STABILITY RESULTS INVOLVING SURFACE AREA MEASURES OF CONVEX BODIES

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Abstract

We strengthen some known stability results from the Brunn-Minkowski theory and obtain new results of similar types. These results concern pairs of convex bodies for which either surface area measures, or counterparts of such measures in the Brunn-Minkowski-Firey theory, or geometrically significant transforms of such measures, are close to each other.
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1 Introduction

In recent decades, several of the classical uniqueness theorems for convex bodies have been turned into quantitative versions, in the form of stability results. The starting point for the present investigation are uniqueness theorems of Minkowski and Aleksandrov, respectively. Minkowski's theorem, in its later general form, says that a d-dimensional convex body is uniquely determined, up to a translation, by its $(d-1)$st surface area measure. A theorem of Aleksandrov, independently proved by Fenchel and Jessen, states the extension of this result to lower order surface area measures. Aleksandrov's projection theorem asserts that a d-dimensional convex body with a given centre of symmetry is uniquely determined by the volumes (or intrinsic volumes of a given positive order) of its $(d-1)$-dimensional orthogonal projections. Also this theorem involves surface area measures, since volumes of $(d-1)$-dimensional orthogonal projections and area measures are related by the cosine transform, see (6).

In the following, we improve some known stability results corresponding to these uniqueness theorems, and we obtain new stability versions of some similar uniqueness assertions.

To formulate a stability version of Minkowski's uniqueness theorem, we denote by $\mathcal{K}^d(r,R)$ the set of convex bodies in Euclidean space $\mathbb{R}^d$ which contain some ball of radius $r>0$ and are contained in some ball of radius $R>r$. Let $K,L\in\mathcal{K}^d(r,R)$ be convex bodies whose surface area measures $S_{d-1}(K,\cdot)$ and $S_{d-1}(L,\cdot)$ satisfy

$$|S_{d-1}(K,\cdot)-S_{d-1}(L,\cdot)|\le\epsilon. \tag{1}$$

A typical stability version of Minkowski's theorem requires to find a number $\alpha>0$, which depends only on d, and a number $c>0$ depending only on d,r,R such that for $\epsilon\ge 0$, inequality (1) implies

$$\delta(K,L+x)\le c\epsilon^{\alpha} \tag{2}$$

with a suitable $x \in \mathbb{R}^d$; here δ denotes the Hausdorff metric. The best result of this type up to now is due to Diskant [4] (Theorem 7.2.2 in [23]); it gives a stability order $\alpha = 1/d$. Under the stronger assumption that K, L are of class C^2_+ and that

$$|S_{d-1}(K, \cdot) - S_{d-1}(L, \cdot)| \leq \epsilon\sigma, \tag{3}$$

where σ denotes the spherical Lebesgue measure, Diskant [6] (with details in [7]) obtained (2) with the better exponent $\alpha = 1/(d-1)$. Our first result, to be proved in Section 2, will achieve (2) with $\alpha = 1/(d-1)$ under the weaker assumption (1), for a large class of convex bodies including polytopes and bodies of class C^2_+. We also show that in this result the exponent $1/(d-1)$ is optimal.

Assumption (1) is essentially equivalent to an assumption on the total variation norm of the difference of the surface area measures of K and L: (1) implies

$$\|S_{d-1}(K, \cdot) - S_{d-1}(L, \cdot)\|_{TV} \leq 2\epsilon, \tag{4}$$

and (4) implies (1) with ϵ replaced by 2ϵ.

Following a suggestion of Wolfgang Weil, we replace this assumption on the total variation distance of measures by a more natural one on the Prohorov distance d_P. The inequality (1) implies

$$d_P(S_{d-1}(K, \cdot), S_{d-1}(L, \cdot)) \leq \epsilon, \tag{5}$$

but not conversely. We show in Section 3 that the weaker assumption (5) is still sufficient to obtain the stability estimate (2) for all convex bodies $K, L \in \mathcal{K}^d(r, R)$ with $\alpha = 1/d$.

For the Aleksandrov-Fenchel-Jessen theorem on lower order surface area measures (Corollary 7.2.5 in [23]), a stability version was obtained by Schneider [22] (Theorem 7.2.6 in [23]). Lutwak's work on the Brunn-Minkowski-Firey theory, where Minkowski sums of convex bodies are replaced by Firey's p-sums, contains also a generalization of the Aleksandrov-Fenchel-Jessen theorem (Corollary (2.3) in [19]). In Section 4 we will give a stability result for this extended theorem. As corollaries of the proof, we obtain stability versions of two inequalities of Lutwak [19].

For a convex body $K \subset \mathbb{R}^d$ and a unit vector $u \in S^{d-1}$, we denote by K^u the image of K under orthogonal projection onto $u^\perp$, the hyperplane through 0 orthogonal to u. We write $V_{d-1}(\cdot)$ for the volume in $(d-1)$-dimensional hyperplanes. Then

$$V_{d-1}(K^u) = \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| S_{d-1}(K, dv), \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbb{R}^d$. Thus, the projection function $u \mapsto V_{d-1}(K^u)$ of K is, up to a constant factor, the cosine transform of the surface area measure $S_{d-1}(K, \cdot)$. A special case of Aleksandrov's projection theorem (e.g., Theorem 3.3.6 in [10]) says that two d-dimensional centrally symmetric convex bodies with the same projection function differ only by a translation. Stability versions of this uniqueness theorem are due to Campi [3] (for $d = 3$) and to Bourgain & Lindenstrauss [2]. In Section 5 we use the method of Bourgain and Lindenstrauss to obtain further stability results of a similar nature. One of these results concerns the sine transform

$$u \mapsto \int_{S^{d-1}} \sqrt{1 - \langle u, v \rangle^2}\, S_{d-1}(K, dv)$$

of the surface area measure, which also has geometric significance. Then we obtain some stability results for convex bodies which are not necessarily centrally symmetric. They refer to various integral transforms, appearing in work of Anikonov & Stepanov [1], Goodey & Weil [12], Schneider [24].

2 Stability for Minkowski's theorem

We work in d-dimensional real vector space $\mathbb{R}^d$ ($d \geq 3$), equipped with the standard Euclidean structure. The set of convex bodies (non-empty compact convex sets) in $\mathbb{R}^d$ is denoted by $\mathcal{K}^d$. For notions from the theory of convex bodies which are not explained here, we refer to [23]. Apart from replacing $\mathbb{E}^n$ by $\mathbb{R}^d$, we use the terminology of that book.

Let $K \in \mathcal{K}^d$ be a convex body, and let $S_{d-1}(K, \cdot)$ be its surface area measure of order $d - 1$. It is a finite Borel measure on the unit sphere S^{d-1}. By Lebesgue's decomposition theorem, it can be decomposed, with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$, into an absolutely continuous part $S^a_{d-1}(K, \cdot)$ and a singular part $S^s_{d-1}(K, \cdot)$. The latter can be decomposed further, by defining

$$S^c_{d-1}(K, \cdot) := \sum_{u \in S^{d-1}} S_{d-1}(K, \{u\}) \delta_u, \tag{7}$$

where δ_u denotes the Dirac measure (unit point mass) at u, and

$$S^n_{d-1}(K, \cdot) := S^s_{d-1}(K, \cdot) - S^c_{d-1}(K, \cdot).$$

Clearly, in (7) at most countably many summands are non-zero. Moreover, $S_{d-1}(K, \{u\}) = \mathcal{H}^{d-1}(F(K, u))$ for $u \in S^{d-1}$, where $F(K, u)$ is the support set of K with outer normal vector u. Thus, we have the decomposition

$$S_{d-1}(K, \cdot) = S^a_{d-1}(K, \cdot) + S^c_{d-1}(K, \cdot) + S^n_{d-1}(K, \cdot) \tag{8}$$

of the surface area measure $S_{d-1}(K, \cdot)$ into an absolutely continuous measure $S^a_{d-1}(K, \cdot)$, a component $S^c_{d-1}(K, \cdot)$ which is an at most countable sum of point masses, and a singular component $S^n_{d-1}(K, \cdot)$ without point masses.

2.1 Theorem. *Let $0 < r < R$. There exists a number c, which depends only on d, r, R, with the following property. If $K, L \in \mathcal{K}^d(r, R)$ are convex bodies satisfying*

$S^n_{d-1}(K,\cdot)=0$, $S^n_{d-1}(L,\cdot)=0$ and the assumption

$$|S_{d-1}(K,\cdot)-S_{d-1}(L,\cdot)|\le\epsilon \tag{9}$$

for some $\epsilon\ge 0$, then

$$\delta(K,L+x)\le c\epsilon^{\frac{1}{d-1}}$$

for a suitable vector $x\in\mathbb{R}^d$.

The condition $S^n_{d-1}(K,\cdot)=0$ is fulfilled, for example, if K is a polytope, or if the surface area measure of K is absolutely continuous. The latter is true, in particular, if the support function of K is of class C^2.

The exponent $1/(d-1)$ in Theorem 2.1 is optimal, at least for the class of polytopes. This can be seen by choosing for K a unit cube and for L the polytope which is obtained from K by cutting off a vertex of K in such a way that the section plane meets K in a regular $(d-1)$-simplex of edge length $\epsilon^{1/(d-1)}$. On the other hand, under the stronger assumption that L is a ball and that (3) holds, Theorem 3.4 in [17] achieves (2) with $\alpha=1$.

The subsequent proof of Theorem 2.1 is a refinement of the approach of Diskant [6], [7] and makes also use of Diskant [5]. The proof does not require the full condition $S^n_{d-1}(K,\cdot)=0=S^n_{d-1}(L,\cdot)$, but only its consequence

$$S^n_{d-1}((1-t)K+tL,\cdot)\ge\max\{S^n_{d-1}(K,\cdot),S^n_{d-1}(L,\cdot)\}\qquad\text{for } 0\le t\le 1; \tag{10}$$

hence we will work under this assumption.

As usual, we write

$$V_1(K,L):=V(K[d-1],L)$$

for $K,L\in\mathcal{K}^d$, where V denotes the mixed volume, thus

$$V_1(K,L)=\frac{1}{d}\int_{S^{d-1}}h(L,u)S_{d-1}(K,du).$$

Here $h(L,\cdot)$ is the support function of L. We also write $V(\cdot)$ for the volume functional in $\mathbb{R}^d$.

We assume that $K,L\in\mathcal{K}^d(r,R)$ are convex bodies satisfying (9) and (10). For $t\in[0,1]$ we set

$$H_t:=(1-t)K+tL,$$

then $H_t\in\mathcal{K}^d(r,R)$. In the following, $c_1,c_2,\dots$ denote positive constants which depend only on d,r,R.

The proof is divided into four steps.

Step I. First we show that

$$|V(H_t)-V(K)|\le c_1\epsilon,\qquad |V(H_t)-V(L)|\le c_1\epsilon \tag{11}$$

for $t \in [0,1]$.

By Lemma 7.2.3 in [23], the estimates

$$|V(L) - V_1(K,L)| \le c_2\epsilon, \tag{12}$$

$$0 \le V_1(K,L) - V(K)^{\frac{d-1}{d}} V(L)^{\frac{1}{d}} \le c_3\epsilon \tag{13}$$

and the corresponding ones with K and L interchanged follow from the assumptions on K and L. From $V_1(K,L)^d \ge V(K)^{d-1}V(L)$ and (12) we deduce that

$$V(L)^d \left(1 + \frac{c_2\epsilon}{V(L)}\right)^d = (V(L) + c_2\epsilon)^d \ge V(K)^{d-1}V(L),$$

hence

$$V(K) \le V(L)\,(1 + c_4\epsilon)^{\frac{d}{d-1}} \le V(L) + c_5\epsilon.$$

By symmetry, we infer that

$$|V(K) - V(L)| \le c_5\epsilon.$$

Since $K, L \in \mathcal{K}^d(r,R)$, this implies

$$\left|V(K)^{1/d} - V(L)^{1/d}\right| \le c_6\epsilon. \tag{14}$$

The function ϕ defined by

$$\phi(t) := V(H_t)^{1/d} - (1-t)V(K)^{1/d} - tV(L)^{1/d}, \qquad t \in [0,1],$$

is concave and satisfies $\phi(0) = \phi(1) = 0$, hence

$$\phi'(0) \ge \phi(t) \ge 0 \qquad \text{for } t \in [0,1]. \tag{15}$$

Using (13) and $K \in \mathcal{K}^d(r,R)$ we get

$$\phi'(0) = \frac{V_1(K,L) - V(K)^{\frac{d-1}{d}} V(L)^{\frac{1}{d}}}{V(K)^{\frac{d-1}{d}}} \le c_7\epsilon. \tag{16}$$

Hence, by (14), (15) and (16) we obtain

$$\left|V(H_t)^{1/d} - V(K)^{1/d}\right| \le \left|V(L)^{1/d} - V(K)^{1/d}\right| + c_7\epsilon \le c_8\epsilon.$$

Since $H_t, K \in \mathcal{K}^d(r,R)$, this implies the first inequality of (11), and the second follows by symmetry.

Step II. Next we show an analogue of (11) for surface areas, namely

$$\left|V_1(H_t,B^d) - V_1(K,B^d)\right| \le c_9\epsilon, \qquad \left|V_1(H_t,B^d) - V_1(L,B^d)\right| \le c_9\epsilon \tag{17}$$

for $t \in [0,1]$; here B^d is the unit ball.

If the support function $h(M,\cdot)$ of the convex body $M \in \mathcal{K}^d$ is second order differentiable at $u \in S^{d-1}$ (which holds for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$), then the eigenvalues of the second order differential $d^2h(M,\cdot)|_{u^\perp}$ at u are the principal radii of curvature of ∂M at the point with outer normal vector u. Their product is denoted by $D_{d-1}h(M,u)$, thus

$$D_{d-1}h(M,u) = \det(d^2h(M,u)|_{u^\perp}).$$

For all $u \in S^{d-1}$ with the property that $h(K,\cdot)$ and $h(L,\cdot)$ are second order differentiable at u, and thus for $\mathcal{H}^{d-1}$ almost all $u \in S^{d-1}$, we set

$$\overline{m}(u) := \min\{D_{d-1}h(K,u), D_{d-1}h(L,u)\}.$$

Minkowski's determinant inequality states that

$$D_{d-1}h(H_t,u)^{\frac{1}{d-1}} \geq (1-t)D_{d-1}h(K,u)^{\frac{1}{d-1}} + tD_{d-1}h(L,u)^{\frac{1}{d-1}},$$

hence

$$D_{d-1}h(H_t,u) \geq \overline{m}(u) \tag{18}$$

for all $t \in [0,1]$. Let ω_1 denote the measurable set of all $u \in S^{d-1}$ such that $h(K,\cdot)$ and $h(L,\cdot)$ are second order differentiable at u and $D_{d-1}h(K,u) > \overline{m}(u)$. Then $\overline{m}(u) = D_{d-1}h(L,u)$ for $u \in \omega_1$ and

$$\begin{aligned} 0 &\leq \int_{S^{d-1}} (D_{d-1}h(K,u) - \overline{m}(u))\mathcal{H}^{d-1}(du) \\ &= \int_{\omega_1} (D_{d-1}h(K,u) - D_{d-1}h(L,u))\mathcal{H}^{d-1}(du) \\ &= S_{d-1}(K,\omega_1) - S_{d-1}(L,\omega_1). \end{aligned}$$

Here we have used the fact that if $M \in \mathcal{K}^d$ and ω is the set of all $u \in S^{d-1}$ such that $h(M,\cdot)$ is second order differentiable at u, then the restriction of the measure $S_{d-1}(M,\cdot)$ to ω is absolutely continuous with respect to $\mathcal{H}^{d-1}$. This can be verified on the basis of [15], [16] (and with the terminology used there): Choose any normal boundary point $x \in \tau(M,\omega)$ of M, that is, $x = \tau_M(u)$ for a uniquely determined $u \in \omega$. Then by an inspection of the proofs of Lemma 3.4 and Lemma 3.1 in [15], we obtain $k_i(x,u) > 0$ for $i \in \{1,\ldots,d-1\}$; moreover, by Lemma 3.1 in [15], we find $k_i(x) = k_i(x,u)$ for $i \in \{1,\ldots,d-1\}$. This shows that $H_{d-1}(M,x) > 0$ for $\mathcal{H}^{d-1}$ almost all $x \in \tau(M,\omega)$. The assertion is then implied by Theorem 3.7 in [16].

Now it follows from (9) that

$$0 \leq \int_{S^{d-1}} (D_{d-1}h(K,u) - \overline{m}(u))\mathcal{H}^{d-1}(du) \leq \epsilon. \tag{19}$$

For $M \in \mathcal{K}^d$ and $u \in S^{d-1}$, we set

$$f(M,u) := V_{d-1}(F(M,u))$$

and

$$m(u) := \min\{f(K,u), f(L,u)\}.$$

The additivity of support sets ([23], Theorem 1.7.5(c)), together with the Brunn-Minkowski theorem, implies that

$$f(H_t,u)^{\frac{1}{d-1}} \geq (1-t)f(K,u)^{\frac{1}{d-1}} + tf(L,u)^{\frac{1}{d-1}}$$

and therefore

$$f(H_t,u) \geq m(u) \tag{20}$$

for $u \in S^{d-1}$ and all $t \in [0,1]$.

Let ω_2 denote the set of all $u \in S^{d-1}$ such that $f(K,u) > m(u)$. Hence, ω_2 is at most countable, and for $u \in \omega_2$ we have $m(u) = f(L,u)$ and

$$f(K,u) - m(u) = S_{d-1}(K,\{u\}) - S_{d-1}(L,\{u\}),$$

thus

$$\begin{aligned} 0 &\leq \sum_{u \in S^{d-1}} (f(K,u) - m(u)) = \sum_{u \in \omega_2} (f(K,u) - m(u)) \\ &= S_{d-1}(K,\omega_2) - S_{d-1}(L,\omega_2). \end{aligned}$$

Therefore, (9) implies

$$0 \leq \sum_{u \in S^{d-1}} (f(K,u) - m(u)) \leq \epsilon. \tag{21}$$

For convex bodies $M, H \in \mathcal{K}^d$, we now make use of the decomposition

$$\begin{aligned} V_1(M,H) &= \frac{1}{d}\int_{S^{d-1}} h(H,u)\left[S^a_{d-1}(M,du) + S^c_{d-1}(M,du) + S^n_{d-1}(M,du)\right] \\ &= V_1^a(M,H) + V_1^c(M,H) + V_1^n(M,H) \end{aligned}$$

with

$$\begin{aligned} V_1^a(M,H) &:= \frac{1}{d}\int h(H,u)D_{d-1}h(M,u)\mathcal{H}^{d-1}(du), \\ V_1^c(M,H) &:= \frac{1}{d}\sum h(H,u)f(M,u), \\ V_1^n(M,H) &:= \frac{1}{d}\int h(H,u)S^n_{d-1}(M,du). \end{aligned}$$

Here, as below, we write $\int$ instead of $\int_{S^{d-1}}$ if the integration is extended over S^{d-1}. Similarly, $\sum$ means $\sum_{u\in S^{d-1}}$, where the summation effectively extends only over countably many summands.

We estimate the expression

$$I := V(H_t) - V_1^n(K,H_t) - \frac{1}{d}\int h(H_t,u)\overline{m}(u)\mathcal{H}^{d-1}(du) - \frac{1}{d}\sum h(H_t,u)m(u)$$

from both sides. Without loss of generality, we assume that $rB^d \subset K, L$. Then the support function of H_t satisfies $r \le h(H_t,\cdot) \le 2R$. First we insert in I the expression

$$-V_1^n(K,H_t) = -V_1(K,H_t) + V_1^a(K,H_t) + V_1^c(K,H_t)$$

and use $H_t = (1-t)K + tL$ together with the estimates (11), (19), (21) to obtain

$$\begin{aligned} I &= V(H_t) - \frac{1}{d}\int h(H_t,u)S_{d-1}(K,du) \\ &\quad +\frac{1}{d}\int h(H_t,u)(D_{d-1}h(K,u)-\overline{m}(u))\mathcal{H}^{d-1}(du) \\ &\quad +\frac{1}{d}\sum h(H_t,u)(f(K,u)-m(u))\mathcal{H}^{d-1}(du) \\ &\le (1-t)(V(H_t)-V(K)) + t(V(H_t)-V_1(K,L)) \\ &\quad +\frac{2R}{d}\left[\int (D_{d-1}h(K,u)-\overline{m}(u))\mathcal{H}^{d-1}(du) + \sum (f(K,u)-m(u))\right] \\ &\le t(V(H_t)-V_1(K,L)) + c_{10}\epsilon. \end{aligned} \tag{22}$$

Next, we insert in I the decomposition

$$V(H_t) = V_1^a(H_t,H_t) + V_1^c(H_t,H_t) + V_1^n(H_t,H_t)$$

and use the fact that $S_{d-1}^n(H_t,\cdot) - S_{d-1}^n(K,\cdot)$ is, by (10), a positive measure. Together with (18) and (20), this gives

$$\begin{aligned} I &= \frac{1}{d}\int h(H_t,u)\left(S_{d-1}^n(H_t,du) - S_{d-1}^n(K,du)\right) \\ &\quad +\frac{1}{d}\int h(H_t,u)\left(D_{d-1}h(H_t,u)-\overline{m}(u)\right)\mathcal{H}^{d-1}(du) \\ &\quad +\frac{1}{d}\sum h(H_t,u)(f(H_t,u)-m(u)) \\ &\ge \frac{r}{d}\big\{S_{d-1}^n\left(H_t,S^{d-1}\right) - S_{d-1}^n(K,S^{d-1}) \end{aligned}$$

$$
\begin{aligned}
& + \int (D_{d-1}h(H_t,u) - \overline{m}(u))\mathcal{H}^{d-1}(du) + \sum (f(H_t,u) - m(u))\big\} \\
\geq\ & \frac{r}{d}\big\{S^n_{d-1}(H_t, S^{d-1}) - S^n_{d-1}(K, S^{d-1}) \\
& + \int (D_{d-1}h(H_t,u) - D_{d-1}h(K,u))\mathcal{H}^{d-1}(du) \\
& + \sum (f(H_t,u) - f(K,u))\big\} \\
=\ & r(V_1(H_t, B^d) - V_1(K, B^d)). \qquad (23)
\end{aligned}
$$

Combining (22) and (23), we find

$$
V_1(H_t, B^d) - V_1(K, B^d) \leq \frac{t}{r}(V(H_t) - V_1(K, L)) + c_{11}\epsilon.
$$

By (11) and (12),

$$
|V(H_t) - V_1(K,L)| \leq |V(H_t) - V(L)| + |V(L) - V_1(K,L)| \leq c_{12}\epsilon
$$

and thus

$$
V_1(H_t, B^d) - V_1(K, B^d) \leq c_{13}\epsilon, \qquad t \in [0,1]. \qquad (24)
$$

On the other hand, from (18), (20), (10), (19), (21) we deduce

$$
\begin{aligned}
& V_1(H_t, B^d) = V_1^a(H_t, B^d) + V_1^c(H_t, B^d) + V_1^n(H_t, B^d) \\
& \geq \frac{1}{d}\int \overline{m}(u)\mathcal{H}^{d-1}(du) + \frac{1}{d}\sum m(u) + \frac{1}{d}S^n_{d-1}(H_t, S^{d-1}) \\
& = \frac{1}{d}\int D_{d-1}h(K,u)\mathcal{H}^{d-1}(du) + \frac{1}{d}\sum f(K,u) + \frac{1}{d}S^n_{d-1}(K, S^{d-1}) \\
& \quad - \frac{1}{d}\int (D_{d-1}h(K,u) - \overline{m}(u))\mathcal{H}^{d-1}(du) \\
& \quad - \frac{1}{d}\sum (f(K,u) - m(u)) + \frac{1}{d}\left(S^n_{d-1}\left(H_t, S^{d-1}\right) - S^n_{d-1}(K, S^{d-1})\right) \\
& \geq V_1(K, B^d) - \frac{2}{d}\epsilon. \qquad (25)
\end{aligned}
$$

Now (24) and (25) yield the first estimate of (17), and the second follows by interchanging K and L.

The next step provides corresponding estimates for projection volumes.

Step III. If $v \in S^{d-1}$ and $t \in [0,1]$, then

$$
|V_{d-1}(H_t^v) - V_{d-1}(K^v)| \leq c_{14}\epsilon, \qquad |V_{d-1}(H_t^v) - V_{d-1}(L^v)| \leq c_{14}\epsilon. \qquad (26)
$$

For the proof, we define

$$\begin{aligned} J &:= V_{d-1}\left(H_t^v\right) - \frac{1}{2}\int |\langle u,v\rangle| S_{d-1}^n(K,du) \\ &\quad -\frac{1}{2}\int |\langle u,v\rangle| \overline{m}(u)\mathcal{H}^{d-1}(du) - \frac{1}{2}\sum |\langle u,v\rangle| m(u). \end{aligned}$$

Since

$$\begin{aligned} J &= \frac{1}{2}\int |\langle u,v\rangle|(D_{d-1}h(H_t,u)-\overline{m}(u))\mathcal{H}^{d-1}(du) \\ &\quad +\frac{1}{2}\sum |\langle u,v\rangle|(f(H_t,u)-m(u)) \\ &\quad +\frac{1}{2}\int |\langle u,v\rangle| \left(S_{d-1}^n(H_t,du) - S_{d-1}^n(K,du)\right) \\ &\geq 0, \end{aligned}$$

we can deduce that

$$\begin{aligned} V_{d-1}\left(H_t^v\right) - V_{d-1}(K^v) &\geq -\frac{1}{2}\int |\langle u,v\rangle|(D_{d-1}h(K,u)-\overline{m}(u))\mathcal{H}^{d-1}(du) \\ &\quad -\frac{1}{2}\sum |\langle u,v\rangle|(f(K,u)-m(u)) \\ &\geq -\epsilon. \end{aligned} \tag{27}$$

On the other hand, by (17), (19), (21)

$$\begin{aligned} & V_{d-1}\left(H_t^v\right) - V_{d-1}(K^v) \leq J \\ &= \frac{1}{2}\int |\langle u,v\rangle| \left(S_{d-1}^n(H_t,du) - S_{d-1}^n(K,du)\right) \\ &\quad +\frac{1}{2}\int |\langle u,v\rangle| \left(D_{d-1}h(H_t,u)-\overline{m}(u)\right)\mathcal{H}^{d-1}(du) \\ &\quad +\frac{1}{2}\sum |\langle u,v\rangle|(f(H_t,u)-m(u)) \\ &\leq S_{d-1}^n(H_t,S^{d-1}) - S_{d-1}^n(K,S^{d-1}) \\ &\quad +\int (D_{d-1}h(H_t,u)-\overline{m}(u))\mathcal{H}^{d-1}(du) + \sum (f(H_t,u)-m(u)) \\ &= d(V_1(H_t,B^d) - V_1(K,B^d)) \end{aligned}$$

$$+\int (D_{d-1}h(K,u)-\overline{m}(u))\mathcal{H}^{d-1}(du)+\sum (f(K,u)-m(u))$$

$$\leq c_{15}\epsilon. \tag{28}$$

The estimates (27) and (28) yield the first estimate in (26), and the second estimate follows by symmetry.

Step IV. The rest of the proof now follows from the work of Diskant [5], [7]. For given $v\in S^{d-1}$, the function defined by

$$\phi_v(K,L,t) := V_{d-1}\left(H_t^v\right)^{\frac{1}{d-1}} - (1-t)V_{d-1}(K^v)^{\frac{1}{d-1}} - tV_{d-1}(L^v)^{\frac{1}{d-1}}$$

for $t\in[0,1]$ can be estimated, in view of (26), by

$$\begin{aligned}
\phi_v(K,L,t) &= (1-t)\left[V_{d-1}\left(H_t^v\right)^{\frac{1}{d-1}} - V_{d-1}(K^v)^{\frac{1}{d-1}}\right] \\
&\quad +t\left[V_{d-1}\left(H_t^v\right)^{\frac{1}{d-1}} - V_{d-1}(L^v)^{d-1}\right] \\
&\leq c_{16}\epsilon.
\end{aligned}$$

If $\lambda>0$ is such that $V_{d-1}(\lambda K^v)=V_{d-1}(L^v)$, one obtains $\phi_v(\lambda K,L,t)\leq c_{17}\epsilon$. Now the main theorem of [5] shows that there exist $\epsilon_0>0$ and c_{18}, depending only on d,r,R, such that

$$\delta(\lambda K^v, L^v + x(v)) \leq c_{18}\epsilon^{\frac{1}{d-1}}$$

for some $x(v)\in v^{\perp}$, if $\epsilon\leq\epsilon_0$, and therefore

$$\delta(K^v, L^v + x(v)) \leq c_{19}\epsilon^{\frac{1}{d-1}}.$$

For $\epsilon>\epsilon_0$, the same inequality holds if the constant c_{19} is adjusted. Thus the assertion of Theorem 2.1 is implied by Theorem 4.3.4 in [10].

3 Stability and Prohorov metric

For a set $A\subset S^{d-1}$ and for $\epsilon>0$, let

$$A_\epsilon := \{y\in S^{d-1} : \|x-y\|<\epsilon \text{ for some } x\in A\},$$

where $\|\cdot\|$ is the Euclidean norm. For finite Borel measures μ,ν on S^{d-1}, let

$$\begin{aligned}
d_P(\mu,\nu) &:= \inf\big\{\epsilon>0 : \mu(A)\leq\nu(A_\epsilon)+\epsilon \text{ and } \nu(A)\leq\mu(A_\epsilon)+\epsilon \\
&\qquad \text{for all Borel sets } A\subset S^{d-1}\big\}.
\end{aligned}$$

This defines the *Prohorov metric* d_P, which metrizes the weak topology (e.g., see [8], Section 11.3, in the case of probability measures).

The following theorem strengthens Diskant's stability theorem (Theorem 7.2.2 in [23]), replacing the assumption (1) by the weaker assumption (5).

3.1 Theorem. *Let* $0 < r < R$. *There exists a number* c*, depending only on* d, r, R*, such that, for* $K, L \in \mathcal{K}^d(r, R)$,

$$\delta(K, L + x) \leq c d_P(S_{d-1}(K, \cdot), S_{d-1}(L, \cdot))^{1/d}$$

for some $x \in \mathbb{R}^d$.

Proof. In the following, the positive constants $c_1, c_2, \ldots$ depend only on d, r, R. Let $K, L \in \mathcal{K}^d(r, R)$. We set $\mu := S_{d-1}(K, \cdot)$, $\nu := S_{d-1}(L, \cdot)$, $\mu_1 := \mu(S^{d-1})$, $\nu_1 := \nu(S^{d-1})$, $\epsilon := d_P(\mu, \nu)$. Then

$$|\mu_1 - \nu_1| \leq \epsilon, \qquad \mu_1, \nu_1 \geq \frac{1}{c_1},$$

and hence

$$\left|\frac{\mu_1}{\nu_1} - 1\right| \leq c_1 \epsilon, \qquad \left|\frac{\nu_1}{\mu_1} - 1\right| \leq c_1 \epsilon.$$

For any Borel set $A \subset S^{d-1}$ we deduce that

$$\frac{\mu(A)}{\mu_1} \leq \frac{\nu_1}{\mu_1}\left(\frac{\nu(A_\epsilon)}{\nu_1} + \frac{\epsilon}{\nu_1}\right) \leq (1 + c_1\epsilon)\left(\frac{\nu(A_\epsilon)}{\nu_1} + \frac{\epsilon}{\nu_1}\right) \leq \frac{\nu(A_\epsilon)}{\nu_1} + c_2\epsilon.$$

By symmetry, we find

$$d_P\left(\frac{\mu}{\mu_1}, \frac{\nu}{\nu_1}\right) \leq c_2 \epsilon.$$

For a function $f : S^{d-1} \to \mathbb{R}$ we set

$$\|f\|_L := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}, \tag{29}$$

$$\|f\|_\infty := \sup_x |f(x)|, \qquad \|f\|_{BL} := \|f\|_L + \|f\|_\infty. \tag{30}$$

It follows from the proof of Corollary 11.6.5 in [8] that

$$\left|\int f d\left(\frac{\mu}{\mu_1} - \frac{\nu}{\nu_1}\right)\right| \leq 2\|f\|_{BL} d_P\left(\frac{\mu}{\mu_1}, \frac{\nu}{\nu_1}\right).$$

Thus, for any function $f : S^{d-1} \to \mathbb{R}$ with $\|f\|_{BL} \leq 1$ we get

$$\begin{aligned} \left|\int f d(\mu - \nu)\right| &\leq \mu_1 \left[\left|\int f d\left(\frac{\mu}{\mu_1} - \frac{\nu}{\nu_1}\right)\right| + \left|\frac{1}{\nu_1} - \frac{1}{\mu_1}\right| \left|\int f\, d\nu\right|\right] \\ &\leq \mu_1(2c_2\epsilon + c_3\epsilon) = c_4\epsilon. \end{aligned}$$

We may assume that $K \subset RB^d$, then $\|h(K,\cdot)\|_{BL} \le 2R$ (cf. [23], Lemma 1.8.10). Therefore,

$$|V(K) - V_1(L,K)| = \left| \frac{1}{d} \int_{S^{d-1}} h(K,u)(\mu - \nu)(du) \right| \le \frac{2R}{d} c_4 \epsilon = c_5 \epsilon,$$

similarly $|V(L) - V_1(K,L)| \le c_5\epsilon$. These estimates correspond to the inequalities (7.2.6) in [23], and the proof can now be completed as the proof of Lemma 7.2.3 and of Theorem 7.2.2 in [23]. Note that the latter proof gives $\delta(K, L+x) \le c\epsilon^{1/d}$ if ϵ is smaller than a certain positive constant ϵ_1 depending only on d, r, R; if $\epsilon \ge \epsilon_1$, then the same inequality is achieved by a suitable choice of c.

4 Stability results in the Brunn-Minkowski-Firey theory

A basic notion of the Brunn-Minkowski theory is the vector addition of convex bodies, which corresponds to the addition of support functions,

$$h(K+L,\cdot) = h(K,\cdot) + h(L,\cdot).$$

For $p \ge 1$, a p-sum of convex bodies $K, L \in \mathcal{K}_0^d$ (the set of convex bodies in $\mathbb{R}^d$ with 0 as interior point) can be defined by

$$h(K +_p L, \cdot) := \left[h(K,\cdot)^p + h(L,\cdot)^p\right]^{1/p},$$

since the right-hand side is again a support function. Such p-means of convex bodies were introduced by Firey [9]. Lutwak [19], [20] has extended large parts of the Brunn-Minkowski theory to this more general combination of convex bodies. In this Brunn-Minkowski-Firey theory, as it is now called, the role of the classical surface area measures $S_m(K,\cdot)$, $m = 0, \dots, d-1$ (see, e.g., Section 4.2 of [23]) is played by measures $S_{p,i}(K,\cdot)$ on the sphere S^{d-1} ($i = 0, \dots, d-1$). The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_{d-1-i}(K,\cdot)$ and has a Radon-Nikodym derivative given by

$$\frac{dS_{p,i}(K,\cdot)}{dS_{d-1-i}(K,\cdot)} = h(K,\cdot)^{1-p}.$$

(Note that in [19] the measure $S_{1,i}(K,\cdot)$ is, unfortunately, denoted by $S_i(K,\cdot)$ and not by $S_{d-1-i}(K,\cdot)$, as usual.) Lutwak's theory contains an analogue of the Aleksandrov-Fenchel-Jessen theorem, Corollary (2.3) of [19]: *Suppose $K, L \in \mathcal{K}_0^d$ and $0 \le i < d$. If $d - i \ne p > 1$ and $S_{p,i}(K,\cdot) = S_{p,i}(L,\cdot)$, then $K = L$.* In the following, we obtain a stability version of this result. Again, we use an assumption on the Prohorov distance of two measures, which is weaker than the corresponding assumption for the total variation distance of the measures.

By $\mathcal{K}_0^d(r,R)$ we denote the set of convex bodies $K \subset \mathbb{R}^d$ which satisfy $rB^d \subset K \subset RB^d$, where $0 < r < R$.

4.1 Theorem. *Let $p > 1$ and $0 < r < R$. Suppose that $K, L \in \mathcal{K}_0^d(r,R)$, $i \in \{0,\dots,d-1\}$, $d-i \neq p$, and*

$$d_P(S_{p,i}(K,\cdot), S_{p,i}(L,\cdot)) \le \epsilon \tag{31}$$

with some $\epsilon \ge 0$. Then

$$\delta(K,L) \le c\epsilon^{q/2} \quad \text{with} \quad q = \frac{1}{(d+1)2^{d-i-2}},$$

where the constant c depends only on d,p,r,R.

Proof. In the following, $c_1, c_2, \dots$ denote positive constants which depend only on d,p,r,R. In the subsequent estimations where such constants occur, we very often tacitly use the facts that $rB^d \subset K, L \subset RB^d$, and that mixed volumes are monotone in each argument.

With K and L as in the theorem, we use the notations (all integrations are over the sphere S^{d-1})

$$\begin{aligned}
W_i(K) &= \frac{1}{d}\int h(K,u) S_{d-1-i}(K,du),\\
W_i(K,L) &= \frac{1}{d}\int h(L,u) S_{d-1-i}(K,du) = V(K[d-1-i], L[1], B^d[i]),\\
W_{p,i}(K,L) &= \frac{1}{d}\int h(L,u)^p S_{p,i}(K,du)\\
&= \frac{1}{d}\int h(L,u)^p h(K,u)^{1-p} S_{d-1-i}(K,du).
\end{aligned}$$

As in [23], p. 398, we write, for some fixed $i \in \{0,\dots,d-1\}$ and for $k \in \{0,\dots,d-i\}$,

$$V_{(k)} := V(K[d-i-k], L[k], B^d[i]),$$

thus $W_i(K) = V_{(0)}$, $W_i(K,L) = V_{(1)}$, $W_i(L) = V_{(d-i)}$. With these notations, Lutwak's [19] inequality (IIp) (p. 132; see also Theorem 1.2 in [19]) reads

$$W_{p,i}(K,L)^{d-i} \ge V_{(0)}^{d-i-p} V_{(d-i)}^p. \tag{32}$$

Interchanging K and L, we get

$$W_{p,i}(L,K)^{d-i} \ge V_{(d-i)}^{d-i-p} V_{(0)}^p. \tag{33}$$

Another inequality proved by Lutwak [19] (p. 137) states that

$$W_{p,i}(K,L) \geq W_i(K,L)^p W_i(K)^{1-p} = V_{(1)}^p V_{(0)}^{1-p}. \tag{34}$$

Using (31), we can estimate as in Section 3 and obtain

$$\begin{aligned} \left|V_{(0)} - W_{p,i}(L,K)\right| &= \frac{1}{d}\left|\int h(K,u)^p [S_{p,i}(K,du) - S_{p,i}(L,du)]\right| \\ &\leq c_1 \|h(K,\cdot)^p\|_{BL}\epsilon, \end{aligned}$$

hence

$$|V_{(0)} - W_{p,i}(L,K)| \leq c_2\epsilon \tag{35}$$

and similarly

$$|W_{p,i}(K,L) - V_{(d-i)}| \leq c_3\epsilon. \tag{36}$$

We write

$$\begin{aligned} & W_{p,i}(K,L) - V_{(0)}^{\frac{d-i-p}{d-i}} V_{(d-i)}^{\frac{p}{d-i}} \\ &= \left(\frac{V_{(d-i)}}{V_{(0)}}\right)^{\frac{p}{d-i}} \left(V_{(d-i)}^{\frac{d-i-p}{d-i}} V_{(0)}^{\frac{p}{d-i}} - W_{p,i}(L,K)\right) \\ &\quad + \left(\frac{V_{(d-i)}}{V_{(0)}}\right)^{\frac{p}{d-i}} \left[W_{p,i}(L,K) - V_{(0)}\right] + \left[W_{p,i}(K,L) - V_{(d-i)}\right]. \end{aligned}$$

By (33), the first term on the right is not positive, hence (35) and (36) give

$$W_{p,i}(K,L) - V_{(0)}^{\frac{d-i-p}{d-i}} V_{(d-i)}^{\frac{p}{d-i}} \leq c_4\epsilon. \tag{37}$$

Now we assume that $i \in \{0,\dots,d-2\}$. We write (37) in the form

$$\left[W_{p,i}(K,L) - V_{(1)}^p V_{(0)}^{1-p}\right] + \left[V_{(1)}^p - V_{(0)}^{\frac{p(d-1-i)}{d-i}} V_{(d-i)}^{\frac{p}{d-i}}\right] V_{(0)}^{1-p} \leq c_4\epsilon.$$

Here both brackets are nonnegative, the first by (34), and the second by the Aleksandrov-Fenchel inequalities. We deduce that

$$W_{p,i}(K,L) - V_{(1)}^p V_{(0)}^{1-p} \leq c_4\epsilon \tag{38}$$

and

$$V_{(1)}^p \leq V_{(0)}^{\frac{p(d-1-i)}{d-i}} V_{(d-i)}^{\frac{p}{d-i}} + c_5\epsilon. \tag{39}$$

Interchanging K and L in (39) gives

$$V_{(d-1-i)}^p \leq V_{(d-i)}^{\frac{p(d-1-i)}{d-i}} V_{(0)}^{\frac{p}{d-i}} + c_5\epsilon. \tag{40}$$

Multiplication of (39) and (40) yields

$$V_{(1)}V_{(d-1-i)} \le V_{(0)}V_{(d-i)} + c_6\epsilon. \tag{41}$$

We are now in the same situation as in the proof of Theorem 7.2.6 in [23]: the inequality there before (7.2.12) is precisely (41), with m replaced by $d-i$ and c_2 replaced by c_6. Hence, the subsequent arguments in [23] (see the Appendix of the present paper) lead to the conclusion that

$$\delta(\overline{K}, \overline{L}) \le c_7\epsilon^q \tag{42}$$

(see also the hint at the end of the proof of Theorem 3.1). Here $\overline{K} = [K - s(K)]/b(K)$, where $s(K)$ is the Steiner point and $b(K)$ is the mean width of K.

We put $\lambda = b(K)/b(L)$ and $t = s(K) - \lambda s(L)$, then (42) implies

$$\delta(K, \lambda L + t) \le c_8\epsilon^q. \tag{43}$$

To derive (34), Hölder's inequality was used. In order to estimate t, we need a sharper version of that inequality. We use a special case of an inequality by Kober [18], namely

$$w_1a_1 + w_2a_2 - a_1^{w_1}a_2^{w_2} \ge w(a_1^{1/2} - a_2^{1/2})^2$$

for $a_1, a_2 \ge 0$ and $w_1, w_2 > 0$ with $w_1 + w_2 = 1$, where $w := \min\{w_1, w_2\}$. Here we put, for $p > 1$ and $a, b > 0$,

$$w_1 = \frac{1}{p}, \quad w_2 = \frac{p-1}{p}, \quad a_1 = a^pb^{1-p}, \quad a_2 = b$$

and obtain

$$a^pb^{1-p} + (p-1)b - pa \ge mb^{1-p}(a^{p/2} - b^{p/2})^2 \tag{44}$$

with $m = \min\{1, p-1\}$.

Write $h(M, \cdot) = h_M$ for $M \in \mathcal{K}^d$ and put

$$I(M) := \frac{1}{d}\int h_M(u)\, S_{d-1-i}(K, du).$$

We apply (44) with

$$a = \frac{h_L(u)}{I(L)}, \quad b = \frac{h_K(u)}{I(K)},$$

where $u \in S^{d-1}$, and integrate over all $u \in S^{d-1}$ with respect to the measure $(1/d)S_{d-1-i}(K, \cdot)$. The result can be written as

$$\frac{W_{p,i}(K,L)}{V_{(1)}^pV_{(0)}^{1-p}} - 1 \ge c_9 \int \left[\left(\frac{h_L}{I(L)}\right)^{p/2} - \left(\frac{h_K}{I(K)}\right)^{p/2}\right]^2 h_K^{1-p}\, dS_{d-1-i}(K, \cdot). \tag{45}$$

The quotient $h_L/I(L)$ is invariant under a dilatation of L, hence on the right-hand side, the body L can be replaced by λL. Therefore, (38) and (45) yield

$$\int \left[\left(\frac{h_{\lambda L}}{I(\lambda L)} \right)^{p/2} - \left(\frac{h_K}{I(K)} \right)^{p/2} \right]^2 dS_{d-1-i}(K,\cdot) \leq c_{10}\epsilon. \tag{46}$$

Since $I(M)$ is invariant under translations of M, inequality (43) shows that

$$|I(\lambda L) - I(K)| \leq c_{11}\epsilon^q.$$

Using this inequality and the mean value theorem, we can estimate, for $u \in S^{d-1}$,

$$\begin{aligned} |h_{\lambda L}(u) - h_K(u)| &\leq c_{12} \left| h_{\lambda L}(u)^{p/2} - h_K(u)^{p/2} \right| \\ &\leq c_{13} \left| \left(\frac{h_{\lambda L}(u)}{I(\lambda L)} \right)^{p/2} - \left(\frac{h_K(u)}{I(K)} \right)^{p/2} \right| + c_{14}\epsilon^q. \end{aligned}$$

Together with (46), this yields an estimate

$$\begin{aligned} &\left(\int |h_{\lambda L} - h_K| \, dS_{d-1-i}(K,\cdot) \right)^2 \\ &\leq c_{15} \int |h_{\lambda L} - h_K|^2 \, dS_{d-1-i}(K,\cdot) \leq c_{16}\epsilon^q. \end{aligned}$$

Since $h_{\lambda L+t}(u) = h_{\lambda L}(u) + \langle u, t\rangle$, it follows from (43) that

$$|\langle u, t\rangle| \leq |h_{\lambda L}(u) - h_K(u)| + c_8\epsilon^q$$

for $u \in S^{d-1}$. Writing $t_1 = t/\|t\|$ if $t \neq 0$, we deduce that

$$\|t\| \int |\langle u, t_1\rangle| \, S_{d-1-i}(K, du) \leq c_{17}\epsilon^{q/2}.$$

Now

$$\int |\langle u, t_1\rangle| \, S_{d-1-i}(K, du) \geq c_{18},$$

since the integral is, up to a factor depending only on d, an intrinsic volume of a projection of K and hence can be estimated from below by a constant depending only on d and r. The conclusion is that

$$\|t\| \leq c_{19}\epsilon^{q/2}. \tag{47}$$

To estimate λ, we first deduce from (36) and (37) that

$$V_{(d-i)} - V_{(0)}^{\frac{d-i-p}{d-i}} V_{(d-i)}^{\frac{p}{d-i}} \leq c_{20}\epsilon,$$

thus

$$V_{(d-i)}^{\frac{d-i-p}{d-i}} - V_{(0)}^{\frac{d-i-p}{d-i}} \le c_{21}\epsilon.$$

Since we have assumed $d-i-p \neq 0$, this implies

$$W_i(L) - W_i(K) \le c_{22}\epsilon.$$

From (43), we get

$$K \subset \lambda L + t + c_8\epsilon^q B^d \subset (1 + c_{23}\epsilon^q)\lambda L + t,$$

hence

$$\begin{aligned} W_i(L) &\le W_i(K) + c_{22}\epsilon \\ &\le W_i((1 + c_{23}\epsilon^q)\lambda L) + c_{22}\epsilon \\ &= [(1 + c_{23}\epsilon^q)\lambda]^{d-i} W_i(L) + c_{22}\epsilon. \end{aligned}$$

This gives $\lambda \ge 1 - c_{24}\epsilon^q$. Interchanging the roles of K and L, we similarly obtain $\lambda^{-1} \ge 1 - c_{25}\epsilon^q$ and hence

$$|\lambda - 1| \le c_{26}\epsilon^q. \tag{48}$$

The inequalities (43), (47) and (48) finally give

$$\delta(K, L) \le c_{27}\epsilon^{q/2}.$$

This completes the proof of Theorem 4.1 in the case where $i \in \{0, \dots, d-2\}$.

Finally, we consider the (simpler) case $i = d-1$. As before, we deduce that $W_{d-1}(L) - W_{d-1}(K) \le c_{22}\epsilon$, and hence by symmetry

$$|\lambda - 1| \le c_{28}\epsilon, \tag{49}$$

where $\lambda = b(K)/b(L)$. Using (37) and (45), we find that

$$\int \left[\left(\frac{h_L}{W_{d-1}(L)} \right)^{p/2} - \left(\frac{h_K}{W_{d-1}(K)} \right)^{p/2} \right]^2 d\sigma \le c_{29}\epsilon$$

and thus

$$\int \left| h_{\lambda L}^{p/2} - h_K^{p/2} \right|^2 d\sigma \le c_{30}\epsilon.$$

An application of the mean value theorem shows that

$$\int |h_{\lambda L} - h_K|^2 \, d\sigma \le c_{31}\epsilon,$$

hence (49) and Corollary 1 in [25] (see also Lemma 6.6.4 in [23]) give $\delta(K, L) \le c_{32}\epsilon^{\frac{1}{d+1}}$. The proof of Theorem 4.1 is now complete.

We remark that the preceding proof also permits us to give stability versions of two inequalities due to Lutwak [19]. The first of these is his inequality (II_p) (which is (32) above).

4.2 Corollary. *Let $p > 1$ and $0 < r < R$. Suppose that $K, L \in \mathcal{K}_0^d(r,R)$, $i \in \{0,\dots,d-1\}$ and*

$$W_{p,i}(K,L) - W_i(K)^{\frac{d-i-p}{d-i}} W_i(L)^{\frac{p}{d-i}} \le \epsilon \tag{50}$$

with some $\epsilon \ge 0$. Then there is a constant c depending only on d, p, r, R such that

$$\delta(K, \lambda L) \le c\epsilon^{q/2},$$

where $\lambda = b(K)/b(L)$ and q is as in Theorem 4.1.

Proof. Assume that $i \in \{0,\dots,d-2\}$. Then the assumption (50) implies that $\delta(\overline{K},\overline{L}) \le c_{33}\epsilon^q$, by the argument after equation (37). The subsequent argument in the proof of Theorem 4.1, which shows that $|s(K) - \lambda s(L)| \le c_{34}\epsilon^{q/2}$, remains the same, hence $\delta(K, \lambda L) \le c_{35}\epsilon^{q/2}$, as stated.

The case $i = d-1$ can be treated as in the proof of Theorem 4.1.

The next result gives a stability version of Lutwak's Corollary (1.3) (using his notations).

4.3 Corollary. *Let $p > 1$, $0 < r < R$ and $\vartheta \in (0,1)$. Suppose that $K, L \in \mathcal{K}_0^d(r,R)$, $i \in \{0,\dots,d-1\}$ and*

$$W_i((1-\vartheta)\cdot K +_p \vartheta \cdot L)^{\frac{p}{d-i}} - (1-\vartheta) W_i(K)^{\frac{p}{d-i}} - \vartheta W_i(L)^{\frac{p}{d-i}} \le \epsilon \tag{51}$$

with some $\epsilon \ge 0$. Then there is a constant c depending only on d, p, r, R such that

$$\delta(K, \tau L) \le c \min\{\vartheta, 1-\vartheta\}^{-q/2} \epsilon^{q/2},$$

where τ is a suitable positive constant and q is as in Theorem 4.1.

Proof. Put $M := (1-\vartheta)\cdot K +_p \vartheta \cdot L$. From the definitions of p-sums and of the functionals $W_{p,i}$, we obtain

$$\begin{aligned} W_i(M) &= W_{p,i}(M, (1-\vartheta)\cdot K +_p \vartheta \cdot L) \\ &= (1-\vartheta) W_{p,i}(M,K) + \vartheta W_{p,i}(M,L). \end{aligned}$$

Since $M \in \mathcal{K}_0^d(r,R)$, we can apply Corollary 4.2 and deduce that, with suitable numbers $\tau_1, \tau_2 > 0$,

$$\begin{aligned} W_i(M) \ \ge\ & (1-\vartheta)\left[c_{36}\delta(M,\tau_1 K)^{\frac{2}{q}} + W_i(M)^{\frac{d-i-p}{d-i}} W_i(K)^{\frac{p}{d-i}}\right] \\ & + \vartheta\left[c_{37}\delta(M,\tau_2 L)^{\frac{2}{q}} + W_i(M)^{\frac{d-i-p}{d-i}} W_i(L)^{\frac{p}{d-i}}\right]. \end{aligned}$$

From this we infer that

$$\begin{aligned}
&W_i(M)^{\frac{p}{d-i}} - (1-\vartheta)W_i(K)^{\frac{p}{d-i}} - \vartheta W_i(L)^{\frac{p}{d-i}} \\
&\geq (1-\vartheta)c_{38}\delta(M,\tau_1 K)^{\frac{2}{q}} + \vartheta c_{39}\delta(M,\tau_2 L)^{\frac{2}{q}} \\
&\geq \min\{\vartheta, 1-\vartheta\}c_{40}\left[\delta(M,\tau_1 K) + \delta(M,\tau_2 L)\right]^{\frac{2}{q}} \\
&\geq \min\{\vartheta, 1-\vartheta\}c_{41}\delta(\tau_1 K,\tau_2 L)^{\frac{2}{q}} \geq \min\{\vartheta, 1-\vartheta\}c_{42}\delta(K,\tau L)^{\frac{2}{q}}.
\end{aligned}$$

5 Stability of inverse integral transforms

The starting point of this section is formula (6),

$$V_{d-1}(K^u) = \frac{1}{2}\int_{S^{d-1}} |\langle u,v\rangle| S_{d-1}(K,dv), \qquad u \in S^{d-1},$$

which expresses the projection function $u \mapsto V_{d-1}(K^u)$ of a convex body K as the cosine transform of its area measure of order $d-1$. The stability result of Bourgain & Lindenstrauss [2] is a quantitative version of the fact that two d-dimensional convex bodies with the same centre of symmetry must be close if their projection functions are close. Groemer's [14] book contains a detailed presentation of this theorem and its proof (Theorem 5.5.7). In the present section, we use the method of Bourgain and Lindenstrauss to obtain stability estimates for the inversion of further integral transforms of area measures occurring in the geometry of convex bodies.

These integral transforms are of the following type. Let $\Phi : [-1,1] \to \mathbb{R}$ be a bounded, Borel measurable function. For a finite signed Borel measure μ on S^{d-1}, let

$$(T_\Phi \mu)(u) := \int_{S^{d-1}} \Phi(\langle u,v\rangle)\mu(dv) \qquad \text{for } u \in S^{d-1}. \tag{52}$$

For a bounded measurable function f on S^{d-1}, the transform $T_\Phi f$ is defined as $T_\Phi \mu$ for the signed measure $\mu = f\sigma$, where σ denotes spherical Lebesgue measure.

We need a few facts about spherical harmonics, which can all be found in [14]. If Y_n is a spherical harmonic of degree n on S^{d-1}, then

$$T_\Phi Y_n = a_{d,n}(\Phi) Y_n$$

with

$$a_{d,n}(\Phi) = \omega_{d-1}\int_{-1}^{1} \Phi(t) P_n^d(t)(1-t^2)^{(d-3)/2}\, dt,$$

where P_n^d is the Legendre polynomial of dimension d and degree n (e.g., [14], Th. 3.4.1). Here $\omega_k = k\kappa_k$ is the area of the k-dimensional unit ball, and κ_k is its volume. The numbers $a_{d,n}(\Phi)$ are called the multipliers of T_Φ.

For $f, g \in L_2(S^{d-1})$, the space of square integrable real functions on S^{d-1}, a scalar product is defined by

$$(f,g) := \int_{S^{d-1}} fg\, d\sigma,$$

and the L_2-norm by $\|f\| := \sqrt{(f,f)}$. Let $\{Y_{nj} : j = 1, \dots, N(d,n)\}$ be an orthonormal basis of the real vector space of spherical harmonics of degree $n \in \mathbb{N}_0$. For $f \in L_2(S^{d-1})$, the relation

$$f \sim \sum_{n=0}^{\infty} Y_n \tag{53}$$

means that

$$Y_n = \sum_{j=1}^{N(d,n)} (f, Y_{nj}) Y_{nj},$$

and the series in (53) is called the condensed harmonic expansion of f ([14], p. 72). Similarly, for a finite signed measure μ on S^{d-1} we write

$$\mu \sim \sum_{n=0}^{\infty} Y_n \tag{54}$$

if

$$Y_n = \sum_{j=1}^{N(d,n)} \left(\int_{S^{d-1}} Y_{nj}\, d\mu \right) Y_{nj}.$$

If (54) holds, then

$$T_\Phi \mu \sim \sum_{n=0}^{\infty} a_{d,n}(\Phi) Y_n. \tag{55}$$

The following theorem is only a slight extension of the result of Bourgain and Lindenstrauss, to general transformations T_Φ. For the reader's convenience, we repeat the essential steps of the proof, in a simplified form, to indicate where changes are necessary. Recall that the norm $\|\cdot\|_{BL}$ was defined by (30) and that $\|\mu\|_{TV}$ denotes the total variation norm of the signed measure μ.

5.1 Theorem. *Assume that the multipliers of the transformation T_Φ satisfy*

$$a_{d,0}(\Phi) \neq 0, \qquad |a_{d,n}(\Phi)^{-1}| \leq bn^\beta \quad \text{for } n \in \mathbb{N} \tag{56}$$

with suitable $b, \beta > 0$. Let μ be a finite signed measure on S^{d-1}, and let $F : S^{d-1} \to \mathbb{R}$ be a Lipschitz function. Then for each $\alpha \in (0, 1/(1+\beta))$ there is a constant c depending only on d, Φ, α such that

$$\left| \int_{S^{d-1}} F\, d\mu \right| \leq c \|F\|_{BL}\, \|\mu\|_{TV}^{1-\alpha}\, \|T_\Phi \mu\|^\alpha.$$

If μ is even and (56) *holds for even n, then the same conclusion can be drawn.*

Proof. We choose b, β (depending on Φ) so that (56) holds. The constants $c_1, c_2, \dots$ in the following depend only on $d, \Phi, b, \beta, \alpha$ and hence only on d, Φ, α.

It was the idea of Bourgain and Lindenstrauss [2] to use the Poisson transform

$$\mu_\tau := \frac{1}{\omega_d} \int_{S^{d-1}} \frac{1-\tau^2}{(1+\tau^2-2\tau\langle u,v\rangle)^{d/2}}\, \mu(dv), \qquad u \in S^{d-1},$$

for $0<\tau<1$. We have (all integrations are over S^{d-1})

$$\int F_\tau \, d\mu = \int F \mu_\tau \, d\sigma$$

and

$$\begin{aligned} \left| \int F \, d\mu \right| &\le \left| \int (F - F_\tau)\, d\mu \right| + \left| \int F_\tau \, d\mu \right| \\ &\le \|F - F_\tau\|_\infty \, \|\mu\|_{TV} + \left| \int F \mu_\tau \, d\sigma \right|. \end{aligned} \tag{57}$$

For $\tau \ge 1/4$,

$$\|F - F_\tau\|_\infty \le 2^{d+1} \frac{\omega_{d-1}}{\omega_d} \|F\|_L (1-\tau) \log \frac{2}{1-\tau} \tag{58}$$

([14], Lemma 5.5.8). Moreover,

$$\left| \int F \mu_\tau \, d\sigma \right| \le \|F\| \, \|\mu_\tau\|. \tag{59}$$

If (54) is the condensed harmonic expansion of μ, then

$$\mu_\tau \sim \sum_{n=0}^\infty \tau^n Y_n.$$

The maximal value of the function $g(x) = x^\beta \tau^x$ for $x>0$ is $(-\beta/e \log \tau)^\beta$, hence

$$n^\beta \tau^n (1-\tau)^\beta \le \left(\frac{\beta}{e}\right)^\beta \left(\frac{1-\tau}{-\log\tau}\right)^\beta \le \left(\frac{\beta}{e}\right)^\beta$$

for $n \in \mathbb{N}$. Therefore, (56) gives

$$\tau^n \le c_1 (1-\tau)^{-\beta} |a_{d,n}(\Phi)|.$$

Together with Parseval's relation, this yields

$$\|\mu_\tau\|^2 = \sum_{n=0}^\infty \tau^{2n} \|Y_n\|^2 \le c_1^2 (1-\tau)^{-2\beta} \sum_{n=0}^\infty |a_{d,n}(\Phi)|^2 \|Y_n\|^2.$$

Now (55) shows that

$$\|\mu_\tau\| \le c_1(1-\tau)^{-\beta}\|T_\Phi\mu\|. \tag{60}$$

From (57), (58), (59), (60) we get

$$\begin{aligned}\left|\int F\, d\mu\right| &\le c_2\|F\|\,\|T_\Phi\mu\|(1-\tau)^{-\beta} + c_3\|F\|_L\|\mu\|_{TV}(1-\tau)\log\frac{2}{1-\tau}\\ &\le c_4\|F\|_{BL}\left[\|T_\Phi\mu\|(1-\tau)^{-\beta} + \|\mu\|_{TV}(1-\tau)\log\frac{2}{1-\tau}\right].\end{aligned}$$

Since Φ is bounded, we have

$$\|T_\Phi\mu\| \le c_5\|\mu\|_{TV}.$$

Therefore, we can find a constant c_6 and a number $\tau \in [\frac{1}{4}, 1)$ such that

$$\|T_\Phi\mu\|(1-\tau)^{-\beta} = c_6\|\mu\|_{TV}(1-\tau)\log\frac{2}{1-\tau}.$$

For this τ and for $\alpha \in (0,1)$ we get

$$\left|\int F\, d\mu\right| \le c_7\|F\|_{BL}(1-\tau)^{1-\alpha(1+\beta)}\left(\log\frac{2}{1-\tau}\right)^{1-\alpha}\|\mu\|_{TV}^{1-\alpha}\|T_\Phi\mu\|^\alpha.$$

If now $\alpha < 1/(1+\beta)$, then we get

$$\left|\int F\, d\mu\right| \le c_8\|F\|_{BL}\,\|\mu\|_{TV}^{1-\alpha}\|\,T_\Phi\mu\|^\alpha.$$

If the signed measure μ is even, then the components in (54) satisfy $Y_n = 0$ for odd n. Therefore, one can conclude as above. This completes the proof of Theorem 5.1.

The geometric applications are of the following type.

5.2 Theorem. *Let Φ and β be as in Theorem* 5.1*, let $0 < r < R$. For $\gamma \in (0, 1/d(1+\beta))$, there is a constant c depending only on d, Φ, γ, r, R with the following property. If $K, L \in \mathcal{K}^d(r,R)$ and*

$$\mu := S_{d-1}(K,\cdot) - S_{d-1}(L,\cdot), \tag{61}$$

then

$$\delta(K, L+x) \le c\|T_\Phi\mu\|^\gamma \tag{62}$$

with a suitable vector $x \in \mathbb{R}^d$.

If K and L are centrally symmetric and (56) *holds for even n, then the same conclusion can be drawn.*

Proof. The constants $c_1, c_2, \dots$ in this proof depend only on d, Φ, γ, r, R. We apply Theorem 5.1 with $\alpha = d\gamma$ to the measure μ given by (61) and to $F = h_K$, the support function of K. Without loss of generality, we assume that $K \subset RB^d$. Since $\|\mu\|_{TV} = S_{d-1}(K, S^{d-1}) + S_{d-1}(L, S^{d-1})$ can be estimated from above by a constant depending only on R and d and the same is true for $\|h_K\|_{BL}$ (cf. [23], Lemma 1.8.10), we get

$$\left| \int F \, d\mu \right| \le c_1 \|T_\Phi \mu\|^\alpha.$$

By the geometric meaning of F and μ, this reads

$$|V(K) - V_1(L, K)| \le c_2 \|T_\Phi \mu\|^\alpha,$$

and interchanging K and L we get

$$|V(L) - V_1(K, L)| \le c_2 \|T_\Phi \mu\|^\alpha.$$

By a result of Diskant [4] (compare the remark at the end of the proof of Theorem 3.1), the two inequalities

$$|V(K) - V_1(L, K)| \le \epsilon, \quad |V(L) - V_1(K, L)| \le \epsilon$$

together imply

$$\delta(K, L + x) \le c_3 \epsilon^{1/d}$$

for suitable $x \in \mathbb{R}^d$, provided that $\epsilon \le \epsilon_0$, where $\epsilon_0 > 0$ is a constant depending only on d, r, R. If $c_2 \|T_\Phi \mu\|^\alpha \le \epsilon_0$, then we get

$$\delta(K, L + x) \le c_4 \|T_\Phi \mu\|^{\alpha/d},$$

and if $c_2 \|T_\Phi \mu\|^\alpha > \epsilon_0$, the same estimate holds if c_4 is chosen suitably.

If K and L are centrally symmetric, then the signed measure μ is even. This completes the proof of Theorem 5.2.

The special case of Theorems 5.1 and 5.2 treated by Bourgain and Lindenstrauss concerned the cosine transform, where $\Phi(t) = \frac{1}{2}|t|$ for $t \in [-1, 1]$. In that case, (56) holds for even n with $\beta = (d+2)/2$. Hence, for convex bodies $K, L \in \mathcal{K}^d(r, R)$ with the same centre of symmetry and for the $(d-1)$st projection function $V_{d-1}(K, u) = V_{d-1}(K^u)$, $u \in S^{d-1}$, one gets

$$\delta(K, L) \le c \|V_{d-1}(K, \cdot) - V_{d-1}(L, \cdot)\|^\gamma \tag{63}$$

for $\gamma \in (0, 2/d(d+4))$.

It is natural to ask for similar results for the ith projection function,

$$V_i(K, u) = V_i(K^u) = \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, S_i(K, dv), \qquad u \in S^{d-1}.$$

By a well-known integral geometric formula, the convex bodies K, L satisfy $V_i(K, \cdot) = V_i(L, \cdot)$ if the projections of K and L on an i-dimensional subspace always have the same i-dimensional volume. For $i = 1$, a strong stability result was proved by Goodey and Groemer [13]. In two books, the question for corresponding generalizations was posed. Groemer [14], p. 222, writes that 'at present such stability estimates exist only in the cases $i = 1$ and $i = d - 1$'. Gardner [10] asks in his Problem 4.7 (p. 157) whether a stability result of the type (63) can be obtained for $1 < i < d - 1$. Curiously, a positive answer on the basis of published results could have been given at the time when those books were written. In fact, the analytic part of the Bourgain-Lindenstrauss [2] proof (just replace μ by $\mu_i := S_i(K, \cdot) - S_i(L, \cdot)$ in the first part of the proof of Theorem 5.2) gives

$$|V_{(0)} - V_{(i)}| \le c_2 \|T_\Phi \mu_i\|^\alpha, \qquad |V_{(i+1)} - V_{(1)}| \le c_2 \|T_\Phi \mu_i\|^\alpha$$

for $\alpha \in (0, 2/(d+4))$ (if $\Phi(t) = \frac{1}{2}|t|$), where

$$V_{(k)} := V(K[i+1-k], L[k], B^d[d-1-i]).$$

As shown in [23] (Proof of Lemma 7.2.3), the inequalities

$$|V_{(0)} - V_{(i)}| \le \epsilon, \qquad |V_{(i+1)} - V_{(1)}| \le \epsilon$$

for $K, L \in \mathcal{K}^d(r, R)$ and some $\epsilon > 0$ imply

$$0 \le V_{(1)} - V_{(0)}^{i/(i+1)} V_{(i+1)}^{1/(i+1)} \le \left(\frac{R}{r} + 1\right) \epsilon.$$

One can now essentially use the proof of a stability result for the Aleksandrov-Fenchel-Jessen theorem ([23], Theorem 7.2.6) to deduce the following.

5.3 Theorem. *Let $i \in \{2, \ldots, d-2\}$ and $0 < r < R$, let $K, L \in \mathcal{K}^d(r, R)$ be convex bodies which are centrally symmetric with the same centre. For*

$$\gamma \in \left(0, \frac{1}{(d+1)(d+4)2^{i-2}}\right),$$

there exists a constant c depending only on d, γ, r, R such that

$$\delta(K, L) \le c \|V_i(K, \cdot) - V_i(L, \cdot)\|^\gamma.$$

We turn to other integral transforms of type T_Φ which have occurred in geometric contexts. The *sine transform* is the transformation T_Φ with $\Phi(t) = \sqrt{1 - t^2}$ for $t \in [-1, 1]$. If $K \in \mathcal{K}^d$ is a convex body and $u \in S^{d-1}$, then

$$\begin{aligned} V^{(d-1)}(K, u) &:= \int_{-\infty}^{\infty} V_{d-2}(K \cap (u^\perp + tu))\, dt \\ &= \frac{1}{2(d+1)} \int_{S^{d-1}} \sqrt{1 - \langle u, v \rangle^2}\, S_{d-1}(K, dv) \end{aligned} \tag{64}$$

(see [21], p. 60); here $2V_{d-2}(K')$ is the $(d-1)$-dimensional surface area of a $(d-1)$-dimensional convex body K'. Thus the functional $V^{(d-1)}(K,\cdot)$, the integrated surface area of parallel hyperplane sections, is, up to a factor, the sine transform of the surface area measure of K. The sine transform S is connected with the cosine transform C and the spherical Radon transform R by the relation $RC = \kappa_{d-2}S$, which is easily obtained by a direct calculation. (For convex bodies this implies, in view of (64), that Radon transforms of projection functions are connected with sections; this interplay was studied in greater generality by Goodey [11].) This relation implies corresponding relations for the multipliers: if

$$f \sim \sum_{n=0}^{\infty} Y_n, \quad Cf \sim \sum_{n=0}^{\infty} \zeta_{d,n} Y_n, \quad Rf \sim \sum_{n=0}^{\infty} \rho_{d,n} Y_n,$$

then

$$\kappa_{d-2} Sf \sim \sum_{n=0}^{\infty} \rho_{d,n}\zeta_{d,n} Y_n.$$

For even n, we have $|\zeta_{d,n}^{-1}| = O\left(n^{(d+2)/2}\right)$, as remarked above, and $|\rho_{d,n}^{-1}| = O\left(n^{(d-2)/2}\right)$ (as follows from [14], Lemma 3.4.7 and (3.4.19)), hence $|\rho_{d,n}^{-1}\zeta_{d,n}^{-1}| = O(n^d)$. Thus, for the function $\Phi(t) = \sqrt{1-t^2}$, assumption (56) holds for even n with $\beta = d$. This gives the following result.

5.4 Theorem. *Let $0 < r < R$, let $K, L \in \mathcal{K}^d(r,R)$ be convex bodies with the same centre of symmetry. For $\gamma \in (0, 1/d(d+1))$, there exists a constant c depending only on d, γ, r, R such that*

$$\delta(K,L) \le c\|V^{(d-1)}(K,\cdot) - V^{(d-1)}(L,\cdot)\|^{\gamma}.$$

The results involving the cosine or sine transform are necessarily restricted to centrally symmetric convex bodies, since a transform $T_\Phi\mu$ with an even function Φ does not contain information on the odd part of the signed measure μ. We turn now to stability versions of some uniqueness theorems for not necessarily symmetric convex bodies.

Anikonov and Stepanov [1] have proposed to consider, for $K \in \mathcal{K}^d$ and $u \in S^{d-1}$, besides the projection volume $V_{d-1}(K,u) = V_{d-1}(K^u)$, also the area $S(K,u)$ of the illuminated portion of K in direction u, that is,

$$S(K,u) := S_{d-1}(K, \{v \in S^{d-1} : \langle u, v\rangle \ge 0\}).$$

They showed that the combined functional

$$F(K,u) := pV_{d-1}(K,u) + qS(K,u), \qquad u \in S^{d-1},$$

with constants p, q ($p, q, 2p\kappa_{d-1} + q\omega_d \neq 0$) determines the convex body K uniquely up to a translation. They also proved a corresponding stability result in $\mathbb{R}^3$. This, however, is rather weak, since it assumes that the difference $F(K,\cdot)-F(L,\cdot)$ is small in a norm that involves derivatives up to order six. A stronger result can be obtained with the aid of Theorem 5.2. In fact, we have $F(K,\cdot)=T_\Phi S_{d-1}(K,\cdot)$ with $\Phi=p\Phi_1+q\Phi_2$, where $\Phi_1(t)=\frac{1}{2}|t|$ and $\Phi_2=\mathbf{1}_{[0,1]}$. Now, $a_{d,n}(\Phi_1)=0$ for odd n, and $|a_{d,n}(\Phi_1)^{-1}|=O(n^{(d+2)/2})$ for even n. On the other hand, $a_{d,n}(\Phi_2)=0$ for even $n>0$, and $|a_{d,n}(\Phi_2)^{-1}|=O(n^{d/2})$ for odd n (see [14], Lemma 3.4.6 and (3.4.20)). It follows that Φ satisfies (56) with $\beta=(d+2)/2$ (note that the assumption $2p\kappa_{d-1}+q\omega_d\neq 0$ ensures that $a_{d,0}(\Phi)\neq 0$). Hence, for any two convex bodies $K,L\in\mathcal{K}^d(r,R)$, we have

$$\delta(K,L+x)\le c\|F(K,\cdot)-F(L,\cdot)\|^\gamma$$

with a suitable vector $x\in\mathbb{R}^d$, for $\gamma\in(0,2/d(d+4))$ and with c depending only on d,p,q,γ,r,R.

The last two transformations to be considered stem from the part of theoretical stereology or geometric tomography where one is interested in obtaining information on convex bodies from lower dimensional sections. The *second mean section body* $M_2(K)$ of a convex body $K\in\mathcal{K}^d$ was introduced by Goodey and Weil [12]. It is defined by

$$h(M_2(K),\cdot)=\int_{\mathcal{E}_2^d}h(K\cap E,\cdot)\,\mu_2(dE).$$

Here $\mathcal{E}_2^d$ is the affine Grassmannian of two-dimensional planes in $\mathbb{R}^d$ and μ_2 is its motion invariant measure, normalized so that $\mu_2(\{E\in\mathcal{E}_2^d: E\cap B^d\neq\emptyset\})=\kappa_{d-2}$. Thus, $M_2(K)$ comprises information about the two-dimensional sections of K, in integrated form. Goodey and Weil showed that two d-dimensional convex bodies K and L with $M_2(K)=M_2(L)$ differ only by a translation, and they mentioned briefly (on p. 429) that a corresponding stability version could be obtained. We will make this more explicit.

For unit vectors u,v, let $\alpha(u,v)\in[0,\pi]$ denote the angle between u and v. Goodey and Weil (*loc. cit.*, Corollary 2) proved that

$$h(M_2(K)-t,u)=\frac{\kappa_2\kappa_{d-2}}{\binom{d}{2}\kappa_d}\int_{S^{d-1}}\alpha(u,v)\sin\alpha(u,v)\,S_{d-1}(-K,dv) \tag{65}$$

for $u\in S^{d-1}$, where t is a suitable translation vector. Their proof (cf. Theorem 2) provides no information about this translation, but a related remark of Goodey [11], p. 165, gives a hint. Denoting by $z_{r+1}(K)$ the intrinsic $(r+1)$st moment vector of K (see [23], p. 304), we have

$$\begin{aligned}\frac{1}{\kappa_d}\int_{S^{d-1}}h(M_2(K)-t,u)\,u\,\sigma(du) &= z_1(M_2(K)-t)=z_1(M_2(K))-t\\ &= \frac{\kappa_2\kappa_{d-2}}{\binom{d}{2}\kappa_d}z_{d-1}(K)-t.\end{aligned}$$

But this must be the zero vector, as we see by using (65), Fubini's theorem, the fact that a vector integral of the form

$$\int_{S^{d-1}} f(\langle u,v\rangle)\, u\, \sigma(du)$$

is invariant under rotations fixing v and hence is a multiple of v, and that $\int_{S^{d-1}} v\, S_{d-1}(-K,dv) = 0$. We deduce that the body

$$M_2'(K) := \frac{\binom{d}{2}\kappa_d}{\kappa_2\kappa_{d-2}} M_2(K) - z_{d-1}(K),$$

which we call the *normalized second mean section body* of K, satisfies

$$h(M_2'(K),u) = \int_{S^{d-1}} \alpha(u,v)\sin\alpha(u,v)\, S_{d-1}(-K,dv) \qquad \text{for } u \in S^{d-1}.$$

Thus, $h(M_2'(K),\cdot) = T_\Phi S_{d-1}(-K,\cdot)$ with $\Phi(t) = (\arccos t)\sqrt{1-t^2}$ for $t \in [-1,1]$. From a computation in [12] (formula (4.10), together with the relation

$$c_n(t) = \binom{n+d-3}{d-3} P_n^d(t)$$

between Gegenbauer and Legendre polynomials, see [14], p. 97) it follows that

$$a_{d,n}(\Phi) = \frac{c(d)}{(n-1)(n+d-1)} \left(\frac{n!\Gamma\left(\frac{1}{2}(n+d)\right)}{(n+d-2)!\Gamma\left(\frac{1}{2}(n+2)\right)} \right)^2,$$

with a constant $c(d)$ depending only on d. From this we deduce that (56) holds with $\beta = d$. For the resulting stability estimate, we may now use the Hausdorff distance also on the right-hand side: For $\gamma \in (0, d(d+1))$, there exists a constant c depending only on d, r, R, γ such that, for $K, L \in \mathcal{K}^d(r,R)$,

$$\delta(K, L+x) \le c\delta(M_2'(K), M_2'(L))^\gamma$$

for a suitable vector $x \in \mathbb{R}^d$.

The origin of our last example is an investigation, [24], on the oriented mean normal measure of a stationary stochastic process of convex particles and its determination from planar sections. There one has reason to consider the function defined by

$$V_+^{(d-1)}(K,u) := \int_{-\infty}^{\infty} \mathcal{H}^{d-2}(\partial^u K \cap (u^\perp + tu))\, dt$$

for $K \in \mathcal{K}^d$ and $u \in S^{d-1}$; here $\mathcal{H}^{d-2}$ denotes the $(d-2)$-dimensional Hausdorff measure and $\partial^u K$ is the 'upper boundary' of K in direction u, that is, the set of all boundary points of K at which there exists an outer unit normal vector v with $\langle u, v\rangle \geq 0$. By formula (17) of [24],

$$V_+^{(d-1)}(K,\cdot) = T_\Phi S_{d-1}(K,\cdot)$$

with $\Phi(t) = \sqrt{1-t^2}\mathbf{1}_{[0,1]}$ (so that T_Φ could be called the *hemispherical sine transform*). For even $n \in \mathbb{N}$, the multipliers of T_Φ are essentially those of the sine transform, namely $a_{d,n}(\Phi) = \frac{1}{2}a_{d,n}(\Psi)$ for $\Psi(t) = \sqrt{1-t^2}$. Hence, as shown before Theorem 5.4, $|a_{d,n}(\Phi)^{-1}| = O(n^d)$ for even n. For odd n, the multipliers have not been determined explicitly, but it has been shown that $a_{d,1}(\Phi) \neq 0$ and, for odd $n \geq 3$,

$$a_{d,n}(\Phi) = \omega_{d-1}\frac{1\cdot 3\cdots(n-2)}{(d-1)(d+1)\cdots(d+n-4)}\frac{f(n)}{(n+d-1)(n+d-3)}$$

with

$$n+d-3 < (-1)^{(n-1)/2}f(n) \leq n+d-1$$

([24], p. 36 together with (20) and (19)). From this, one obtains $|a_{d,n}(\Phi)^{-1}| = O(n^{d/2})$ for odd n. We deduce that (56) holds with $\beta = d$. Hence, for $0 < r < R$ and $\gamma \in (0, 1/d(d+1))$ there exists a constant c depending only on d, γ, r, R such that, for $K \in \mathcal{K}^d(r,R)$,

$$\delta(K, L+x) \leq c\|V_+^{(d-1)}(K,\cdot) - V_+^{(d-1)}(L,\cdot)\|^\gamma$$

with a suitable vector $x \in \mathbb{R}^d$.

6 Appendix

In the proofs of Theorems 4.1 and 5.3 we have referred to the proof of Theorem 7.2.6 in [23], which in turn relies on inequality (6.4.9) of [23] (p. 335). The proof of (6.4.9) given there is not complete, as A. Giannopoulos has kindly pointed out. We take this opportunity to correct the error (using the same notations). The proof of (6.4.9) as given is correct if $U_{12}U_{00} - U_{01}U_{02} < 0$; observe that

$$U_{01}^2 - U_{00}U_{11} \geq 0, \quad U_{02}^2 - U_{00}U_{22} \geq 0. \tag{66}$$

Now, for $\lambda_1, \lambda_2 \geq 0$ also

$$\begin{aligned} 0 \leq\ & V(\lambda_1 K_1 + \lambda_2 K_2, K_0, \mathcal{C})^2 \\ & -V(\lambda_1 K_1 + \lambda_2 K_2, \lambda_1 K_1 + \lambda_2 K_2, \mathcal{C})V(K_0, K_0, \mathcal{C}) \\ =\ & \lambda_1^2(U_{01}^2 - U_{00}U_{11}) + \lambda_2^2(U_{02}^2 - U_{00}U_{22}) - 2\lambda_1\lambda_2(U_{12}U_{00} - U_{01}U_{02}). \end{aligned}$$

If $U_{12}U_{00} - U_{01}U_{02} > 0$, we can deduce (6.4.9) from this inequality. If $U_{12}U_{00} - U_{01}U_{02} = 0$, (6.4.9) holds by (66).

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