to consider the case $K \in \mathscr{K}_o^d$, since the curvature measures are locally defined. Furthermore, since all notions involved in Theorems 6.1 and 3.3 are invariant with respect to translations, we can assume that $o \in \operatorname{int} K$. Consider the maps

$$\eta:\ \operatorname{bd} K \to S^{d-1}, \qquad x \mapsto |x|^{-1}x$$

and

$$f^*: S^{d-1} \to \operatorname{bd} K, \qquad u \mapsto \rho(K,u)u,$$

which are bi-Lipschitz homeomorphisms that are inverse to each other. Let $\omega := \eta(\operatorname{bd} K \cap \beta)$. Then $\omega \subset S^{d-1}$ is an open subset of S^{d-1}. The assumptions of Theorems 3.3(a) and 3.1, applied to K^*, imply that $S_1(K^*,\cdot)\llcorner\omega \leqslant S_0(K^*,\cdot)\llcorner\omega$. Moreover, an application of Corollary 5.1 to K^* yields, for $\mathscr{H}^{d-1}$ almost all $x \in \operatorname{bd} K$ and $l \in \{0,\ldots,d-1\}$, that

$$\left\langle \frac{x}{|x|}, \sigma_K(x) \right\rangle^{l+2} D_l h\left(K^*, \frac{x}{|x|}\right) \leqslant H_l(K,x) \leqslant D_l h\left(K^*, \frac{x}{|x|}\right), \tag{7}$$

where σ_K denotes the *spherical image map* of K, which is defined for all regular, and hence for $\mathscr{H}^{d-1}$ almost all boundary points of K. Let $r, R \in (0, \infty)$ be such that $B^d(o, r) \subset K \subset B^d(o, R)$. Then, for $(x, u) \in \mathscr{N}(K)$,

$$\left(\left\langle \frac{x}{|x|}, u \right\rangle\right)^{-1} \leqslant \frac{R}{r} =: c.$$

Using Lemma 3.1 from [17], we hence obtain, for $l \in \{0, \ldots, d-1\}$ and $q > 0$, that

$$\begin{aligned}
&\int_\omega D_l h(K^*, u)^q \mathscr{H}^{d-1}(du) \\
&\leqslant c^{(l+2)q} \int_\omega H_l(K, f^*(u))^q \mathscr{H}^{d-1}(du) \\
&= c^{(l+2)q} \int_\omega H_l(K, f^*(u))^q \frac{\langle u, \sigma_K(f^*(u))\rangle}{\rho(K,u)^{d-1}} J_{d-1} f^*(u) \mathscr{H}^{d-1}(du) \\
&= c^{(l+2)q} \int_{\beta \cap \mathrm{bd}\, K} H_l(K, x)^q \frac{\langle \frac{x}{|x|}, \sigma_K(x)\rangle}{|x|^{d-1}} \mathscr{H}^{d-1}(dx) \\
&\leqslant \frac{c^{(l+2)q}}{r^{d-1}} \int_{\beta \cap \mathrm{bd}\, K} H_l(K, x)^q \mathscr{H}^{d-1}(dx)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_{\beta \cap \mathrm{bd}\, K} H_l(K, x)^q \mathscr{H}^{d-1}(dx) \\
&\leqslant \int_{\beta \cap \mathrm{bd}\, K} D_l h\left(K^*, \frac{x}{|x|}\right)^q \mathscr{H}^{d-1}(dx) \\
&= \int_\omega D_l h(K^*, u)^q J_{d-1} f^*(u) \mathscr{H}^{d-1}(du) \\
&\leqslant c R^{d-1} \int_\omega D_l h(K^*, u)^q \mathscr{H}^{d-1}(du).
\end{aligned}$$

These two estimates show that

$$D_l h(K^*, \cdot) \in \mathrm{L}^q(\omega) \quad \Leftrightarrow \quad H_l(K, \cdot) \in \mathrm{L}^q(\beta \cap \mathrm{bd}\, K). \tag{8}$$

Hence, we get that $D_1 h(K^*, \cdot) \in \mathrm{L}^p(\omega)$, and thus Weil's result (Theorem 6.1) yields, for $j \in \{1, \ldots, [p]\}$, that

$$S_j(K^*, \cdot) \llcorner \omega \ll S_0(K^*, \cdot) \llcorner \omega \qquad \text{and} \qquad D_j h(K^*, \cdot) \in \mathrm{L}^{[p/j]}(\omega).$$

Again from Theorem 3.1 we conclude, for $j \in \{1, \ldots, [p]\}$, that

$$C_{d-1-j}(K, \cdot) \llcorner \beta \ll C_{d-1}(K, \cdot) \llcorner \beta$$

and another application of Eq. (8) then completes the proof of (a).

The proof of (b) follows from (a) and from Newton's inequalities for elementary symmetric functions (see [15, 23]). ∎

EXAMPLE 6.1. The result of Theorem 6.1(a) cannot be improved in general. To see this, let $r, R > 0$ and define $X : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$X((\vartheta, \varphi)) := ((R + \sin \vartheta) \cos \varphi, (R + \sin \vartheta) \sin \varphi, \cos \vartheta).$$

Then $K := \text{conv}(X([0, \pi] \times [0, 2\pi]))$ is the convex hull of a torus. From Theorem 2.2 it easily follows that $S_1(K, \cdot)$ is absolutely continuous with respect to $S_0(K, \cdot)$. In addition, $D_1 h(K, \cdot) \in \mathrm{L}^p(S^2)$ for each $p \in [1, 2)$. In fact, the principal radii of curvature of K are given by

$$r_1(T(\vartheta, \varphi)) = r, \qquad r_2(T(\vartheta, \varphi)) = \frac{R + r \sin \vartheta}{\sin \vartheta},$$

where $T : (0, \pi) \times [0, 2\pi] \to \mathbb{R}^3$ is defined by

$$T(\vartheta, \varphi) := (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta).$$

Now we obtain

$$\begin{aligned} I(p) &:= \int_{S^2} (r_1(u) + r_2(u))^p \mathscr{H}^2(du) \\ &= 2\pi \int_0^\pi [2r(\sin \vartheta)^{1/p} + R(\sin \vartheta)^{(1-p)/p}]^p \, d\vartheta. \end{aligned}$$

If $p \in [1, 2)$, it follows that

$$I(p) \leqslant 2^p \pi \int_0^\pi [(2r)^p \sin \vartheta + R^p (\sin \vartheta)^{1-p}] \, d\vartheta.$$

The integral on the right-hand side is finite, since $p - 1 < 1$ and

$$\lim_{\vartheta \to 0} [(\sin \vartheta)^{1-p} \vartheta^{p-1}] = 1.$$

But for $p \geqslant 2$ one obtains $I(p) = \infty$. On the other hand, $S_2(K, \cdot)$ even has point masses.

By polarity a corresponding example for curvature measures is obtained. In fact, this follows from the first transfer principle and Eq. (8).

We now turn to the stability theorem which was mentioned in the Introduction and stated as Theorem 3.4 in Section 3. A familiar way of establishing stability and uniqueness results for balls is to use symmetrization techniques. This is also the method which was used by Diskant in order to prove stability results for convex bodies K for which $S_{d-1}(K,\cdot)$ or $C_0(K,\cdot)$ are close to the corresponding measures of the unit ball $B^d(o,1)$. It is surprising, however, that it is possible to improve Diskant's result for the $(d-1)$th surface area measure by means of Diskant's stability result for the Gauss curvature measure $C_0(K,\cdot)$.

Proof of Theorem 3.4. We can assume that $\varepsilon > 0$. The assumption of Theorem 3.4 implies that $\operatorname{int} K \neq \emptyset$ and

$$1-\varepsilon \leqslant D_{d-1}h(K,u) \leqslant 1+\varepsilon, \tag{9}$$

for $\mathscr{H}^{d-1}$ almost all $u \in S^{d-1}$. By Theorem 2.3 in [20], the left-hand side of (9) yields that

$$C_0(K,\cdot) \ll C_{d-1}(K,\cdot); \tag{10}$$

moreover, the density function is given by $H_{d-1}(K,\cdot)$. Let $\omega_0 \subset S^{d-1}$ be the set of all $u \in S^{d-1}$ such that h_K is not second-order differentiable at u or (9) is not satisfied. Hence, we get

$$0 = S_0(K,\omega_0) \geqslant (1+\varepsilon)^{-1} S_{d-1}(K,\omega_0) = (1+\varepsilon)^{-1}\mathscr{H}^{d-1}(\tau(K,\omega_0)) \geqslant 0.$$

Let $\mathscr{M}(K)$ denote the set of normal boundary points of K. Then, for $x \in \mathscr{M}(K)\setminus\tau(K,\omega_0)$, and hence for $\mathscr{H}^{d-1}$ almost all $x \in \operatorname{bd} K$, we obtain that

$$H_{d-1}(K,x) D_{d-1}h(K,\sigma_K(x)) = 1; \tag{11}$$

see Remark 2 after Lemma 2.5 in [16]. From (9) and (11) we deduce

$$1-\varepsilon \leqslant (1+\varepsilon)^{-1} \leqslant H_{d-1}(K,x) \leqslant (1-\varepsilon)^{-1} \leqslant 1+2\varepsilon, \tag{12}$$

since $0<\varepsilon<\frac{1}{2}$. Thus (10) and (12) imply

$$(1-2\varepsilon) C_{d-1}(K,\cdot) \leqslant C_0(K,\cdot) \leqslant (1+2\varepsilon) C_{d-1}(K,\cdot).$$

Now the proof is completed by applying Theorem 1 of Diskant [7]; cf. [26, Theorem 7.2.11]. ∎

Remark 6.1. Let $K \coloneqq B^d(o,(1+\varepsilon)^{1/(d-1)})$, $0<\varepsilon<\frac{1}{4}$. Then the assumptions of Theorem 3.4 are fulfilled, but the Hausdorff distance of K to an arbitrary unit ball is greater than or equal to

$$\frac{1}{d-1}\left(\frac{4}{5}\right)^{(d-2)/(d-1)}\varepsilon.$$

Therefore, the exponent of ε (namely 1) in the conclusion of Theorem 3.4 cannot be improved in general.

The proof of Theorem 3.4 also suggests the following consequence, which we include for the sake of completeness.

COROLLARY 6.1. *Let $K \in \mathcal{K}^d_o$, let $\omega \subset S^{d-1}$ be Borel measurable, and let $0 < \alpha \leqslant \beta < \infty$. Then the following conditions are equivalent*:

(a) $\alpha S_0(K,\cdot) \llcorner \omega \leqslant S_{d-1}(K,\cdot) \llcorner \omega \leqslant \beta S_0(K,\cdot) \llcorner \omega$;
(b) $\frac{1}{\beta} C_{d-1}(K,\cdot) \llcorner \tau(K,\omega) \leqslant C_0(K,\cdot) \llcorner \tau(K,\omega) \leqslant \frac{1}{\alpha} C_{d-1}(K,\cdot) \llcorner \tau(K,\omega)$.

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