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Almost Transversal Intersections of Convex Surfaces and Translative Integral Formulae

By Daniel Hug*) of Freiburg, Peter Mani–Levitska of Bern and Reiner Schätzle of Bonn

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Abstract. This work consists of three parts, the first and second of which are concerned with generalized forms of two conjectures by Wm. J. Firey (1978). Let $K, L \subset \mathbb{R}^n$ be compact convex sets which have common interior points and intersect almost transversally. Let $K_r,\ L_r,\ r \geq 0$, denote the outer parallel bodies of K, L at distance r and set $S_r := \partial K_r \cap \partial L_r$, $H_r := \partial K_r \cap L_r$. It has been conjectured by Firey that the Euler characteristic of these sets is independent of $r > 0$. More generally, it has been shown in [10] that S_r and S_0 as well as H_r and H_0 are homotopy equivalent for all $r \geq 0$. In the present work, we prove that, for all $r \geq 0$, S_r and S_0 as well as H_r and H_0 are in fact bi–lipschitz homeomorphic lipschitz submanifolds of $\mathbb{R}^n$. In the second part, we establish translative integral geometric formulae involving such intersections for arbitrary pairs of compact convex sets. Formulae of this type have recently been proved in [10] for compact convex sets with interior points, while very special cases of these have been conjectured by Firey. Finally, the last part is devoted to a theoretical study of general convex surfaces in stochastic geometry.

1. Introduction

In 1978, Wm. J. Firey contributed two conjectures to a collection of open problems in geometric convexity (see [9]). The setting for these conjectures is the Euclidean space $\mathbb{R}^n$, $n \geq 1$, with scalar product $\langle \cdot\, , \cdot \rangle$ and norm $|\cdot|$. Let $K, L \subset \mathbb{R}^n$ be compact convex sets with nonempty interiors and boundaries ∂K, ∂L. For $r \geq 0$ let K_r, L_r denote the set of points of $\mathbb{R}^n$ whose respective distance from K, L is r at the most. Then Firey conjectured that the *Euler characteristics* $\chi(\partial K_r \cap \partial g L_r)$ and $\chi(\partial K_r \cap g L_r)$, defined as in singular homology theory, are independent of $r > 0$, at least for μ almost all $g \in \mathbf{G}(n)$, where μ is a Haar measure on the group $\mathbf{G}(n)$ of proper rigid motions of $\mathbb{R}^n$ and $n \geq 2$. Firey also suggested to use such a result in

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*) Corresponding author/daniel.hug@math.uni-freiburg.de

order to obtain a conjectured extension of the *principal kinematic formula* of integral geometry to general *convex surfaces*; see [10] for further background information.

Both conjectures have recently been established in generalized forms in [10]. In order to state precise results, we need some definitions. Let $K, L \subset \mathbb{R}^n$ be compact, convex sets. We say that K and L *intersect almost transversally* if K and L have common interior points and if, for all $p \in \partial K \cap \partial L$, the normal cone of K at p, $N(K,p)$, intersects $N(L,p)$ only in the zero vector (see [15] for further definitions). Let $\mathcal{H}^n$ denote n–dimensional Hausdorff measure in $\mathbb{R}^n$. It is an important fact that K and $L+t$ intersect almost transversally, for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$ such that $K \cap (L+t) \neq \emptyset$; see [10], Lemma 3.1. Now we set $S_r := \partial K_r \cap \partial L_r$ and $H_r := \partial K_r \cap L_r$ for $r \geq 0$. Whereas Firey's first conjecture concerned the Euler characteristics of S_r and H_r, for $r > 0$, it was proved in [10], Theorems 1.3 and 1.4, that the *homotopy types*, and therefore the Euler characteristics, of these intersections are independent of $r \geq 0$ if K and L intersect almost transversally. The crucial points here are that the case $r = 0$ is included and the almost transversal intersection of two convex surfaces is a natural and convenient geometric condition. Indeed, these points are involved in an essential way in the derivation of translative integral geometric formulae.

The topological results established in [10], and the additional information available from the corresponding proofs (described subsequently), suggest some further study which is carried out in the first part of the present contribution. In the following, we always assume that $K, L \subset \mathbb{R}^n$ are compact, convex sets which intersect almost transversally. Under this assumption, it is shown in [10] that S_r, for $r \geq 0$, is an $(n-2)$–dimensional compact *lipschitz submanifold* of $\mathbb{R}^n$ without boundary, S_r is even a C^1 submanifold for $r > 0$, and S_r is homeomorphic to S_s via a *bi–lipschitz map* for all $r, s > 0$. Moreover, it is proved that H_r, for $r \geq 0$, is an $(n-1)$–dimensional compact lipschitz submanifold of $\mathbb{R}^n$ with $\partial H_r = S_r$, H_r is a C^1 submanifold for $r > 0$, and H_r is homeomorphic to H_s via a bi–lipschitz map for all $r, s > 0$. Therefore the following questions naturally arise.

Are S_r and S_s diffeomorphic of class C^1 for all $r, s > 0$?

Is S_r homeomorphic to S_0 via a bi–lipschitz map for all $r > 0$?

These questions and their analogues for intersections of boundaries and bodies are solved affirmatively in Section 2, where we prove the following two main theorems (see also Corollaries 2.12 and 2.13). Here we write $X \cong Y$ in order to indicate that the topological spaces X and Y are homeomorphic.

Theorem 1.1. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets with common interior points which intersect almost transversally. Define the intersections*

$$S_r \ := \ \partial K_r \cap \partial L_r \,, \quad r \ \geq \ 0 \,.$$

Then all S_r are bi–lipschitz homeomorphic to each other; in particular,

$$S_r \ \cong \ S_s \quad \text{for all} \quad r\,, s \ \geq \ 0 \,.$$

For the intersections of boundaries and bodies, we shall prove:

Theorem 1.2. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets with common interior points which intersect almost transversally. Define the intersections*

$$H_r \;:=\; \partial K_r \cap L_r\,, \quad r \;\geq\; 0\,.$$

Then all H_r are bi–lipschitz homeomorphic to each other; in particular,

$$H_r \;\cong\; H_s \quad \textit{for all} \quad r\,,\, s \;\geq\; 0\,.$$

Compared to the results in [10], the main novelty here is the inclusion of the case $r = 0$ and thus a positive answer to the second question raised before. In addition, our method of proof will lead to an affirmative resolution of the preceding question concerning the diffeomorphy of the sets S_r (resp. H_r), for $r > 0$.

A new element in our approach to these results is the construction of a *strongly transversal pair of lipschitz vector fields* for S_0 (a definition is given in Section 2). The related notion of a field of transverse planes to a lipschitz manifold has been studied by J. H. C. Whitehead in [25]. Such a pair of lipschitz vector fields is used to obtain a lipschitz parametrization of a *tubular neighbourhood* of S_0. As two additional transverse parameters we finally introduce the *signed distance functions* of ∂K and ∂L. To accomplish the required transition between different descriptions of a tubular neighbourhood of S_0 we use tools from *nonsmooth analysis* such as *inverse* and *implicit function theorems* for lipschitz maps.

Transverse fields appeared in the context of early smoothing theory. In particular, refining and extending previous work by Whitney [26] and Cairns [2], J. H. C. Whitehead showed that the existence of a transverse field for some topological manifold M in $\mathbb{R}^n$ implies that M has a smooth atlas, although not necessarily as a submanifold of $\mathbb{R}^n$. It follows, from our results, that every almost transversal intersection of two general convex surfaces has a smooth atlas. We do not know whether a similar statement remains true if the number of convex surfaces increases to three, or more.

A major motivation to study topological properties of intersections of convex surfaces comes from an attempt to extend global kinematic integral geometric formulae involving the Euler characteristic to arbitrary convex surfaces. To be more specific, for $n \geq 2$ Firey conjectured the formula

$$\int_{\mathbf{G}(n)} \chi(\partial K \cap gL)\,\mu(dg) \;=\; \frac{1}{\kappa_n}\sum_{k=0}^{n-1}\binom{n}{k}\big(1-(-1)^{n-k}\big)W_{n-k}(K)W_k(L)\,,$$

together with its counterpart for intersections of two boundaries, where κ_n denotes the volume of the n–dimensional Euclidean unit ball B^n and the functionals $W_0,\ldots,W_n$ are the classical *quermassintegrals* (*Minkowski functionals*). The Minkowski functionals are very special examples of *mixed volumes* (see [15], and [10] for a short exposition). The latter are in fact involved in the following more general *translative integral geometric formula*,

$$\begin{aligned}(1.1)\qquad &\int_{\mathbb{R}^n} \chi(\partial K \cap (L+t))\,\mathcal{H}^n(dt)\\ &= \sum_{i=0}^{n}\binom{n}{i}\big\{V(K[n-i],-L[i]) + (-1)^{i-1}V(K[n-i],L[i])\big\}\,,\end{aligned}$$

which has been established in [10] for compact, convex sets $K, L \subset \mathbb{R}^n$ with nonempty interiors and $n \geq 2$. There also the following integral formula has been obtained, under the same assumptions, for the intersections of two convex surfaces:

$$\begin{aligned}(1.2)\qquad &\int_{\mathbb{R}^n} \chi(\partial K \cap (\partial L + t))\, \mathcal{H}^n(dt) \\ &= (1 + (-1)^n) \sum_{i=0}^{n} \binom{n}{i} \left\{ V(K[i], -L[n-i]) + (-1)^{i-1} V(K[i], L[n-i]) \right\}.\end{aligned}$$

Equation (1.2) can be deduced from equation (1.1), once the former has been established and one knows that H_0 and S_0 are lipschitz submanifolds of $\mathbb{R}^n$ with $\partial H_0 = S_0$ (compare [6], Corollary 8.8). In [10], the proof of both results is based on a combination of analytic, topological and integral geometric arguments. The topological results require the restriction to sets with nonempty interiors. In this paper, we extend both equations to arbitrary compact, convex sets by means of some additional integral geometric considerations. In retrospect, it would have been possible to derive these extensions in [10], on the basis of the results and tools provided there.

Henceforth, we write ∂K for the *relative boundary* of a compact convex set $K \subset \mathbb{R}^n$.

Theorem 1.3. *Let $K, L \subset \mathbb{R}^n$, $n \geq 1$, be compact convex sets, and set $k := \dim K$, $l := \dim L$. Then the map $t \mapsto \chi(\partial K \cap (\partial L + t))$ is integrable with respect to $\mathcal{H}^n$ on $\mathbb{R}^n$ and*

$$\begin{aligned}(1.3)\qquad &\int_{\mathbb{R}^n} \chi(\partial K \cap (\partial L + t))\, \mathcal{H}^n(dt) \\ &= \left(1 + (-1)^{k+l-n}\right) \sum_{i=0}^{n} \binom{n}{i} \left\{ V(K[i], -L[n-i]) + (-1)^{n-l+i-1} V(K[i], L[n-i]) \right\}.\end{aligned}$$

Theorem 1.4. *Let $K, L \subset \mathbb{R}^n$, $n \geq 1$, be compact convex sets, and set $k := \dim K$. Then the map $t \mapsto \chi(\partial K \cap (L + t))$ is integrable with respect to $\mathcal{H}^n$ on $\mathbb{R}^n$ and*

$$\begin{aligned}(1.4)\qquad &\int_{\mathbb{R}^n} \chi(\partial K \cap (L + t))\, \mathcal{H}^n(dt) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left\{ V(K[i], -L[n-i]) + (-1)^{k-i-1} V(K[i], L[n-i]) \right\}.\end{aligned}$$

The range of the indices in (1.1) and (1.2) is extended formally, in comparison with formulae (1.5) and (1.4) in [10], but the additional summands all vanish. In a similar way, the summation in Theorem 1.4 can be restricted to $i \in \{0, \ldots, k-1\}$. The proof of Theorem 1.4 will be given in Section 3, Theorem 1.3 can be established by similar arguments. However, in the case of lower–dimensional compact, convex sets, it does not seem to be possible to deduce Theorem 1.3 directly from Theorem 1.4. In Section 3, we shall also treat *iterated translative integral formulae* and an extension of Theorem 1.4 to elements L of the *convex ring*. Such additional results are important prerequisites for applications in *stochastic geometry*.

The final section is designed to provide a theoretical study of convex surfaces in stochastic geometry. Our intention is to demonstrate in an exemplary way how the integral geometric results obtained in Section 3 can be used to extend known results and methods to admit the treatment of convex surfaces as basic particles of point processes or as test sets. This is not completely straightforward, since the Euler characteristic is not a *locally bounded* functional (cf. [24], p. 332, or [18], p. 184) in the present setting.

2. Homeomorphy via transversal fields

Throughout this section we consider two compact convex sets $K, L \subset \mathbb{R}^n$ with common interior points, that is

$$\operatorname{int} K \cap \operatorname{int} L \neq \emptyset\,, \tag{2.1}$$

which intersect almost transversally, that is which satisfy

$$N(K,p) \cap N(L,p) = \{0\} \tag{2.2}$$

and

$$N(K,p) \cap (-N(L,p)) = \{0\} \tag{2.3}$$

for all $p \in \partial K \cap \partial L$. Clearly, condition (2.3) is implied by (2.1).

We put $M := \partial K \cap \partial L$ and know from [10] that M is an $(n-2)$–dimensional lipschitz manifold without boundary. (The reader should be warned that the symbol M appears with a different meaning in [10].)

Subsequently, $B^n(a,r)$ denotes the Euclidean unit ball of radius $r \geq 0$ centred at $a \in \mathbb{R}^n$, and $S^{n-1} := \partial B^n$. Further notation will be introduced when necessary.

2.1. Construction of strongly transversal fields

We define the open convex tangent cone (inner cone) of K at $p \in K$ by

$$I(K,p) := \{\lambda(y-p) \mid y \in \operatorname{int} K,\ \lambda > 0\}\,.$$

Roughly speaking, normal cones and tangent cones are connected by duality of convex cones. This connection will be used in the proof of the following lemma.

Lemma 2.1. *Let $K, L \subset \mathbb{R}^n$ be two compact convex sets with nonempty interiors, and let $p \in \partial K \cap \partial L$. Then* (2.2) *is satisfied if and only if*

$$I(K,p) \cap (-I(L,p)) \neq \emptyset\,,$$

and (2.3) *is satisfied if and only if*

$$I(K,p) \cap I(L,p) \neq \emptyset\,.$$

Proof. We start with some preliminary remarks. Let us define the support cone $S(K,p)$ of K at p as in [15] by setting

$$S(K,p) := \operatorname{cl}\{\lambda(y-p) \mid y \in K,\ \lambda > 0\}\,,$$

where "cl" designates the formation of the topological closure. Obviously, we have $I(K,p) \subseteq S(K,p)$ and $\operatorname{cl} I(K,p) = S(K,p)$. By Theorem 1.1.14 in [15] we thus see that $I(K,p) = \operatorname{int} S(K,p)$.

It is sufficient to prove the first assertion. Obviously, condition (2.2) is violated if and only if there is some $u \in \mathbb{R}^n \setminus \{0\}$ such that

$$\{\lambda u \mid \lambda \geq 0\} \subseteq N(K,p) \cap N(L,p)$$

or equivalently

$$(N(K,p) \cap N(L,p))^* \subseteq \{z \in \mathbb{R}^n \mid \langle z, u\rangle \leq 0\}\,, \tag{2.4}$$

where "$*$" denotes the formation of the dual cone; see [15, p. 34]. By Theorem 1.6.3 and equation (2.2.1) in [15], it follows that

$$(N(K,p) \cap N(L,p))^* \;=\; S(K,p) + S(L,p)\,,$$

and hence condition (2.4) is satisfied if and only if

$$S(K,p) + S(L,p) \subseteq \{z \in \mathbb{R}^n \mid \langle z, u\rangle \leq 0\} \tag{2.5}$$

for some $u \in \mathbb{R}^n \setminus \{0\}$. It can easily be inferred from the separation Theorem 1.3.8 in [15] that condition (2.5) is fulfilled if and only if $I(K,p) \cap (-I(L,p)) = \emptyset$, which completes the proof. □

In [25], Whitehead defined the notion of a field of transverse planes to a manifold. Specifying a basis for each of the transverse planes, we find the following definition useful.

Definition 2.2. A pair of continuous mappings $x, y : M \to \mathbb{R}^n$ such that, for each $p \in M$,

$$x(p) \in I(K,p) \cap I(L,p) \quad \text{and} \quad y(p) \in I(K,p) \cap (-I(L,p)) \tag{2.6}$$

is called a *strongly transversal pair of vector fields* (for M). If x and y are lipschitz mappings, then the pair x, y is said to be a *strongly transversal pair of lipschitz vector fields* (for M).

The existence of such vector fields is shown in the next proposition.

Proposition 2.3. *There exists a strongly transversal pair of lipschitz vector fields. Moreover, let $x_0 \in \operatorname{int} K \cap \operatorname{int} L$ be arbitrarily fixed. Then x can be chosen such that*

$$x(p) \;=\; x_0 - p\,, \quad p \in M\,. \tag{2.7}$$

Proof. Clearly, a map x defined as in (2.7) is lipschitz and

$$x(p) \in I(K,p) \cap I(L,p)$$

for all $p \in M$.

Next we construct the map y. We consider any $p_0 \in M$. Since K and L intersect almost transversally and by Lemma 2.1, we can select

$$y_0 \in I(K,p_0) \cap (-I(L,p_0)) \;\neq\; \emptyset\,.$$

By definition we know that there exist $\lambda, \mu > 0$ such that

$$p_0 + \lambda y_0 \in \operatorname{int} K \quad \text{and} \quad p_0 - \mu y_0 \in \operatorname{int} L\,.$$

As $\operatorname{int} K$ and $\operatorname{int} L$ are open und nonempty, there exists an open neighbourhood $U(p_0)$ of p_0 in $\mathbb{R}^n$ such that

$$p + \lambda y_0 \in \operatorname{int} K \quad \text{and} \quad p - \mu y_0 \in \operatorname{int} L\,,$$

for all $p \in U(p_0) \cap M$. Therefore

$$\begin{aligned} y_0 &= \lambda^{-1}(p + \lambda y_0 - p) \in I(K,p)\,, \\ -y_0 &= \mu^{-1}(p - \mu y_0 - p) \in I(L,p)\,, \end{aligned}$$

and hence

$$\overline{y}(p_0) \;:=\; y_0 \in I(K,p) \cap (-I(L,p))\,, \tag{2.8}$$

for all $p \in U(p_0) \cap M$. As M is compact, we may select a finite cover

$$M = \bigcup_{i=1}^{N} (U(p_i) \cap M)$$

and a subordinated partition of unity $\varphi_i \in C_0^\infty(U(p_i))$, $i \in \{1, \ldots, N\}$, with $0 \le \varphi_i \le 1$ and

$$\sum_{i=1}^{N} \varphi_i = 1 \quad \text{on} \quad M\,.$$

Then we define

$$y(p) \;:=\; \sum_{i=1}^{N} \varphi_i(p)\overline{y}(p_i)\,, \quad p \in M\,.$$

Clearly, $y : M \to \mathbb{R}^n$ is lipschitz. Now let $p \in M$ be fixed for the moment. Then we have indices $1 \le i_1 < \ldots < i_l \le N$ such that

$$p \in U(p_i)\,, \quad i \in \{i_1, \ldots, i_l\}\,,$$

and

$$p \notin U(p_i)\,, \quad i \notin \{i_1, \ldots, i_l\}\,.$$

Hence (2.8) implies that

$$\overline{y}\big(p_{i_j}\big) \in I(K,p) \cap (-I(L,p))\,, \quad j \in \{1, \ldots, l\}\,,$$

and

$$\sum_{j=1}^{l} \varphi_{i_j}(p) = 1\,.$$

This yields

$$y(p) = \sum_{j=1}^{l} \varphi_{i_j}(p)\overline{y}\big(p_{i_j}\big) \in I(K,p) \cap (-I(L,p))\,,$$

since the open tangent cones are convex. This proves the assertion, since $p \in M$ was arbitrarily chosen. □

Remark 2.4. The proof of Proposition 2.3 is similar to the proof of MICHAEL'S selection theorem (see [12], Theorem 3.2″, or Theorem 1.6 in [1]). Instead of trying to apply such an abstract result, we decided to proceed in a more direct (and more elementary) way, which also yields additional information. In fact, if K and L are of class C^1, then M is of class C^1 and so are the lipschitz vector fields constructed in Proposition 2.3.

2.2. Lipschitz parametrization of a tubular neighbourhood

The strongly transversal pair of lipschitz vector fields of the previous subsection will be used to parametrise a tubular neighbourhood of M. We recall that M has codimension 2. As additional two parameters we can choose the signed distance functions of ∂K and ∂L (see Proposition 2.9), where the signed distance function $d_{\partial Z}$ of a nonempty set $Z \subset \mathbb{R}^n$ is defined by

$$d_{\partial Z}(x) \;:=\; d(Z, x) - d(\mathbb{R}^n \setminus Z, x)\,, \quad x \in \mathbb{R}^n\,.$$

We start with a preparatory definition.

Definition 2.5. Let $x, y : M \to \mathbb{R}^n$ be a strongly transversal pair of lipschitz vector fields for M. Then lipschitz maps Φ, F are defined by

$$\Phi \;:\; M \times \mathbb{R}^2 \longrightarrow \mathbb{R}^n\,, \quad \Phi(p, \lambda, \mu) \;:=\; p + \lambda x(p) + \mu y(p)\,, \tag{2.9}$$

and

$$F \;:\; M \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2\,, \quad F(p, \lambda, \mu) \;:=\; (d_{\partial K}, d_{\partial L})(\Phi(p, \lambda, \mu))\,. \tag{2.10}$$

The desired parametrization will be constructed by using the inverse and implicit function theorems for lipschitz maps (see [4], Section 7) and the maps Φ and F. To this end, we have to calculate several generalized Jacobians in the next three lemmas. We first turn to Φ which gives a parametrization of an open neighbourhood in terms of the transverse parameters λ, μ.

Lemma 2.6. *For each $p_0 \in M$ there exists an open neighbourhood $V \subseteq M \times \mathbb{R}^2$ of $(p_0, 0) \in M \times \mathbb{R}^2$ such that Φ maps V homeomorphically and bi–lipschitz onto an open neighbourhood of $p_0 \in \mathbb{R}^n$.*

Proof. Let $p_0 \in M$ be fixed. Recall that M is an $(n-2)$–dimensional lipschitz manifold without boundary (see [10]).

First, we shall choose a simple bi–lipschitz chart onto an open neighbourhood of $p_0 \in M$. Second, we compute the generalized Jacobian of Φ in $(p_0, 0)$, according to [4], and we show that this Jacobian has full rank. Finally, the lemma is deduced by the application of an inverse function theorem for lipschitz maps; see [4], Theorem 7.1.1.

Following the proof of Proposition 2.2 in [10], we obtain more precise information about the form of a particular chart. To simplify the notation, we assume that

$$x(p_0) \;=\; \gamma e_n \quad \text{and} \quad y(p_0) \;=\; \alpha e_1 + \beta e_n\,, \tag{2.11}$$

where $\gamma > 0$, $\alpha < 0$ and $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

By the definition of the inner cone, there is some $\lambda > 0$ such that $p_0 + \lambda x(p_0) =: x_0 \in \operatorname{int} K \cap \operatorname{int} L$. Thus we see that there are $\delta > 0$, $x_0 = (y_0, t_0)$ and convex lipschitz maps

$$f,\, g \;:\; U(y_0, \delta) \;\longrightarrow\; \mathbb{R}\,,$$

where $U(y_0, \delta) \subset \mathbb{R}^{n-1}$ denotes an open ball of radius δ centred at y_0, such that

$$\partial K \cap (U(y_0, \delta) \times (-\infty, t_0)) \;=\; \operatorname{graph}(f | U(y_0, \delta))$$

and

$$\partial L \cap (U(y_0, \delta) \times (-\infty, t_0)) \;=\; \operatorname{graph}(g | U(y_0, \delta))\,.$$

Clearly, for vectors $v \in \partial f(y_0)$ and $w \in \partial g(y_0)$ in the generalized subgradient (subdifferential) of f, g, respectively, we get

$$(v, -1) \in N(K, p_0) \quad \text{and} \quad (w, -1) \in N(L, p_0)\,.$$

Since $y(p_0) \in I(K, p_0) \cap (-I(L, p_0))$, we see that

$$-\langle (v, -1), y(p_0) \rangle \;\geq\; \mu_0\, |(v, -1)| \;\geq\; \mu_0 \;>\; 0$$

and

$$\langle (w, -1), y(p_0) \rangle \;\geq\; \mu_0\, |(w, -1)| \;\geq\; \mu_0 \;>\; 0$$

for some $\mu_0 > 0$ independent of the particular choice of v, w. Then (2.11) implies that

$$\alpha \langle v, e_1 \rangle \;\leq\; -\mu_0 + \beta \;\leq\; \alpha \langle w, e_1 \rangle - 2\mu_0\,.$$

In particular, as $\mu_0 > 0 > \alpha$, there is $\mu > 0$ such that

$$\langle v - w, e_1 \rangle \;\geq\; \mu \quad \text{for} \quad v \in \partial f(y_0)\,, \quad w \in \partial g(y_0)\,.$$

Putting

$$\Lambda(y) \;:=\; f(y) - g(y)\,,$$

we get as in [10], Proposition 2.2, (see also [4], Theorem 2.5.1) that

$$\langle \xi, e_1 \rangle \;\geq\; \mu \quad \text{for} \quad \xi \in \partial \Lambda(y_0)\,;$$

in particular $\xi \neq 0$. Here the generalized gradient $\partial \Lambda(y_0)$ of the lipschitz map Λ at y_0 is defined as in [4], p. 27.

Now applying the implicit function theorem (see [4], Corollary 7.1.3), we get as in [10], Proposition 2.2, that locally

$$[f = g] \;=\; [\Lambda = 0] \;=\; \{(\zeta(z), z)\}$$

for some lipschitz function $\zeta : U(z_0) \subseteq \mathbb{R}^{n-2} \to \mathbb{R}$ which is defined on an open neighbourhood $U(z_0)$ of some $z_0 \in \mathbb{R}^{n-2}$. Putting

$$h(z) := (\zeta(z), z, f(\zeta(z), z)) = (\zeta(z), z, g(\zeta(z), z)),$$

we obtain the bi–lipschitz chart

$$h : U(z_0) \subseteq \mathbb{R}^{n-2} \longrightarrow U(p_0) \subseteq M, \quad h(z_0) = p_0,$$

we were looking for. Note that here and in the following we do not strictly distinguish between row and column vectors. We define

$$\hat{\Phi}(z, \lambda, \mu) := \Phi(h(z), \lambda, \mu) = h(z) + \lambda x(h(z)) + \mu y(h(z)),$$

and we have to prove that $\hat{\Phi}$ maps an open neighbourhood of $(z_0, 0, 0)$ onto an open neighbourhood of $p_0 \in \mathbb{R}^n$. To this end, we want to invoke the inverse function theorem [4], Theorem 7.1.1. We have to establish that each member

$$A \in \partial\hat{\Phi}(z_0, 0, 0)$$

of the generalized Jacobian of $\hat{\Phi}$ has full rank. Actually, to avoid unnecessary complications on sets of measure zero, we consider the smaller set

$$\partial_S\hat{\Phi}(z_0, 0, 0) := \operatorname{conv}\Big\{ \lim_{i\to\infty} D\hat{\Phi}(z_i, \lambda_i, \mu_i) \mid (z_i, \lambda_i, \mu_i) \to (z_0, 0, 0), \quad h, x \circ h, y \circ h \text{ are differentiable at } z_i \Big\}, \tag{2.12}$$

where "conv" denotes the formation of the convex hull; see [4], Proposition 2.6.4. We recall that lipschitz functions are differentiable almost everywhere by Rademacher's theorem.

We will show that

$$Av \neq 0 \quad \text{for} \quad A \in \partial_S\hat{\Phi}(z_0, 0, 0), \quad v \in \mathbb{R}^n \setminus \{0\}. \tag{2.13}$$

We consider (z_i, λ_i, μ_i) as in (2.12) and calculate

$$B \longleftarrow D\hat{\Phi}(z_i, \lambda_i, \mu_i) = \begin{pmatrix} \nabla\zeta(z_i) & & \\ I_{n-2} & x(h(z_i)) & y(h(z_i)) \\ (\nabla_z f(\zeta(\cdot), \cdot))(z_i) & & \end{pmatrix} + \lambda_i\, (D(x \circ h)(z_i), 0, 0) + \mu_i\, (D(y \circ h)(z_i), 0, 0).$$

Since $\nabla\zeta, \nabla_z f(\zeta(\cdot), \cdot), D(x \circ h), D(y \circ h)$ are bounded and $\lambda_i, \mu_i \to 0$, we see that

$$B = \begin{pmatrix} \xi & 0 & \alpha \\ I_{n-2} & 0 & 0 \\ \eta & \gamma & \beta \end{pmatrix}$$

for some $\xi, \eta \in \mathbb{R}^{n-2}$.

Since this form of B is closed under taking convex combinations, we infer that all $A \in \partial_S \hat{\Phi}(z_0, 0, 0)$ have the form above. In particular A is nonsingular, as $\alpha, \gamma \neq 0$, which implies (2.13), and the lemma is proved. □

To switch from the transverse parameters λ, μ to the signed distance functions, we calculate the generalized derivative of the signed distance functions. Although the following lemma may be known, we shall give the proof for the reader's convenience.

Lemma 2.7. *Let $p_0 \in \partial K$ and $c \in I(K, p_0)$. Then*

$$\langle \partial d_{\partial K}(p_0), c \rangle \subseteq (-\infty, 0)\,. \tag{2.14}$$

Proof. Assume that (2.14) is not true. Then by Theorem 2.5.1 in [4], we can find a sequence z_j, $j \in \mathbb{N}$, such that $z_j \notin \partial K$, $z_j \to p_0$ as $j \to \infty$, $d_{\partial K}$ is differentiable at z_j, and

$$\lim_{j \to \infty} \langle \nabla d_{\partial K}\big(z_j\big), c \rangle \;\geq\; 0\,. \tag{2.15}$$

By Proposition 2.5.4 in [4], for each $j \in \mathbb{N}$ there is some $p_j \in \partial K$ satisfying $\big|z_j - p_j\big| = d\big(\partial K, z_j\big) > 0$ and

$$\begin{aligned} v_j \;&:=\; \nabla d_{\partial K}\big(z_j\big) \;=\; \frac{z_j - p_j}{\big|z_j - p_j\big|} \quad \text{if} \quad z_j \notin K\,, \\ v_j \;&:=\; \nabla d_{\partial K}(z_j) \;=\; -\frac{z_j - p_j}{\big|z_j - p_j\big|} \quad \text{if} \quad z_j \in K\,. \end{aligned}$$

Now if $z_j \notin K$, then $K \subseteq \big\{z \in \mathbb{R}^n \mid \big\langle v_j, z - p_j \big\rangle \leq 0\big\}$ and $v_j \in N\big(K, p_j\big)$. If $z_j \in \operatorname{int} K$, then we set $\varrho_j := \big|z_j - p_j\big|$ and obtain $B^n\big(z_j, \varrho_j\big) \subseteq K$ with $p_j \in \partial B^n\big(z_j, \varrho_j\big)$. Therefore

$$v_j \;=\; \frac{p_j - z_j}{\big|p_j - z_j\big|} \in N\big(B^n\big(z_j, \varrho_j\big), p_j\big) \;=\; N(K, p_j)\,.$$

Taking a subsequence, we obtain $v_j \to v \in S^{n-1}$. As clearly $\big|z_j - p_j\big| \leq \big|z_j - p_0\big| \to 0$, and hence $p_j \to p_0$, we deduce that $v \in N(K, p_0) \cap S^{n-1}$. Since $c \in I(K, p_0)$ and $I(K, p_0)$ is open, this yields $\langle v, c \rangle < 0$. On the other hand, we obtain from (2.15) that $\langle v, c \rangle \geq 0$, which is a contradiction. □

With the previous lemma, we are now able to compute the generalized derivatives of the function F defined in (2.10). This allows us to switch from the transverse parameters to the signed distance functions.

Lemma 2.8. *For any $p_0 \in M$, there exist an open neighbourhood $U(p_0) \subseteq M$ of $p_0 \in M$, $\delta_0 > 0$, an open neighbourhood W of p_0 in $\mathbb{R}^n$, and a lipschitz map*

$$\Psi \;:\; U(p_0) \times (-\delta_0, \delta_0)^2 \;\longrightarrow\; W$$

that maps $U(p_0) \times (-\delta_0, \delta_0)^2$ homeomorphically and bi–lipschitz onto W in such a way that

$$d_{\partial K}(\Psi(p, \varrho_1, \varrho_2)) \;=\; \varrho_1 \quad \textit{and} \quad d_{\partial L}(\Psi(p, \varrho_1, \varrho_2)) \;=\; \varrho_2$$

for $p \in U(p_0)$ *and* $|\varrho_1|, |\varrho_2| < \delta_0$.

Proof. We shall define

$$\Psi(p, \varrho_1, \varrho_2) \;:=\; \Phi(p, \lambda(p, \varrho_1, \varrho_2), \mu(p, \varrho_1, \varrho_2)) \tag{2.16}$$

where $\lambda, \mu : U(p_0) \times (-\delta_0, \delta_0)^2 \to \mathbb{R}$ are lipschitz maps obtained by an application of the implicit function theorem for lipschitz maps (compare [4], Corollary 7.1.3) to the map $\widetilde{F} : \mathbb{R}^{n-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\widetilde{F}(p, \varrho_1, \varrho_2, \lambda, \mu) \;:=\; (d_{\partial K}, d_{\partial L}) \circ \Phi(p, \lambda, \mu) - (\varrho_1, \varrho_2)\,.$$

The functions $\lambda(\cdot)$, $\mu(\cdot)$ will then satisfy

$$F(p, \lambda(p, \varrho_1, \varrho_2), \mu(p, \varrho_1, \varrho_2)) \;=\; (\varrho_1, \varrho_2) \tag{2.17}$$

for $p \in U(p_0)$ and $|\varrho_1|, |\varrho_2| < \delta_0$ after choosing suitable $U(p_0) \subseteq M$ and $\delta_0 > 0$. This will establish the conclusion of the lemma.

To apply the implicit function theorem, we have to verify that the (projected) generalized Jacobian (cf. [4], p. 256)

$$\pi_{\lambda,\mu} \partial \widetilde{F}(p_0, 0, 0) \subseteq \mathbb{R}^{2,2}$$

has full rank in the sense of [4], p. 253. In order to verify this condition on the rank, we define the lipschitz functions

$$\varphi(p, \lambda, \mu) \;:=\; d_{\partial K}(\Phi(p, \lambda, \mu)) \;=\; d_{\partial K}(p + \lambda x(p) + \mu y(p))$$

and

$$\psi(p, \lambda, \mu) \;:=\; d_{\partial L}(\Phi(p, \lambda, \mu)) = d_{\partial L}(p + \lambda x(p) + \mu y(p))\,.$$

By the definition of the generalized Jacobian (compare [4], Proposition 2.6.2), we get

$$\pi_{\lambda,\mu} \partial \widetilde{F}(p_0, 0, 0) \subseteq \begin{pmatrix} \partial_\lambda \varphi(p_0, 0, 0) & \partial_\mu \varphi(p_0, 0, 0) \\ \partial_\lambda \psi(p_0, 0, 0) & \partial_\mu \psi(p_0, 0, 0) \end{pmatrix}.$$

Here the right–hand side is defined as the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2,2}$ with $a \in \partial_\lambda \varphi(p_0, 0, 0)$, $b \in \partial_\mu \varphi(p_0, 0, 0)$, $c \in \partial_\lambda \psi(p_0, 0, 0)$ and $d \in \partial_\mu \psi(p_0, 0, 0)$. Now by the chain rule of nonsmooth analysis (see [4], Theorem 2.6.6), and by Lemma 2.7, we get

$$\partial_\lambda \varphi(p_0, 0, 0) \subseteq \operatorname{conv}\left(\langle \partial d_{\partial K}(p_0), x(p_0) \rangle \right) \subseteq (-\infty, 0)\,,$$

and likewise

$$\begin{aligned} \partial_\mu \varphi(p_0, 0, 0) &\subseteq \operatorname{conv}\left(\langle \partial d_{\partial K}(p_0), y(p_0) \rangle \right) \subseteq (-\infty, 0)\,, \\ \partial_\lambda \psi(p_0, 0, 0) &\subseteq \operatorname{conv}\left(\langle \partial d_{\partial L}(p_0), x(p_0) \rangle \right) \subseteq (-\infty, 0)\,, \\ \partial_\mu \psi(p_0, 0, 0) &\subseteq \operatorname{conv}\left(\langle \partial d_{\partial L}(p_0), y(p_0) \rangle \right) \subseteq (0, \infty)\,, \end{aligned}$$

where the last inclusion is due to the fact that $-y(p_0) \in I(L, p_0)$.

This indeed yields that $\pi_{\lambda,\mu} \partial \widetilde{F}(p_0, 0, 0)$ has full rank. □

Lemma 2.6 and Lemma 2.8 both yield parametrizations locally in an open neighbourhood of $(p_0, 0) \in M \times \mathbb{R}^2$. By standard compactness and covering arguments, we now extend these results to global parametrizations of a tubular neighbourhood of M in $\mathbb{R}^n$.

Proposition 2.9. *Let $x, y : M \to \mathbb{R}^n$ be a strongly transversal pair of lipschitz vector fields for M. Then there exist an open neighbourhood $V \subseteq M \times \mathbb{R}^2$ of $M \times \{0\}$ and $\delta > 0$ such that*

(i) *the map Φ, defined in* (2.9)*, maps V homeomorphically and bi–lipschitz onto $U_\delta(M) := \{x \in \mathbb{R}^n \mid |d_{\partial K}(x)|, |d_{\partial L}(x)| < \delta\}$, that is $V \cong U_\delta(M) \subseteq \mathbb{R}^n$;*

(ii) *there is a lipschitz map*

$$\Psi : M \times (-\delta, \delta)^2 \longrightarrow U_\delta(M)$$

that maps $M \times (-\delta, \delta)^2$ homeomorphically and bi–lipschitz onto $U_\delta(M)$ in such a way that

$$d_{\partial K}(\Psi(p, \varrho_1, \varrho_2)) = \varrho_1 , \quad d_{\partial L}(\Psi(p, \varrho_1, \varrho_2)) = \varrho_2 \tag{2.18}$$

for $p \in M$ and $|\varrho_1|, |\varrho_2| < \delta$;

(iii) *for all $p \in M$,*

$$\begin{aligned} B_p :&= \Phi(\{(q, \lambda, \mu) \in V \mid q = p\}) \\ &= \Psi(\{(q, \varrho_1, \varrho_2) \mid q = p, |\varrho_1|, |\varrho_2| < \delta\}) \end{aligned} \tag{2.19}$$

is a two–dimensional open topological ball contained in $E_p := p + \operatorname{lin}\{x(p), y(p)\}$.

Proof. We have already seen that Φ is a local, bi–lipschitz homeomorphism in an open neighbourhood of $M \times \{0\}$. By [25], Lemma 4.1, there is an open neighbourhood $V \subseteq M \times \mathbb{R}^2$ of $M \times \{0\}$ that is mapped homeomorphically by Φ onto an open neighbourhood $U \subseteq \mathbb{R}^n$ of M. If $\delta > 0$ is small enough, then $U_\delta(M) \subseteq U$ and we can choose $V := \Phi^{-1}(U_\delta(M))$.

Likewise, bi–lipschitz mappings satisfying (2.18) were obtained locally in Lemma 2.8. Since M is compact, we can select a finite cover of open neighbourhoods $U(p_j) \subseteq M$, open neighbourhoods W_j of p_j in $\mathbb{R}^n$, $\delta_j > 0$ and bi–lipschitz mappings

$$\Psi_j : U(p_j) \times (-\delta_j, \delta_j)^2 \longrightarrow W_j$$

as in Lemma 2.8, which satisfy (2.18). Choosing $U(p_j)$ and $\delta < \min_j \delta_j$ small enough such that the Ψ_j coincide on the intersections of $U(p_j)$ (compare (2.16) and (2.17)), we can paste together a lipschitz map $\Psi : M \times (-\delta, \delta)^2 \to \mathbb{R}^n$ which is a local, bi–lipschitz homeomorphism in $\mathbb{R}^n$. Note that by construction $\Psi(p, 0, 0) = p$ for all $p \in M$. Hence Lemma 4.1 in [25] can be applied again. The assertions (i) and (ii) now follow if $\delta > 0$ is possibly reduced further.

The remaining assertion (iii) follows from (2.16) and (2.17). □

Remark 2.10. If K and L are parallel bodies of compact convex sets C, D, that is, $K = C_r$, $L = D_s$ for some $r, s > 0$, then the previous constructions show that Φ and Ψ can be chosen to be of class C^1 (compare Remark 2.4). In fact, in this case the distance functions $d_{\partial K}$ and $d_{\partial L}$ are differentiable in a neighbourhood of ∂K and ∂L, respectively.

Lemma 2.11. *Let $\gamma : [0,1] \to E_p \cap U_\delta(M)$ be continuous, and assume that $\gamma(0) \in B_p$. Then $\gamma([0,1]) \subseteq B_p$.*

Proof. Define $A := \{t \in [0,1] \mid \gamma(t) \in B_p\}$. Note that $A \neq \emptyset$ and that A is open, since γ is continuous and B_p is open in $E_p \cap U_\delta(M)$. To complete the proof, we show that A is also closed. Hence let $t_j \in A$ for $j \in \mathbb{N}$ and $t_j \to t \in [0,1]$ as $j \to \infty$. Then $\gamma\big(t_j\big) \in B_p$, for all $j \in \mathbb{N}$, and therefore there are $\lambda_j, \mu_j \in \mathbb{R}$ such that $\gamma\big(t_j\big) = \Phi\big(p, \lambda_j, \mu_j\big)$ and $\big(p, \lambda_j, \mu_j\big) \in V$. Since $\gamma(t) \in U_\delta(M)$ for all $t \in [0,1]$, we obtain $\big(p, \lambda_j, \mu_j\big) = \Phi^{-1}\big(\gamma\big(t_j\big)\big)$. But $\Phi : V \to U_\delta(M)$ is a homeomorphism, and thus

$$\Phi^{-1}(\gamma(t)) \;=\; \lim_{j\to\infty} \Phi^{-1}\big(\gamma\big(t_j\big)\big) \;=\; \lim_{j\to\infty} \big(p, \lambda_j, \mu_j\big) \;=\; (p, \lambda, \mu)$$

for some $\lambda, \mu \in \mathbb{R}$ with $(p, \lambda, \mu) \in V$. This shows that $\gamma(t) = \Phi(p, \lambda, \mu) \in B_p$. □

2.3. Proof of bi–lipschitz homeomorphy

In this subsection, we prove Theorems 1.1 and 1.2 on the homeomorphy of intersections of convex bodies (compact convex sets with nonempty interiors) and boundaries of convex bodies.

Proposition 2.9 already provides all tools which are needed for the proof of Theorem 1.1.

Proof of Theorem 1.1. From [10], Equation (2.7), we know that

$$S_r \cong S_s \quad \text{for} \quad r\,,\, s \;>\; 0\,.$$

Moreover, the homeomorphism was provided by a bi–lipschitz map. Therefore, it suffices to prove that S_0 can be mapped onto S_r by a bi–lipschitz homeomorphism, for some (small) $r > 0$.

We consider the lipschitz map

$$T \;:\; M \;=\; S_0 \;\longrightarrow\; S_r\,,$$

defined by

$$T(p) \;:=\; \Psi(p, r, r)\,,$$

where Ψ is from Proposition 2.9 and $0 < r < \delta$. From (2.18), we get

$$(d_{\partial K}, d_{\partial L})(T(p)) \;=\; (r, r)\,,$$

hence $T(p) \in S_r$.

On the other hand, if $x \in S_r \subseteq U_\delta(M)$, where $U_\delta(M)$ is as defined in Proposition 2.9, there exists $(p, \varrho_1, \varrho_2) \in M \times (-\delta, \delta)^2$ such that

$$\Psi(p, \varrho_1, \varrho_2) \;=\; x\,,$$

since $\Psi : M \times (-\delta, \delta)^2 \to U_\delta(M)$ is bijective. Since $x \in S_r$ and using (2.18) again, we find

$$(r, r) \;=\; (d_{\partial K}, d_{\partial L})(x) \;=\; (d_{\partial K}, d_{\partial L})(\Psi(p, \varrho_1, \varrho_2)) \;=\; (\varrho_1, \varrho_2)\,,$$

hence $\varrho_1 = \varrho_2 = r$ and $x = T(p)$.

This proves that T maps S_0 onto S_r. Since T is clearly injective, as Ψ is injective, and S_0 and S_r are compact, T is a bi–lipschitz homeomorphism as required. □

The difficulty in the case where a body intersects with a boundary is that we have to consider also points which do not lie in the tubular neighbourhood $U_\delta(M)$ which we have parametrised in Proposition 2.9 with the signed distance functions. Clearly, we have to consider points x with $d_{\partial L}(x) < -\delta$.

Therefore we establish the existence of the homeomorphism for Theorem 1.2 in two steps. First, we apply a homeomorphism $\partial K \to \partial K_r$ induced by normal fields, and then we "stretch smoothly out" on ∂K_r using the parametrization of Proposition 2.9 in order to get that the image of H_0 is indeed H_r. All these transformations will also be bi–lipschitz.

Proof of Theorem 1.2. From [10], Equation (2.17), we know that

$$H_r \;\cong\; H_s \quad \text{for} \quad r\,,\, s \;>\; 0\,.$$

In fact, the proof in [10] even shows that H_r and H_s can be transformed into each other by a bi–lipschitz map, for all $r, s > 0$. Therefore, it suffices to prove that H_0 can be mapped onto H_r by a bi–lipschitz homeomorphism, for some (small) $r > 0$.

Fix $x_0 \in \operatorname{int} K \cap \operatorname{int} L$ and set $x(p) := x_0 - p$ for $p \in \mathbb{R}^n$. By Proposition 2.3 there is a strongly transversal pair (x, y) of lipschitz vector fields on M. Furthermore, Proposition 2.9 ensures the existence of $\delta_0 \in (0,1)$ and a map $\Psi : M \times (-\delta_0, \delta_0)^2 \to U_{\delta_0}(M)$ which is bi–lipschitz and satisfies (2.18) and (2.19), where δ is replaced by δ_0 now.

We define a map $\Gamma : \partial K \times [0, \infty) \to \mathbb{R}^n \setminus \operatorname{int} K$ by

$$\Gamma(q, r) \;:=\; q + \lambda(q, r)(q - x_0)\,,$$

where $\lambda(q, r) \geq 0$ is specified by the requirement that $q + \lambda(q, r)(q - x_0) \in \partial K_r$. Thus $\Gamma|\partial K \times [0, 1]$ defines a bi–lipschitz correspondence. In particular,

$$\Gamma_r \;:=\; \Gamma(\,\cdot\,, r) \;:\; \partial K \;\longrightarrow\; \partial K_r$$

is a bi–lipschitz map, for each $r \geq 0$. Hence, for any $r \geq 0$, we see that H_0 and $\Gamma_r(H_0) \subset \partial K_r$ are homeomorphic via a bi–lipschitz map. It remains to prove that $\Gamma_r(H_0)$ and $H_r = \partial K_r \cap L_r$ can be transformed into each other by a bi–lipschitz homeomorphism, for some $r > 0$.

Let B_p, for $p \in M$, be defined as in Proposition 2.9. For $p \in M$ and $q \in B_p \cap \partial K$ we observe that $q - x_0 = (q - p) - x(p) \in \operatorname{lin}\{x(p), y(p)\}$, hence

(2.20) $$\Gamma_r(q) \in E_p\,.$$

By the lipschitz property of Γ, there is a positive constant $c > 0$ such that

$$|\Gamma_r(p) - p| \;\leq\; c\,r\,,$$

for all $p \in \partial K$ and $r \in [0, 1]$.

We fix $0 < r < \delta_0$ sufficiently small such that for $\varrho_1 = \delta_0/4$ and for all $0 \leq t \leq 1$ the following conditions are satisfied:

(i) if $p \in H_0$, then $d_{\partial L}(\Gamma_{tr}(p)) < \varrho_1$;

(ii) if $p \in \partial K \cap [d_{\partial L} \geq 0]$, then $d_{\partial L}(\Gamma_{tr}(p)) > -\varrho_1/4$;

(iii) if $p \in \partial K \cap [d_{\partial L} < -2\varrho_1]$, then $d_{\partial L}(\Gamma_{tr}(p)) < -\varrho_1$;

(iv) if $p \in \partial K \cap [d_{\partial L} \geq -2\varrho_1]$, then $d_{\partial L}(\Gamma_{tr}(p)) > -4\varrho_1$.

From (i) we infer

$$\Gamma_r(H_0) \subseteq [d_{\partial L} < \varrho_1]\,,$$

and (ii) implies

$$\partial K_r \cap [d_{\partial L} \leq -\varrho_1/4] \subseteq \Gamma_r(H_0)\,. \tag{2.21}$$

Subsequently, we construct a bi–lipschitz homeomorphism

$$\Xi \;:\; \Gamma_r(H_0) \;\longrightarrow\; \partial K_r \cap L_r \;=\; H_r\,.$$

From (2.21) we deduce that

$$\partial K_r \cap [d_{\partial L} \leq -\varrho_1/4] \;=\; \Gamma_r(H_0) \cap [d_{\partial L} \leq -\varrho_1/4]\,.$$

Set

$$\Gamma^1 \;:=\; \Gamma_r(H_0) \cap [d_{\partial L} \leq -3\varrho_1/4] \quad \text{and} \quad \Gamma^2 \;:=\; \Gamma_r(H_0) \cap [d_{\partial L} \geq -\varrho_1]\,.$$

Then we define

$$\Xi|\Gamma^1 \;:=\; \mathrm{id}_{\Gamma^1}\,.$$

It remains to define Ξ on Γ^2 in a consistent way. As a first step towards such a definition, we deduce a description of Γ^2 in terms of Ψ. To this end, we assert that

$$\Gamma^2 \;=\; \{\Gamma_r(\Psi(p,0,\varrho)) \mid p \in M,\;\varrho \in [-2\varrho_1, 0]\} \cap [d_{\partial L} \geq -\varrho_1]\,. \tag{2.22}$$

In fact, since $\Psi(p,0,\varrho) \in H_0$, for any $p \in M$ and $\varrho \in [-2\varrho_1, 0]$, the set on the right–hand side of (2.22) is contained in Γ^2. Conversely, let $y \in \Gamma^2$. Hence $d_{\partial L}(y) \geq -\varrho_1$ and $y = \Gamma_r(x)$ for some $x \in H_0$. By (iii) we obtain $d_{\partial L}(x) \geq -2\varrho_1$, and thus $x \in \partial K \cap [-2\varrho_1 \leq d_{\partial L} \leq 0] \subseteq U_{\delta_0}(M)$. From this we conclude that $x = \Psi(p,0,\varrho)$ for some $p \in M$ and $\varrho \in [-2\varrho_1, 0]$, which proves (2.22).

Next we consider $\gamma(t) := \Gamma_{tr}(\Psi(p,0,\varrho))$, $t \in [0,1]$, for fixed $p \in M$ and $\varrho \in [-2\varrho_1, 0]$. Then $\gamma(0) = \Psi(p,0,\varrho) \in B_p \cap \partial K$ and γ is continuous. In addition, since $\gamma(t) \in \partial K_{tr}$, $d_{\partial L}(\gamma(t)) < \varrho_1$ and $d_{\partial L}(\gamma(t)) > -4\varrho_1$ (compare (i) and (iv)), we infer that $\gamma(t) \in U_{\delta_0}(M)$. From (2.20), we see that $\gamma(t) \in E_p$. Lemma 2.11 now implies that $\Gamma_r(\Psi(p,0,\varrho)) \in B_p$, for all $p \in M$ and $\varrho \in [-2\varrho_1, 0]$.

We define the lipschitz map $\zeta : M \times [-2\varrho_1, 0] \to (-\delta_0, \delta_0)$ by putting

$$\zeta(p,\varrho) \;:=\; d_{\partial L}(\gamma(1)) \;=\; d_{\partial L}(\Gamma_r(\Psi(p,0,\varrho)))\,.$$

We see that $\zeta(p,\varrho) \in (-4\varrho_1, \varrho_1)$, since $-4\varrho_1 < d_{\partial L}(\gamma(1)) < \varrho_1$, and

$$\Gamma_r(\Psi(p,0,\varrho)) \;=\; \Psi(p,r,\zeta(p,\varrho))\,.$$

Let $p \in M$ be fixed for the moment. Since $\Gamma_r(\Psi(p,0,\cdot\,))$ is an injective map on $[-2\varrho_1, 0]$, the map $\zeta(p,\cdot\,)$ must be strictly monotone. By (iii) and (ii) we obtain

$$\zeta(p,-2\varrho_1) \;=\; d_{\partial L}(\Gamma_r(\Psi(p,0,-2\varrho_1))) \;\le\; -\varrho_1$$

and

$$\zeta(p,0) \;=\; d_{\partial L}(\Gamma_r(\Psi(p,0,0))) \;>\; -\frac{\varrho_1}{4}\,.$$

Hence $\zeta(p,\cdot\,)$ is increasing and therefore

$$\zeta(p,[-2\varrho_1,0]) \;=\; [\zeta(p,-2\varrho_1),\zeta(p,0)]\,.$$

Defining the lipschitz map $h : M \to [-\varrho_1/4, \varrho_1]$ by $h(p) := \zeta(p,0)$, we finally obtain from (2.22) that

$$\Gamma^2 \;=\; \{\Psi(p,r,\varrho) \mid p \in M,\ \varrho \in [-\varrho_1, h(p)]\}\,. \tag{2.23}$$

Next we define a bi–lipschitz map

$$\alpha(p,\cdot\,) \;:\; [-\varrho_1, h(p)] \;\longrightarrow\; [-\varrho_1, r]$$

which satisfies

$$\alpha(p,\cdot\,)|[-\varrho_1,-\varrho_1/2] \;=\; \mathrm{id}_{[-\varrho_1,-\varrho_1/2]}$$

and

$$\alpha(p,t) \;:=\; \frac{\varrho_1+2r}{\varrho_1+2h(p)}\,t + \frac{\varrho_1(r-h(p))}{\varrho_1+2h(p)}\,, \qquad t \in [-\varrho_1/2, h(p)]\,.$$

Thus, in particular

$$\alpha(p,-\varrho_1/2) \;=\; -\varrho_1/2 \quad \text{and} \quad \alpha(p,h(p)) \;=\; r\,.$$

Furthermore, we set $G(p,r,t) := (p,r,\alpha(p,t))$ if $p \in M$ and $t \in [-\varrho_1, h(p)]$.

Finally, we define

$$\Xi|\Gamma^2 \;:=\; \Psi \circ G \circ \Psi^{-1}|\Gamma^2\,.$$

By definition, $\Psi \circ G \circ \Psi^{-1}$ is the identity map on $\Gamma_r(H_0) \cap [-\varrho_1 \le d_{\partial L} \le -\varrho_1/2]$, and thus Ξ is well–defined. It is also easy to see that Ξ is injective and bi–lipschitz. Clearly,

$$\mathrm{im}\big(\Xi|\Gamma^1\big) \subseteq H_r \quad \text{and} \quad \mathrm{im}\big(\Xi|\Gamma^2\big) \;=\; \{\Psi(p,r,\varrho) \mid p \in M, \varrho \in [-\varrho_1, r]\} \subseteq H_r\,.$$

On the other hand,

$$\mathrm{im}\big(\Xi|\Gamma^1\big) \supseteq \partial K_r \cap [d_{\partial L} \le -3\varrho_1/4]$$

and for any $y \in \partial K_r \cap [-\varrho_1 \le d_{\partial L} \le r] \subseteq U_{\delta_0}(M)$ there is some $p \in M$ and some $\varrho \in [-\varrho_1, r]$ such that $y = \Psi(p,r,\varrho)$; thus

$$\mathrm{im}\big(\Xi|\Gamma^2\big) \supseteq \partial K_r \cap [-\varrho_1 \le d_{\partial L} \le r]\,.$$

This shows that Ξ maps onto H_r. □

2.4. Some consequences

The proofs of Propositions 2.2 and 2.3 in [10] show that $S_r = \partial K_r \cap \partial L_r$ and $H_r = \partial K_r \cap L_r$ are even manifolds of class C^1 if $r > 0$. Therefore it is natural to ask whether S_r and S_s can be transformed into each other by a diffeomorphism of class C^1 if $r, s > 0$, and a similar question arises for the intersection of boundary and body. The proof given in [10] for the existence of homeomorphisms between any two such sets actually yields that the map constructed there is a bi–lipschitz homeomorphism, but only for $r, s > 0$ in contrast to the present more general results which include the essential case $r = 0$. However, it does not seem to follow from the approach in [10] that there exist smooth transformations. The existence of such smooth transformations is now implied by Remarks 2.4 and 2.10 and by the proof of Theorems 1.1 and 1.2. Here we have to observe that the definition of the map $\alpha(p, \cdot)$, in the proof of Theorem 1.2, can certainly be modified to yield a smooth function.

Corollary 2.12. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets with common interior points which intersect almost transversally. Then S_r and S_s can be transformed into each other by a diffeomorphism of class C^1 if $r, s > 0$.*

Corollary 2.13. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets with common interior points which intersect almost transversally. Then H_r and H_s can be transformed into each other by a diffeomorphism of class C^1 if $r, s > 0$.*

The next two consequences should be seen in analogy to Corollary 3.10 in [13].

Theorem 2.14. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets, and assume that $\delta := \min\{|x - y| : x \in K, y \in L\} > 0$. Then $\partial K_r \cap \partial L_r$ is*
(a) *$\emptyset$ if $r < \delta/2$;*
(b) *bi–lipschitz homeomorphic to the k–dimensional Euclidean unit ball B^k, for some $k \in \{0, \ldots, n-1\}$, if $r = \delta/2$;*
(c) *diffeomorphic (of class C^1) to the $(n-2)$–dimensional unit sphere S^{n-2} if $r > \delta/2$.*

Theorem 2.15. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact convex sets, and assume that $\delta := \min\{|x - y| : x \in K, y \in L\} > 0$. Then $\partial K_r \cap L_r$ is*
(a) *$\emptyset$ if $r < \delta/2$;*
(b) *bi–lipschitz homeomorphic to the k–dimensional Euclidean unit ball B^k, for some $k \in \{0, \ldots, n-1\}$, if $r = \delta/2$;*
(c) *diffeomorphic (of class C^1) to the Euclidean unit ball B^{n-1} if $r > \delta/2$.*

Proof of Theorems 2.14 and 2.15. Let $r_0 := \delta/2$. Since K_{r_0} and L_{r_0} can be separated by a hyperplane, we get $\partial K_{r_0} \cap \partial L_{r_0} = \partial K_{r_0} \cap L_{r_0} = K_{r_0} \cap L_{r_0}$, which implies (a) and (b) of Theorems 2.14 and 2.15.

To prove statement (c) of Theorems 2.14 and 2.15, we first establish the assertion in the case where $r \in (r_0, 2r_0)$ by a direct argument. The general case then follows by an application of the preceding Corollaries 2.12 and 2.13, since K_r and L_r intersect almost transversally for all $r > r_0$. In fact, condition (2.3) is clearly satisfied. Assume that $x \in \partial K_r \cap \partial L_r$ and $\nu(K_r, x) = \nu(L_r, x)$ are common exterior unit normal vectors of K_r, L_r at x. But this implies that $r^{-1}(x - p(K, x)) = r^{-1}(x - p(L, x))$, and hence $p(K, x) = p(L, x) \in K \cap L$, where $p(K, x)$, $p(L, x)$ are the metric projections of x onto K, L, respectively. This is a contradiction to the assumption, and therefore condition (2.2) is also satisfied.

Subsequently, we assume that $r \in (r_0, 2r_0)$. Let us denote by $D(K_r)$, $D(L_r)$ the orthogonal projections of K_r, L_r onto the common support plane H of K_{r_0} and L_{r_0}. We can assume that $H = \{x \in \mathbb{R}^n \mid \langle x, e_n \rangle = 0\}$, $K \subset H^+$ and $L \subset H^-$, where $H^+ := \{x \in \mathbb{R}^n \mid \langle x, e_n \rangle \geq 0\}$ and $H^- := \{x \in \mathbb{R}^n \mid \langle x, e_n \rangle \leq 0\}$. Further, we define functions

$$f(y, r) := \min\{t \in \mathbb{R} \mid y + te_n \in K_r\}, \quad y \in D(K_r),$$

and

$$g(y, r) := \max\{t \in \mathbb{R} \mid y + te_n \in L_r\}, \quad y \in D(L_r),$$

which parametrise the “lower” and the “upper” part of ∂K_r and ∂L_r, respectively.

Now let $x \in \partial K_r \cap \partial L_r$. Let $\nu(K_r, x)$ denote the uniquely determined exterior unit normal vector of K_r at x. Then $\langle \nu(K_r, x), e_n \rangle < 0$. To see this, assume that $\langle p(K_{r_0}, x) - x, e_n \rangle \leq 0$. Since $\langle x, e_n \rangle \leq r - r_0$ (because $x \in L_r$), we obtain

$$\begin{aligned} \langle p(K_{r_0}, x) - r_0 \nu(K_r, x) - r_0 e_n, e_n \rangle &\leq \langle x, e_n \rangle - r_0 - r_0 \langle \nu(K_r, x), e_n \rangle \\ &\leq r - 2r_0 \\ &< 0 \end{aligned}$$

which contradicts

$$p(K_{r_0}, x) - r_0 \nu(K_r, x) - r_0 e_n \in B^n(p(K_{r_0}, x) - r_0 \nu(K_r, x), r_0) \subseteq K_{r_0} \subseteq H^+ .$$

Similarly, we get $\langle \nu(L_r, x), e_n \rangle > 0$. Hence, if $x = (y, z) \in \partial K_r \cap \partial L_r$ with $y \in H$ and $z \in \mathbb{R}$, then $y \in \operatorname{int} D(K_r) \cap \operatorname{int} D(L_r) = \operatorname{int}(D(K_r) \cap D(L_r))$ and $x = (y, f(y, r)) = (y, g(y, r))$. Therefore, we obtain

$$\partial K_r \cap \partial L_r = \{(y, f(y, r)) \mid y \in D(K_r) \cap D(L_r),\ f(y, r) - g(y, r) = 0\}. \tag{2.24}$$

Let $y \in \partial D(K_r) \cap D(L_r)$. Setting $x := (y, f(y, r))$, we get $\langle \nu(K_r, x), e_n \rangle = 0$ (since K_r is smooth) and hence

$$\begin{aligned} f(y, r) &= \langle x, e_n \rangle \\ &= \langle x - (r - r_0)\nu(K_r, x), e_n \rangle \\ &= \langle p(K_{r_0}, x), e_n \rangle \\ &= \langle p(K_{r_0}, x) - r_0 \nu(K_r, x) - r_0 e_n, e_n \rangle + \langle r_0 e_n, e_n \rangle \\ &\geq r_0 . \end{aligned}$$

This implies that $g(y, r) \leq r - r_0 < r_0 \leq f(y, r)$. The same estimate follows if $y \in D(K_r) \cap \partial D(L_r)$. Thus we have shown that $(f - g)(y) > 0$ for all $y \in \partial(D(K_r) \cap D(L_r))$. The convex function $f - g$ is lower semicontinuous on the convex set $D(K_r) \cap D(L_r)$, and therefore $f-g \geq \epsilon > 0$ in a neighbourhood of $\partial(D(K_r)\cap D(L_r))$ in $D(K_r)\cap D(L_r)$. This shows that

$$\begin{aligned} &\{y \in \operatorname{int}(D(K_r) \cap D(L_r)) \mid f(y,r) - g(y,r) = 0\} \\ = \ &\partial\{y \in D(K_r) \cap D(L_r) \mid f(y,r) - g(y,r) \leq 0\} \end{aligned}$$

is the boundary of an $(n-1)$–dimensional compact convex subset of H. Since this boundary is a submanifold of class C^1 (recall that $f(\cdot\,,r)$ and $g(\cdot\,,r)$ are of class C^1 and $\nabla f(\cdot\,,r) \neq \nabla g(\cdot\,,r)$), the assertion (c) of Theorem 2.14 follows from (2.24).

For the proof of statement (c) of Theorem 2.15, we proceed similarly. Let $x = (y,z) \in \partial K_r \cap L_r$. As before it follows that $\langle \nu(K_r,x), e_n\rangle < 0$; hence $y \in \operatorname{int} D(K_r)$ and $x = (y, f(y,r))$. If $x \in \partial L_r$, then $y \in \operatorname{int} D(L_r)$ and $x = (y, g(y,r)) = (y, f(y,r))$; if $x \in \operatorname{int} L_r$, then $y \in \operatorname{int} D(L_r)$ and $g(y,r) > f(y,r)$. Conversely, let $y \in \operatorname{int}(D(K_r)\cap D(L_r))$ and $f(y,r) \leq g(y,r)$. Then $(y, f(y,r)) \in \partial K_r$ and we have to show that $(y, f(y,r)) \in L_r$. Assume that this is not the case. Then there is some $z \in \mathbb{R}$ such that $f(y,r) < z < g(y,r)$ and $\bar{x} := (y,z) \in \partial L_r$. Consequently, $\langle \nu(L_r,\bar{x}), e_n\rangle \leq 0$, and $B^n(\bar{x} - 2(r-r_0)\nu(L_r,\bar{x}), r-r_0) \subset L_{r_0}$. This yields $\bar{x} - 2(r-r_0)\nu(L_r,\bar{x}) + (r-r_0)e_n \in L_{r_0}$, and hence $\langle \bar{x}, e_n\rangle \leq r_0 - r + 2(r-r_0)\langle \nu(L_r,\bar{x}), e_n\rangle \leq r_0 - r$. But then $B^n((y, f(y,r)), r-r_0) \subset \operatorname{int} H^-$, which contradicts $K_{r_0} \subset H^+$. Thus we have shown that

$$\partial K_r \cap L_r \ = \ \{(y, f(y,r)) \mid y \in \operatorname{int}(D(K_r)\cap D(L_r)),\ f(y,r) - g(y,r) \leq 0\}\,.$$

The assertion now follows from arguments involved in the first part of the proof. □

3. Translative integral formulas

This section is mainly devoted to the proof of Theorem 1.4 stated in the introduction. In addition, we present generalizations of this theorem concerning iterated translative integrals and extensions to the convex ring. In the course of the proofs we shall have to deal with questions of measurability and integrability. Such items will also become relevant in the last section.

Proof of Theorem 1.4. If $\dim K = 0$, then $\partial K = \emptyset$; thus both sides of the asserted equation vanish. Henceforth, we can assume that $\dim K \geq 1$. The proof is now accomplished in two steps.

Step I. We assume that $\dim K = n \geq 1$.

First, we observe that (1.4) is true if $n = 1$ and $\dim K = \dim L = 1$, since then both sides of the asserted equation are equal to $2V(L)$.

If $\dim L = 0$, then both sides of (1.4) vanish.

If $n \geq 2$ and $\dim L = 1$, then both sides of (1.4) are equal to

$$2\mathcal{H}^1(L)\mathcal{H}^{n-1}\big(\Pi_{L^\perp} K\big)\,,$$

where $\Pi_{L^\perp} K$ denotes the orthogonal projection of K onto the orthogonal complement $L^\perp$ of the linear subspace which is parallel to the affine hull of L and where $\mathcal{H}^j$ denotes the j–dimensional Hausdorff measure in $\mathbb{R}^n$.

Therefore it remains to check the cases $l := \dim L \in \{2, \ldots, n-1\}$. We can assume that $L \subset U \in \mathbf{G}(n,l)$. Further, we define

$$U(K)^{\perp} \;:=\; \left\{y \in U^{\perp} : (y+U) \cap \operatorname{int} K \neq \emptyset\right\}.$$

Subsequently, for $y \in U(K)^{\perp}$ we apply Theorem 1.2 of [10] to $K \cap (y+U)$ and to $L+y$ in the affine subspace $y+U$. To this end, we have to use Fubini's theorem. The required measurability property can be seen from the following observations. First, the measurability of the map $t \mapsto \chi(\partial K \cap (L+t))$ is easy to check in the subcases considered so far. We define A_1 as the set of all $(x,y) \in U \times U^{\perp}$ for which $(U+y) \cap K \neq \emptyset$ and $(U+y) \cap \operatorname{int} K = \emptyset$. We let A_2 be the set of all $(x,y) \in U \times U(K)^{\perp}$ for which $(K-y) \cap U$ and $L+x$ do not intersect almost transversally with respect to U, and we set $A := \left(U \times U^{\perp}\right) \setminus (A_1 \cup A_2)$. Then A_1 and A_2 are Borel sets of $\mathcal{H}^n$ measure zero (compare Lemma 3.1 in [10]). Therefore it suffices to prove the measurability of the restriction of the map $(x,y) \mapsto \chi(\partial K \cap (L+x+y))$ to A. Setting $B^U := B^n \cap U$, we get, for $(x,y) \in A$,

$$\begin{aligned}
\chi(\partial K \cap (L+x+y)) &= \chi(\partial((K-y) \cap U) \cap (L+x)) \\
&= \chi\big(\partial\big([(K-y) \cap U] + B^U\big) \cap \big(L+x+B^U\big)\big) \\
&= C_0\big(\partial\big([K-y) \cap U] + B^U\big) \cap \big(L+x+B^U\big), \mathbb{R}^n \times \mathbb{R}^n\big),
\end{aligned}$$

where $C_0(R, \cdot\,)$ denotes the Gauss curvature measure of a set $R \subset \mathbb{R}^n$ of positive reach; see [8], [10]. The second equality above is implied by Theorem 1.2 (alternatively, by Theorem 1.4 in [10]), applied in U. The third equality follows from the fact that a set $R \subset U$ with positive reach in U has positive reach in $\mathbb{R}^n$ as well and from the known relation $C_0(R, \mathbb{R}^n \times \mathbb{R}^n) = \chi(R)$, which holds for any compact set $R \subset \mathbb{R}^n$ with positive reach (cf. the discussion in [10], Section 3). Hence the measurability assertions are implied by Proposition 1.1.1 and Theorem 2.1.2 (1) in [27] and by Satz 1.1.6 in [18] (compare also [11]). The required integrability of the Euler characteristic can be deduced from the following equations by a repetition of arguments which were involved in the proofs of Lemma 3.2 and Theorem 1.2 in [10]. In fact, it follows from these results that

$$\begin{aligned}
&\int_U |\, \chi(\partial K \cap (L+y+x)) \,|\; \mathcal{H}^l(dx) \\
\leq\; &\sum_{j=1}^{l} \binom{l}{j} \Big\{ v^{(l)}(K \cap (U+y)[l-j], -(L+y)[j]) \\
&\qquad\qquad + v^{(l)}(K \cap (U+y)[l-j], (L+y)[j]) \Big\}
\end{aligned}$$

if $y \in U(K)^{\perp}$, where we write $v^{(l)}$ for the mixed volume relative to affine subspaces which are parallel to the linear subspace U (see [15] for notation and results concerning mixed volumes). Thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \chi(\partial K \cap (L+t))\, \mathcal{H}^n(dt) \\
=\; &\int_{U^{\perp}} \int_U \chi(\partial K \cap (L+y+x))\, \mathcal{H}^l(dx)\, \mathcal{H}^{n-l}(dy) \;=
\end{aligned}$$

$$= \int_{U(K)^\perp}\int_U \chi(\partial(K\cap(U+y))\cap(L+y+x))\,\mathcal{H}^l(dx)\,\mathcal{H}^{n-l}(dy)$$

$$= \int_{U(K)^\perp}\sum_{j=1}^{l}\binom{l}{j}\Big\{v^{(l)}([K\cap(U+y)][l-j],-(L+y)[j]) + (-1)^{j-1}v^{(l)}([K\cap(U+y)][l-j],(L+y)[j])\Big\}\,\mathcal{H}^{n-l}(dy)\,.$$

Using a formula from p. 293 in [15], we can infer that

$$\int_{\mathbb{R}^n}\chi(\partial K\cap(L+t))\,\mathcal{H}^n(dt)$$

$$= \sum_{j=1}^{l}\binom{n}{j}\{V(K[n-j],-L[j])+(-1)^{j-1}V(K[n-j],L[j])\}$$

$$= \sum_{j=1}^{n}\binom{n}{j}\{V(K[n-j],-L[j])+(-1)^{j-1}V(K[n-j],L[j])\}\,;$$

moreover,

$$\int_{\mathbb{R}^n}|\chi(\partial K\cap(L+t))|\,\mathcal{H}^n(dt)$$

$$\le \sum_{j=1}^{n}\binom{n}{j}\{V(K[n-j],-L[j])+V(K[n-j],L[j])\}\,.$$

This completes the first step of the proof.

Step II. Let $\dim K = k \ge 1$ and $K\subset V\in\mathbf{G}(n,k)$. We write $v^{(k)}$ for the mixed volume relative to affine subspaces which are parallel to the linear subspace V. Applying the result of Step I in the linear subspace V, and slightly extending the previous considerations concerning questions of measurability, we thus obtain

$$\int_{\mathbb{R}^n}\chi(\partial K\cap(L+t))\,\mathcal{H}^n(dt)$$

$$= \int_{V^\perp}\int_V \chi(\partial K\cap(L+y+x))\,\mathcal{H}^k(dx)\,\mathcal{H}^{n-k}(dy)$$

$$= \int_{V^\perp}\int_V \chi(\partial K\cap([(L+y)\cap V]+x))\;\mathcal{H}^k(dx)\,\mathcal{H}^{n-k}(dy)$$

$$= \int_{V^\perp}\sum_{j=1}^{k}\binom{k}{j}\Big\{v^{(k)}(K[k-j],-[(L+y)\cap V][j]) + (-1)^{j-1}v^{(k)}(K[k-j],[(L+y)\cap V][j])\Big\}\,\mathcal{H}^{n-k}(dy)$$

$$= \sum_{j=1}^{k}\binom{n}{k-j}\{V(-K[k-j],L[n-k+j])+(-1)^{j-1}V(K[k-j],L[n-k+j])\},$$

where again we have used [15, p. 293]. In addition, we deduce

$$\begin{aligned} &\int_{\mathbb{R}^n} |\chi(\partial K \cap (L+t))| \, \mathcal{H}^n(dt) \\ &\leq \sum_{j=1}^{k} \binom{n}{k-j} \{V(-K[k-j], L[n-k+j]) + V(K[k-j], L[n-k+j])\} . \end{aligned} \tag{3.1}$$

This proves the assertions of the theorem. □

Corollary 3.1. *The assertions of Theorem* 1.4 *remain true if* $L \subset \mathbb{R}^n$ *is an element of the convex ring.*

For the proof of Corollary 3.1 we need two lemmas.

Lemma 3.2. *Let* $K, L \subset \mathbb{R}^n$ *be compact convex sets. Then, for* $\mathcal{H}^n$ *almost all* $t \in \mathbb{R}^n$*,* $\partial K \cap (L+t)$ *is a lipschitz submanifold or the empty set.*

Proof. We proceed as in the proof of Theorem 1.4 and adopt the same notation. First, observe that if $\dim K = 0$, then there is nothing to prove.

Step 1. If $\dim K = \dim L = n = 1$, then $\partial K \cap (L+t)$ is empty, a one–pointed set or a two–pointed set.

If $\dim K = n \geq 1$ and $\dim L = 0$, then $\partial K \cap (L+t)$ is empty or a one–pointed set.

If $\dim K = n \geq 2$ and $\dim L = 1$, then $\partial K \cap (L+t)$ is empty, a one–pointed set, a two–pointed set or a segment.

Step 2. Let $\dim K = n$ and $\dim L \geq 2$. For $\mathcal{H}^{n-l}$ almost all $y \in U^\perp$ with $(U+y) \cap K \neq \emptyset$ we have $(U+y) \cap \operatorname{int} K \neq \emptyset$. Let y be chosen such that the latter condition is fulfilled. Then, for $\mathcal{H}^l$ almost all $x \in U$, we infer from Proposition 2.3 and Lemma 3.1 in [10] that

$$\partial(K \cap (U+y)) \cap (L+y+x) \; = \; \partial K \cap (L+y+x)$$

is a lipschitz submanifold of $U+y$ or the empty set. Thus we have confirmed the assertion of the lemma in the case where $\dim K = n \geq 1$ if we recall the arguments concerning measurability from the proof of Theorem 1.4.

Step 3. Let K be arbitrarily chosen. For $x \in V$ and $y \in V^\perp$ we have

$$\partial K \cap (L+y+x) \; = \; \partial K \cap ([(L+y) \cap V] + x) .$$

Hence, by the result of Step 2, we obtain for any $y \in V^\perp$ and for $\mathcal{H}^k$ almost all $x \in V$ that $\partial K \cap ([(L+y) \cap V] + x)$ is a lipschitz submanifold or the empty set. This finally yields the desired conclusion. □

Remark 3.3. Let $K, L \subset \mathbb{R}^n$, $n \geq 1$, be compact convex sets. Let aff(Z) denote the affine hull of a nonempty set $Z \subset \mathbb{R}^n$. Then we say that K and L intersect almost transversally if and only if $K \cap L = \emptyset$ or $\dim K = 0$ or one of the following conditions is satisfied:

(a) $\dim K = \dim L = n = 1$ or ($\dim K = n \geq 1$ and $\dim L = 0$) or ($\dim K = n \geq 2$ and $\dim L = 1$);

(b) $\dim K = n$, $\dim L \geq 2$, $\operatorname{aff}(L) \cap \operatorname{int} K \neq \emptyset$ and the sets $K \cap \operatorname{aff}(L)$ and L intersect almost transversally with respect to $\operatorname{aff}(L)$.

(c) $1 \leq \dim K \leq n-1$ and the sets K and $L \cap \operatorname{aff}(K)$ satisfy (a) or (b) with respect to $\operatorname{aff}(K)$ as the surrounding space.

In this terminology, the proof of Lemma 3.2 shows that for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$ the sets K and $L + t$ intersect almost transversally.

The second lemma is of a purely topological nature. First, we recall two definitions adjusted to our purposes.

A closed set $X \subset \mathbb{R}^n$ is called a *neighbourhood retract* if there is a neighbourhood U of X in $\mathbb{R}^n$ and a continuous map $r : U \to X$ such that $r|X = \mathrm{id}_X$.

A topological space X is called an *absolute neighbourhood retract* if for every metric space Y, every closed subset $A \subset Y$, and every continous map $f : A \to X$ there exists a continuous extension of f to a neighbourhood of A in Y.

Clearly, a closed set $X \subset \mathbb{R}^n$ which is an absolute neighbourhood retract is also a neighbourhood retract.

Lemma 3.4. *Let $X_1, \ldots, X_r \subset \mathbb{R}^n$ be closed sets such that all nonempty intersections of these sets are neighbourhood retracts in $\mathbb{R}^n$. Then $X := X_1 \cup \ldots \cup X_r$ is a neighbourhood retract in $\mathbb{R}^n$.*

Proof. The lemma is proved by induction over r. Using Tietze's extension theorem it is easy to check that if a closed set $X \subset \mathbb{R}^n$ is a neighbourhood retract, then it is also an absolute neighbourhood retract. Hence, by Theorem III.14.7 in [5] we can conclude that $X_1 \cup X_2$ is an absolute neighbourhood retract, and hence also a neighbourhood retract. This confirms the assertion of the lemma for $r = 2$.

Now assume that the assertion has been proved for r sets, and let $X_1, \ldots, X_{r+1} \subset \mathbb{R}^n$ satisfy the assumptions of the lemma. Then $X_1 \cup \ldots \cup X_r$ is a neighbourhood retract by the inductive assumption. Moreover, by the same reason

$$(X_1 \cup \ldots \cup X_r) \cap X_{r+1} \;=\; (X_1 \cap X_{r+1}) \cup \ldots \cup (X_r \cap X_{r+1})$$

is a neighbourhood retract, since any nonempty intersection of the r sets $X_1 \cap X_{r+1}, \ldots, X_r \cap X_{r+1}$ is a neighbourhood retract. By the case $r = 2$ of the lemma, which has already been established, $X_1 \cup \ldots \cup X_r \cup X_{r+1}$ is a neighbourhood retract, which completes the induction. □

The following proof is inspired by work of M. Zähle [28] and J. Rataj & M. Zähle [14], in the context of sets of positive reach.

Proof of Corollary 3.1. It is easy to check that the corollary is true if $n = 1$. Henceforth we assume that $n \geq 2$.

It is well–known that the right–hand side of the equation in Theorem 1.4 is additive with respect to L; see, e. g., [15, p. 280]. Therefore it remains to show that the same assertion is true for the left–hand side.

Let $L_1, \ldots, L_r \subset \mathbb{R}^n$, $r \in \mathbb{N}$, be compact convex sets. We define

$$M(t) \;:=\; \partial K \cap ((L_1 \cup \ldots \cup L_r) + t) \;=\; \bigcup_{i=1}^{r} (\partial K \cap (L_i + t))\,.$$

Let $I \subseteq \{1, \ldots, r\}$ be nonempty. Then by Lemma 3.2, for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$, the intersection

$$\partial K \cap \bigcap_{i \in I} (L_i + t) \;=\; \partial K \cap \left[\left(\bigcap_{i \in I} L_i \right) + t \right]$$

is a compact lipschitz submanifold of $\mathbb{R}^n$ or the empty set. Hence, by Proposition IV.8.12 in [6], all these intersections are neighbourhood retracts. Now Lemma 3.4 yields that $M(t)$ is a neighbourhood retract, for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$.

Consequently, if $L, L' \subset \mathbb{R}^n$ are in the convex ring, then by Exercise VIII.6.28.4 in [6] it follows that $(\mathbb{R}^n, \partial K \cap (L+t), \partial K \cap (L'+t))$ is an excisive triad, for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$ (compare also [7], Theorem 11.4 in Chapter VII.11). The remaining assertion thus follows from [6], Propositions V.5.8 and V.4.11. □

In view of applications in stochastic geometry it is of interest to have an extension of Theorem 1.4 to an iterated translative integral formula. In order to describe such an extension we need certain mixed functionals

$$V^{(0)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k)\,, \tag{3.2}$$

where $k \geq 1$, $m_1, \ldots, m_k \in \{0, \ldots, n\}$ with $m_1 + \ldots + m_k = (k-1)n$, and $K_1, \ldots, K_k \subset \mathbb{R}^n$ are nonempty compact convex sets. These functionals and their measure theoretic generalizations are introduced in [20], as important quantities and tools in translative integral geometry.

Theorem 3.5. *Let $K \subset \mathbb{R}^n$ be compact and convex, let $L_1, \ldots, L_r \subset \mathbb{R}^n$ be in the convex ring, and set $k := \dim K$. Then*

$$\begin{aligned} &\int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \chi(\partial K \cap (L_1 + t_1) \cap \ldots \cap (L_r + t_r))\, \mathcal{H}^n(dt_1) \ldots \mathcal{H}^n(dt_r) \\ &= \sum_{i=1}^{k} \sum_{\substack{m_1,\ldots,m_r = n-k+i \\ m_1 + \ldots + m_r = rn-k+i}}^{n} \Big\{ V^{(0)}_{k-i,m_1,\ldots,m_r}(-K, L_1, \ldots, L_r) \\ &\qquad + (-1)^{i-1} V^{(0)}_{k-i,m_1,\ldots,m_r}(K, L_1, \ldots, L_r) \Big\}. \end{aligned}$$

Proof. The integrability of the integrand can easily be deduced from the following lines by means of the estimate (3.1); moreover, the required measurability properties can be checked as in the proof of Theorem 1.4.

Using the translation invariance of $\mathcal{H}^n$, Fubini's theorem and Corollary 3.1, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \chi(\partial K \cap (L_1 + t_1) \cap \ldots \cap (L_r + t_r))\, \mathcal{H}^n(dt_1) \ldots \mathcal{H}^n(dt_r) \\ &= \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \chi(\partial K \cap (L_1 + t_1 + t_r) \cap \ldots \cap (L_{r-1} + t_{r-1} + t_r) \cap (L_r + t_r)) \\ &\qquad \times \mathcal{H}^n(dt_1) \ldots \mathcal{H}^n(dt_r) \end{aligned}$$

$$= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \chi\,(\partial K \cap ([(L_1+t_1)\cap \dots \cap (L_{r-1}+t_{r-1})\cap L_r] + t_r)) \times \mathcal{H}^n(dt_r)\mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-1})$$

$$= \sum_{i=1}^{k} \binom{n}{k-i} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \times \big\{ V(K[k-i], -[(L_1+t_1)\cap \dots \cap (L_{r-1}+t_{r-1})\cap L_r][n-k+i]) + (-1)^{i-1} V(K[k-i], [(L_1+t_1)\cap \dots \cap (L_{r-1}+t_{r-1})\cap L_r][n-k+i]) \big\} \times \mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-1})\,.$$

In the last step we have used the comments subsequent to the statement of Theorem 1.4. Next we replace the mixed volumes by special of the mixed functionals in (3.2). In fact, it is well–known that

$$V^{(0)}_{m,n-m}(M_1, M_2) \;=\; \binom{n}{m} V(M_1[m], -M_2[n-m])$$

is satisfied for all compact convex sets $M_1, M_2 \subset \mathbb{R}^n$ and $m \in \{0,\dots,n\}$. Therefore, it is sufficient to prove that

$$\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} V^{(0)}_{k-i,n-k+i}(K, (L_1+t_1)\cap \dots \cap (L_{r-1}+t_{r-1})\cap L_r) \times \mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-1})$$
$$= \sum_{\substack{m_1,\dots,m_r=n-k+i \\ m_1+\dots+m_r=rn-k+i}}^{n} V^{(0)}_{k-i,m_1,\dots,m_r}(K, L_1, \dots, L_r)\,.$$

Using a similar argument as in the beginning of the proof, we deduce

$$\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} V^{(0)}_{k-i,n-k+i}(K, (L_1+t_1)\cap \dots \cap (L_{r-1}+t_{r-1})\cap L_r) \times \mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-1})$$
$$= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} V^{(0)}_{k-i,n-k+i}(K, [((L_1+t_1)\cap \dots \cap (L_{r-2}+t_{r-2})\cap L_{r-1}) + t_{r-1}]\cap L_r) \times \mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-1})$$
$$= \sum_{\substack{m_{r-1},m_r=n-k+i \\ m_{r-1}+m_r=2n-k+i}}^{n} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} V^{(0)}_{k-i,m_r,m_{r-1}}(K, L_r, (L_1+t_1)\cap \dots \dots \cap (L_{r-2}+t_{r-2})\cap L_{r-1})\,\mathcal{H}^n(dt_1)\dots\mathcal{H}^n(dt_{r-2})\,,$$

where we used Corollary 3.5 in [20] for the last step.

After $r-2$ repetitions of this argument, we finally arrive at the desired equation by means of the symmetry properties of the mixed functionals (see [20], Corollary 3.2).□

The formula of Theorem 3.5 simplifies considerably if we perform additional integrations over the rotation group. In other words, we shall now consider kinematic formulae where one integrates over the group $\mathbf{G}(n)$ with respect to the Haar measure μ on $\mathbf{G}(n)$. The measure μ will be normalized in such a way that

$$\mu(\{g \in \mathbf{G}(n) : g(0) \in B^n\}) \;=\; \mathcal{H}^n(B^n)\,;$$

compare [15], p. 227.

Corollary 3.6. *Let $K \subset \mathbb{R}^n$ be a compact convex set, let $L_1, \dots, L_r \subset \mathbb{R}^n$ be in the convex ring, and set $k := \dim K$. Then*

$$\int_{\mathbf{G}(n)} \dots \int_{\mathbf{G}(n)} \chi(\partial K \cap g_1 L_1 \cap \dots \cap g_r L_r)\, \mu(dg_1) \dots \mu(dg_r)$$
$$= \sum_{i=1}^{k} \sum_{\substack{m_1,\dots,m_r = n-k+i \\ m_1+\dots+m_r = rn-k+i}}^{n} c_{n[r]}^{k-i,m_1,\dots,m_r} \left(1 + (-1)^{i-1}\right) V_{k-i}(K) V_{m_1}(L_1) \dots V_{m_r}(L_r),$$

where

$$c_{n[r]}^{k-i,m_1,\dots,m_r} \;:=\; (k-i)!\,\kappa_{k-i} \prod_{l=1}^{r} \left(\frac{m_l!\,\kappa_{m_l}}{n!\,\kappa_n} \right).$$

Proof. This follows from Theorem 3.5 together with Corollary 3.6 in [20]. □

Remark 3.7. More generally, one can also derive iterated formulas where some of the integrals are extended over the motion group and some are extended over $\mathbb{R}^n$. Such formulas can be deduced from Theorem 3.5 together with Theorem 7 in [22].

4. Some applications to stochastic geometry

In this section, we want to demonstrate how the results of the previous section can be applied to stochastic geometry. We refer to the recent book [18] for notation and terminology not defined here.

We first consider a *stationary point process* X on the convex ring $\mathcal{R}^n$ of $\mathbb{R}^n$, defined over an abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For simplicity we assume that the point processes, which we study, are *simple*. Therefore we can identify the random counting measure X with a locally finite random set. In particular, we can write $C \in X$ instead of $X(\{C\}) = 1$ for a closed set $C \subset \mathbb{R}^n$. The intensity measure $\Theta(\cdot) := \mathbb{E}[X(\cdot)]$ of X is assumed to satisfy $\Theta \not\equiv 0$ and to be locally finite, that is

$$\Theta(\mathcal{F}_C) \;<\; \infty \tag{4.1}$$

for all compact sets $C \subset \mathbb{R}^n$, where $\mathcal{F}_C$ denotes the system of all closed subsets of $\mathbb{R}^n$ which have nonempty intersection with C. Let $\mathcal{R}^n_o$ be the set of all members of the convex ring which have the centre of their circumsphere at the origin. By stationarity we then obtain for any measurable function $f : \mathcal{R}^n \to [0, \infty)$ that the decomposition

$$\int_{\mathcal{R}^n} f\, d\Theta \;=\; \gamma \int_{\mathcal{R}^n_o} \int_{\mathbb{R}^n} f(x + C)\, \mathcal{H}^n(dx)\, \mathbb{P}_o(dC) \tag{4.2}$$

is satisfied with a constant $\gamma > 0$, referred to as the *intensity* of X, and a probability measure $\mathbb{P}_o$ over $\mathcal{R}^n_o$, which is called the *shape distribution* of X. Note that under the assumption of stationarity, Θ is locally finite if and only if

$$\int_{\mathcal{R}^n_o} \mathcal{H}^n(C + B^n)\,\mathbb{P}_o(dC) \;<\; \infty\,. \tag{4.3}$$

For $L \in \mathcal{R}^n$ we denote by $N(L)$ the minimal number $N \in \mathbb{N}$ such that L can be represented as the union of N compact convex sets. It is known that the map $N : \mathcal{R}^n \to \mathbb{N}$ is measurable; see [24], Lemma 1, or [18], Lemma 4.4.1. Now we can state the condition

$$\int_{\mathcal{R}^n_o} \mathcal{H}^n(C + B^n) 2^{N(C)}\,\mathbb{P}_o(dC) \;<\; \infty\,, \tag{4.4}$$

which is stronger than (4.3).

Our first aim is to exhibit relationships between the expectation

$$\mathbb{E}\left[\sum_{C\in X} \chi(\partial K \cap C)\right]$$

and certain *quermassdensities* or, more generally, *densities of mixed volumes* associated with X. The densities of mixed volumes of X which we consider here are defined by

$$\overline{V}(X[i], K[n-i]) \;:=\; \gamma \int_{\mathcal{R}^n_o} V(L[i], K[n-i])\,\mathbb{P}_o(dL)\,,$$

for $i \in \{0, \ldots, n\}$ and an arbitrary compact convex set $K \subset \mathbb{R}^n$, provided that (4.4) is satisfied (see Satz 5.1.4 in [18]).

Before we can state our next theorem, we have to give a proper definition of the map

$$\mathcal{R}^n \;\longrightarrow\; \mathbb{R}\,, \quad C \;\longmapsto\; \chi(\partial K \cap C)\,.$$

It is sufficient to define this map on the measurable subset $\mathcal{R}^n_k := \{C \in \mathcal{R}^n \mid N(C) = k\}$, $k \in \mathbb{N}$, of $\mathcal{R}^n$. Measurability here refers to the σ–algebra which is induced by the Matheron–Fell topology on the closed subsets of $\mathbb{R}^n$ (see [11] and [18]). Let $\mathcal{K}^n$ denote the set of nonempty compact convex subsets of $\mathbb{R}^n$ (*convex bodies*). By Theorem III.6 in [3] (see also p. 191 in [17]) there exists a measurable map $\xi_k : \mathcal{R}^n_k \to (\mathcal{K}^n)^k$ such that

$$\bigcup_{i=1}^{k} \{\xi_k(C)_i\} \;=\; C\,, \quad C \in \mathcal{R}^n_k\,.$$

Further, let $S(k)$ denote the set of all nonempty subsets of $\{1, \ldots, k\}$, and for $v \in S(k)$ define the map $f_v : (\mathcal{K}^n)^k \to \mathcal{K}^n$ by

$$f_v(K_1, \ldots, K_k) \;:=\; \bigcap_{i\in v} K_i\,.$$

Now let $K \subset \mathbb{R}^n$ be compact and convex, and let $C \in \mathcal{R}^n$. Then we say that K and C intersect almost transversally if, for all $v \in S(N(C))$, K and $f_v \circ \xi_{N(C)}(C)$ intersect almost transversally in the sense of Remark 3.3. In this case, $\chi(\partial K \cap C)$ is well–defined in the sense of singular homology theory, as the proof of Corollary 3.1 shows; moreover, setting $k := N(C)$, we have

$$\chi(\partial K \cap C) \;=\; \sum_{v \in S(k)} (-1)^{|v|-1} \chi(\partial K \cap f_v \circ \xi_k(C))\,.$$

If K and C do not intersect almost transversally, then we set $\chi(\partial K \cap C) := 0$.

Theorem 4.1. *Let X be a stationary point process on the convex ring $\mathcal{R}^n$, and let (4.4) be satisfied. Let $K \subset \mathbb{R}^n$ be a compact convex set, and set $k := \dim K$. Then, $\mathbb{P}$ almost surely K and C intersect almost transversally for all $C \in X$; moreover,*

$$\mathbb{E}\left[\sum_{C \in X} \chi(\partial K \cap C)\right] \;=\; \sum_{i=0}^{k-1} \binom{n}{i} \left\{ \overline{V}(X[n-i], -K[i]) + (-1)^{k-i-1} \overline{V}(X[n-i], K[i]) \right\}$$

and

$$\lim_{r \to \infty} \mathbb{E}\left[\sum_{C \in X} \chi(\partial (rK) \cap C)\right] \Big/ r^{k-1}$$
$$= \binom{n}{k-1} \left\{ \overline{V}(X[n-k+1], -K[k-1]) + \overline{V}(X[n-k+1], K[k-1]) \right\}.$$

Proof. The first assertion concerning the almost transversal intersection follows from the assumption of stationarity and from the proof of Corollary 3.1.

The basic tool, which we need for the proof of the asserted relations, is Campbell's theorem (see Satz 3.1.5 in [18] or [19], p. 103). The required measurability follows from the considerations preceding the statement of the theorem. The assumption (4.4) is used to verify the necessary integrability condition. The following argument is a variation of the proof for Satz 5.1.4 in [18]; the main additional difficulty here is that the functional $\chi(\partial K \cap \cdot\,)$ is not locally bounded in general.

We write $C^n := [0,1]^n$ for the unit cube and $\partial^+ C^n$ for its upper right boundary

$$\partial^+ C^n \;:=\; \left\{ x = (x_1, \ldots, x_n) \in C^n \;\Big|\; \max_{1 \le i \le n} x_i = 1 \right\}.$$

We have to show that

$$\int_{\mathcal{R}^n_o} \int_{\mathbb{R}^n} |\chi(\partial K \cap (L+t))| \, \mathcal{H}^n(dt) \, \mathbb{P}_o(dL) \;<\; \infty\,. \tag{4.5}$$

First, we keep $L \in \mathcal{R}^n_o$ fixed. Setting

$$Z \;:=\; \{ z \in \mathbb{Z}^n : L \cap (C^n + z) \neq \emptyset \}$$

and proceeding as in the proofs of Corollary 3.1 and Lemma 5.1.1 in [18], we obtain for $\mathcal{H}^n$ almost all $t \in \mathbb{R}^n$ that

$$\chi(\partial K \cap (L+t))$$
$$= \sum_{z \in Z} \left\{ \chi(\partial K \cap [L \cap (C^n + z) + t]) - \chi(\partial K \cap [L \cap (\partial^+ C^n + z) + t]) \right\}.$$

Therefore, setting

$$\mathrm{I}(L) \;:=\; \sum_{z\in Z}\int_{\mathbb{R}^n} |\chi(\partial K\cap[L\cap(C^n+z)+t])|\,\mathcal{H}^n(dt)$$

and

$$\mathrm{II}(L) \;:=\; \sum_{z\in Z}\int_{\mathbb{R}^n} |\chi(\partial K\cap[L\cap(\partial^+C^n+z)+t])|\,\mathcal{H}^n(dt)\,,$$

we can estimate

$$\int_{\mathbb{R}^n} |\chi(\partial K\cap(L+t))|\,\mathcal{H}^n(dt) \;\le\; \mathrm{I}(L)+\mathrm{II}(L)\,.$$

Subsequently, we show how $\mathrm{II}(L)$ can be estimated; the proof of a similar estimate for $\mathrm{I}(L)$ is easier. By the definition of $N(\cdot)$ we have

$$L \;=\; L_1\cup\ldots\cup L_N \quad\text{and}\quad \partial^+C^n \;=\; F_1\cup\ldots\cup F_n\,,$$

where $N = N(L)$, $L_1,\ldots,L_N$ are compact convex sets and $F_1,\ldots,F_n$ are facets of C^n. For subsets $I\subseteq\{1,\ldots,N\}$ and $J\subseteq\{1,\ldots,n\}$ we set

$$L_I \;:=\; \bigcap_{i\in I} L_i \quad\text{and}\quad F_J \;:=\; \bigcap_{j\in J} F_j\,.$$

Then, using the additivity properties of the Euler characteristic, which are available for $\mathcal{H}^n$ almost all $t\in\mathbb{R}^n$, we obtain for $z\in Z$ that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\chi(\partial K\cap[L\cap(\partial^+C^n+z)+t])|\,\mathcal{H}^n(dt)\\
\le\;& \sum_{j=1}^{n}\sum_{|J|=j}\sum_{i=1}^{N(L)}\sum_{|I|=i}\int_{\mathbb{R}^n} |\chi(\partial K\cap(L_I+t)\cap(F_J+z+t))|\,\mathcal{H}^n(dt)\\
=\;& \sum_{j=1}^{n}\sum_{|J|=j}\sum_{i=1}^{N(L)}\sum_{|I|=i}\int_{\mathbb{R}^n} |\chi(\partial K\cap[L_I\cap(F_J+z)+t])|\,\mathcal{H}^n(dt)\,.
\end{aligned}$$

But now inequality (3.1) and the monotonicity and translation invariance of mixed volumes yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\chi(\partial K\cap[L_I\cap(F_J+z)+t])|\,\mathcal{H}^n(dt)\\
=\;& \sum_{l=0}^{k-1}\binom{n}{l}\{V(-K[l],L_I\cap(F_J+z)[n-l])+V(K[l],L_I\cap(F_J+z)[n-l])\}\\
\le\;& \sum_{l=0}^{k-1}2\binom{n}{l}V(K[l],C^n[n-l])\\
=:\;& c_1(K,n)\,,
\end{aligned}$$

where $c_1(K,n)$ is a constant independent of L, and hence

$$\int_{\mathbb{R}^n} |\chi(\partial K \cap [L \cap (\partial^+ C^n + z) + t])| \, \mathcal{H}^n(dt) \;\le\; 2^n \, 2^{N(L)} \, c_1(K,n) \;=:\; c_2(K,n) \, 2^{N(L)} ;$$

moreover, we have

$$\operatorname{card}(Z) \;\le\; \mathcal{H}^n\big(L + \sqrt{n}\, B^n\big) \;\le\; c(n) \, \mathcal{H}^n(L + B^n) .$$

Thus we obtain

$$\mathrm{II}(L) \;\le\; c_3(K,n) \, 2^{N(L)} \, \mathcal{H}^n(L + B^n) .$$

A similar estimate for $\mathrm{I}(L)$ finally shows that (4.5) is valid.

Therefore we can apply Campbell's theorem and Theorem 1.4 to obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{M\in X} \chi(\partial K \cap M)\right] &= \gamma \int_{\mathcal{R}^n_o} \int_{\mathbb{R}^n} \chi(\partial K \cap (L+t)) \, \mathcal{H}^n(dt) \, \mathbb{P}_o(dL) \\
&= \gamma \sum_{i=0}^{k-1} \binom{n}{i} \int_{\mathcal{R}^n_o} \{ V(-K[i], L[n-i]) \\
&\qquad + (-1)^{k-i-1} V(K[i], L[n-i]) \} \, \mathbb{P}_o(dL) .
\end{aligned} \tag{4.6}$$

Using the homogeneity properties of mixed volumes and the integrability condition (4.5), we can easily deduce the remaining assertion of the theorem from Equation (4.6). □

For the following discussion of the preceding theorem and its consequences, we adopt the notation and the assumptions of Theorem 4.1.

1. If $\dim K = 2$, then Theorem 4.1 yields

$$\mathbb{E}\left[\sum_{C\in X} \chi(\partial K \cap C)\right] = \int_{S^{n-1}} h(DK, u) \overline{S}_{n-1}(X, du) \tag{4.7}$$

where $DK := K + (-K)$ and

$$\overline{S}_{n-1}(X, \cdot\,) \;:=\; \gamma \int_{\mathcal{R}^n_o} S_{n-1}(L, \cdot\,) \, \mathbb{P}_o(dL)$$

is the mean surface area measure of order $n-1$ of X. It is nonnegative, since $S_{n-1}(L, \cdot\,)$ is nonnegative; compare [21]. Let us denote the left–hand side of Equation (4.7) by $F(K)$. Then Equation (4.7) shows that the even part of $\overline{S}_{n-1}(X, \cdot)$, that is the unoriented mean normal measure (compare [16]), is, for instance, determined by the values $F(\vartheta \triangle_0)$, $\vartheta \in \mathbf{SO}(n)$, where $\triangle_0$ is a triangle one of whose angles is an irrational multiple of π. Of course, it is sufficient to consider a dense set of such rotations. In particular, if $\mathbb{P}_o$ is symmetric with respect to reflections in the origin, then $\overline{S}_{n-1}(X, \cdot\,)$ itself is completely determined by these values of F.

2. If $K = rB^n$, $r > 0$, and $S_r^{n-1} := r\partial B^n$, then

$$\mathbb{E}\left[\sum_{C\in X} \chi\big(S_r^{n-1} \cap C\big)\right] \;=\; \sum_{i=1}^{n} \kappa_{n-i} r^{n-i} \left(1 + (-1)^{i-1}\right) \overline{V}_i(X)\,. \tag{4.8}$$

Here the density $\overline{V}_i(X)$ is defined by

$$\overline{V}_i(X) \;:=\; \gamma \int_{\mathcal{R}_o^n} V_i(L)\, \mathbb{P}_o(dL)\,, \quad i \in \{0, \dots, n\}\,,$$

where

$$\frac{\kappa_{n-i}}{\binom{n}{i}} V_i(L) \;:=\; V(L[i], B^n[n-i]) \;=\; W_{n-i}(L)\,.$$

Equation (4.8) shows that the densities $\overline{V}_i(X)$ are determined, for i odd, if

$$\mathbb{E}\left[\sum_{C\in X} \chi\big(S_r^{n-1} \cap C\big)\right]$$

is known for $\left[\frac{n+1}{2}\right]$ different positive radii.

3. Now let $K \subset \mathbb{R}^n$ be an arbitrary compact convex set, and let X be a stationary and *isotropic point process* on $\mathcal{R}^n$. Then we obtain that

$$\mathbb{E}\left[\sum_{C\in X} \chi(\partial K \cap C)\right] \;=\; \frac{1}{\kappa_n} \sum_{i=0}^{k-1} \frac{\kappa_i \kappa_{n-i}}{\binom{n}{i}} \left(1 + (-1)^{k-i-1}\right) V_i(K) \overline{V}_{n-i}(X)\,. \tag{4.9}$$

Various special formulae can be derived from this relation. We just mention that, for $k = 2$,

$$\overline{V}_{n-1}(X) \;=\; \frac{\mathbb{E}\left[\sum_{C\in X} \chi(\partial K \cap C)\right]}{2b(K)}\,,$$

where $b(K)$ is the mean width of K (this can also be deduced from (4.7)); moreover,

$$\frac{2}{\kappa_n} \frac{\kappa_{k-1}\kappa_{n-k+1}}{\binom{n}{k-1}} \overline{V}_{n-k+1}(X) \;=\; \lim_{r\to\infty} \frac{\mathbb{E}\left[\sum_{C\in X} \chi(\partial(rK) \cap C)\right]}{V_{k-1}(rK)}\,,$$

for $k = 1, \dots, n$.

Next we consider a stationary point process X on $\mathcal{K}^n$ whose locally finite intensity measure $\Theta \not\equiv 0$ is concentrated on n–dimensional convex bodies. Let $L \in \mathcal{R}^n$ be arbitrarily chosen. Then we obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{K\in X} \chi(\partial K \cap L)\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} \left\{\overline{V}(X[j], -L[n-j]) + (-1)^{n-j-1} \overline{V}(X[j], L[n-j])\right\}. \end{aligned} \tag{4.10}$$

Again the proof follows by an application of Campbell's theorem. In the present situation, the required integrability of the random variable

$$\omega \longmapsto \sum_{K\in X(\omega)} |\chi(\partial K\cap L)|$$

can be checked more easily than the corresponding statement in the proof of Theorem 4.1; in particular, it is sufficient to assume (4.3) instead of the stronger condition (4.4). As a special case, we find for the intensity of X the representation

$$\lim_{r\to\infty} \frac{\mathbb{E}\left[\sum_{K\in X}\chi(\partial K\cap rL)\right]}{\mathcal{H}^n(rL)} = \begin{cases} 2\gamma & \text{if } n \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$$

provided that L has positive volume.

In the present framework and for a locally bounded, translation invariant and additive functional one can establish an individual ergodic theorem; see Theorem 5.2.1 in [18] for a detailed statement of such a result. The next theorem states a statistical ergodic theorem for a functional which is not locally bounded. The proof follows by modifications and combinations of the proofs of Lemma 5.1.2, Satz 5.1.4, Satz 5.2.1 and Satz 5.2.5 in [18], and by some additional arguments.

Theorem 4.2. *Let X be an ergodic stationary point process on $\mathcal{K}^n$ whose intensity measure is non–zero, locally finite and concentrated on convex bodies $K\in\mathcal{K}^n$ with $\dim K = n$. Then, for any $L\in\mathcal{K}^n$ with positive volume,*

$$\lim_{r\to\infty} \frac{\sum_{K\in X}\chi(\partial K\cap rL)}{\mathcal{H}^n(rL)} = \begin{cases} 2\gamma & \textit{if } n \textit{ is odd}, \\ 0 & \textit{otherwise}, \end{cases}$$

in L^1 as $r\to\infty$.

Our final result concerns a stationary Boolean model with grains in the convex ring, which is obtained from a stationary Poisson process on $\mathcal{R}^n$ by taking the union set

$$Z_X := \bigcup_{C\in X} C.$$

We state the following theorem without proof, since the basic approach is similar to the arguments provided in [18] and [23], if one uses the results of Section 3, in particular Theorem 3.5, and arguments analogous to those employed in the proof of Theorem 4.1.

Theorem 4.3. *Let X be a stationary Poisson process on $\mathcal{R}^n$ with intensity $\gamma>0$ and shape distribution $\mathbb{P}_0$. Assume that condition (4.4) is satisfied. Let $K\in\mathcal{K}^n$ and set $k := \dim K$. Then*

$$\begin{aligned} &\mathbb{E}\left[\chi(Z_X\cap\partial K)\right] \\ &= \left(1+(-1)^{k-1}\right)\left(1-e^{-\overline{V}_n(X)}\right) + e^{-\overline{V}_n(X)} \sum_{i=1}^{k-1}\sum_{l=1}^{k-i} \sum_{\substack{m_1,\ldots,m_l=n-k+i\\ m_1+\ldots+m_l=ln-k+i}} \\ &\quad \times\left\{\overline{V}^{(0)}_{k-i,m_1,\ldots,m_l}(-K,X,\ldots,X) + (-1)^{i-1}\overline{V}^{(0)}_{k-i,m_1,\ldots,m_l}(K,X,\ldots,X)\right\}, \end{aligned}$$

where

$$\overline{V}^{(0)}_{k-i,m_1,\dots,m_l}(K,X,\dots,X) := \gamma^l \int_{\mathcal{R}_0^n} \dots \int_{\mathcal{R}_0^n} V^{(0)}_{k-i,m_1,\dots,m_l}(K,K_1,\dots,K_l)\,\mathbb{P}_0(dK_1)\dots\mathbb{P}_0(dK_l)\,.$$

We conclude with several remarks.

1. If $k = 1$, then

$$\mathbb{E}\left[\chi(Z_X \cap \partial K)\right] = 2\left(1 - e^{-\overline{V}_n(X)}\right).$$

2. If $k = 2$, then

$$\mathbb{E}\left[\chi(Z_X \cap \partial K)\right] = e^{-\overline{V}_n(X)}\overline{V}^{(0)}_{1,n-1}(DK,X) = \frac{1}{n}\, e^{-\overline{V}_n(X)}\mathbb{E}\left[\sum_{C\in X}\chi(\partial K \cap C)\right].$$

3. If X is also isotropic, then the mixed expressions in Theorem 4.3 can be expressed as products of densities of intrinsic volumes; compare Corollary 3.6.

4. If X is a stationary Poisson process on $\mathcal{K}^n$, then it is sufficient to assume that the intensity measure of X is locally finite (condition (4.4) can be dropped).

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Mathematisches Institut
Albert–Ludwigs–Universität
Eckerstrasse 1
D – 79104 Freiburg
Germany
E – mail:
daniel.hug@math.uni-freiburg.de

Mathematisches Institut
Universität Bern
Sidlerstrasse 5
CH – 3012 Bern
Switzerland
E – mail:
mani@math-stat.unibe.ch

Mathematisches Institut
Rheinische Friedrich–Wilhelms–Universität Bonn
Beringstrasse 6
D – 53115 Bonn
Germany
E – mail:
schaetz@math.uni-bonn.de