Distance measurements on processes of flats*

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Dedicated to Joseph Mecke on the occasion of his 65th birthday

Abstract

Distance measurements are useful tools in Stochastic Geometry. For Boolean models Z in $\mathbb{R}^d$, the classical contact distribution functions allow the estimation of important geometric parameters of Z. In two previous papers, several types of generalized contact distributions have been investigated and applied to stationary and nonstationary Boolean models. Here, we consider random sets Z which are generated as the union sets of Poisson processes X of k-flats, $k \in \{0, \dots, d-1\}$, and study distances from a fixed point or a fixed convex body to Z. In addition, we also consider the distances from a given flat or a flag consisting of flats to the individual members of X and investigate the associated process of nearest points in the flats of X. In particular, we discuss to which extent the directional distribution of X is determined by this point process. Some of our results are presented for more general stationary processes of flats.

1 Introduction

Processes of flats (in particular Poisson processes) have a long history in Stochastic Geometry, beginning with the fundamental thesis of Miles [18] from 1961. Matheron [16], in his 1975 book, was the first to characterize the random closed sets Z in $\mathbb{R}^d, d \geq 2$, which arise as union sets of (stationary) Poisson flat processes X. He also started a program which aims at determining characteristic properties of X from observations of transformed images of Z. More precisely, for a stationary process X of k-flats ($k \in \{1, \dots, d-1\}$) and a $(d-k)$-flat E in general position (w.r.t. X), he considered the intensity $\gamma(E)$ of the intersection process $X \cap E$ (which is a process of ordinary points) as a function of E and showed that, for line and hyperplane processes (i.e., $k \in \{1, d-1\}$), this function $\gamma(\cdot)$ determines the intensity and the directional distribution of X uniquely. For a Poisson process X this means that X is determined in distribution. The corresponding uniqueness

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problem for general $k \in \{2, \dots, k-2\}$, formulated also by Matheron, was answered in the negative by Goodey and Howard [5], [6] (see also [7]).

In the early eighties, Joseph Mecke and his students pushed the theory of stationary (Poisson) k-flat processes X further by considering, for $k \geq \frac{d}{2}$, intersection densities of X, that is, intensities of flat processes generated as intersections of two or more flats from X. The main goals of these investigations were inequalities for intersection densities and characterizations of the extremal processes. The surprising fact was that, for $k < d-1$, the isotropic Poisson processes are no longer extremal (which they are for $k = d-1$, i.e., for hyperplane processes). The description of all extremal processes turned out to be a difficult problem, but in some important cases they could be characterized completely. The results of this period were presented by Joseph Mecke in a seminar in Neresheim 1989, jointly organized by him with Rolf Schneider, Dietrich Stoyan and Wolfgang Weil (Daniel Hug was one of the participants), and documented in [17]. This was an exciting time, both mathematically and politically. Joseph Mecke and Dietrich Stoyan were for the first time allowed to give a course in West Germany and, during the week of the seminar, the exciting political development in East Germany and the news from the German embassy in Prague raised the hope for a process of democratization in East Germany or even for the end of the separation of Germany. At that time, Joseph Mecke and Dietrich Stoyan seemed to have some doubts about real changes. Later, both took active part in the process of reorganizing their universities.

Mecke's investigation of intersection densities were later complemented by Schneider [20] who considered, for processes of k-flats with $1 \leq k < \frac{d}{2}$, the intensity of pairs of points which realize the distance between respective pairs of flats of the process. This seems to be the first attempt to use distance methods in the analysis of flat processes. As a more recent work, influenced also by Joseph Mecke, we mention Spodarev [24], [26], who used Schneider's concept of proximity for results on the distances of the flats of X to a given flat M.

Whereas all the results mentioned so far concern stationary (Poissonian or non-Poissonian) processes of flats, recent progress has been made with nonstationary random structures. As a starting point, the work of Fallert [2], [3] should be mentioned, who studied uniqueness problems for nonstationary flat processes with intensity measures satisfying a certain smoothness condition. Whereas later considerations mostly concern nonstationary Boolean models (see, e.g., the references listed in [21]), the investigation of nonstationary processes of flats is continued in the recent work of Schneider [21] in this volume. He generalized Fallert's smoothness assumptions to what he called *translation regular* processes of flats. These are point processes on $\mathcal{E}_k^d$, the space of all k-flats (k-dimensional affine subspaces) in $\mathbb{R}^d$, for which the intensity measure Θ has a density with respect to some translation invariant measure on $\mathcal{E}_k^d$.

In the following, we investigate the extent to which distance measurements can be used to determine characteristic quantities of a stationary or translation regular process X of k-flats, $k \in \{0, \dots, d-1\}$. As far as distances to the union set Z of X are concerned, we will concentrate on Poisson processes X, although there are similar results for the intensity measures of quite general flat processes, involving Palm distributions. Our main tool in this part is a suitable generalization of the contact distribution function. Generalized contact distributions were studied in [11] and [12] for Boolean models and more general germ-grain models. A survey with additional results, which mainly concern cluster processes and cluster models, is given in [13]. Here we present corresponding results for Poisson flat processes and for their union sets. Whereas generalized contact distributions are based on distances from a given test body K (which may shrink to a point), we also discuss the situation where the distances from a given test flat E or, more generally, from a partial flag (of flats) to the individual flats of X are determined. The latter extension leads to new uniqueness results. In particular, we shall describe the intensity measure of the corresponding process X^E of projection points. In this part of our work, we do not need a Poisson assumption.

Since there are some similarities in the derivation of the theorems and since we do not aim to copy the material from [11] and [12], we refer to these papers at some instances in the following.

After collecting the basic geometric notations in the next section, we present the general formula for generalized contact distributions in Section 3. We then consider, in Section 4, as a special case, the joint distribution of the distance and the direction of a given point z to the flat process X (equivalently, to the union set Z), and discuss which distributional properties of the Poisson process X are determined by this information. In Section 5, we replace the point z by a fixed flat E and study the point process X^E consisting of the points in F, $F \in X$, which are nearest to E, without the Poisson assumption. Section 6 then continues the investigations for Poisson processes from Section 4, with flats E as test sets. Finally, in Section 7, the single flat E is replaced by a flag $\mathcal{E}$ consisting of flats and a corresponding point process $X^{\mathcal{E}}$ of nearest points is considered.

Although our general results hold for processes of k-flats, with $k \in \{0, \dots, d-1\}$, we have uniqueness theorems mainly for line processes (and for processes of points or hyperplane processes, but there the results are trivial) as long as distances from a single flat are measured. This is due to the fact that, as in Matheron's original problem and its answer by Goodey *et al.*, Radon-type integral transforms on k-flats come in, which are only injective in these particular cases. The situation changes and we obtain additional uniqueness results when a single flat is replaced by a flag consisting of flats. This progress is again based on uniqueness results for integral transforms investigated in [6], [7].

2 Basic notations

As was already mentioned, we work in Euclidean d-space $\mathbb{R}^d$, $d \geq 2$, and denote by B^d the unit ball in $\mathbb{R}^d$ and by S^{d-1} the unit sphere. We write $\mathcal{H}^k$ for the k-dimensional Hausdorff measure in $\mathbb{R}^d$. Let $\kappa_d := \mathcal{H}^d(B^d)$ and $\omega_d := d\kappa_d = \mathcal{H}^{d-1}(S^{d-1})$. Let $\langle \cdot, \cdot \rangle$ be the usual scalar product on $\mathbb{R}^d$ and $\| \cdot \|$ the Euclidean norm. For a set $A \subset \mathbb{R}^d$, we denote by ∂A the boundary and by $\dim A$ the dimension of A (defined as the dimension of the affine hull of A).

Let $\mathcal{K}^d$ denote the set of all *convex bodies* in $\mathbb{R}^d$ (nonempty compact convex sets) and let $\mathcal{S}^d$ be the *extended convex ring* (locally finite unions of convex bodies). For notions and results from convex geometry which we use without further explanation, we refer to [19]. Besides of the space $\mathcal{E}^d_k$ of k-flats, $k \in \{0, \dots, d\}$, which was mentioned already, we make use of the Grassmannian $\mathcal{L}^d_k$ (the k-flats which contain the origin 0). For $F \in \mathcal{E}^d_k$, we denote by $F^\circ \in \mathcal{L}^d_k$ the linear subspace parallel to F and by $F^\perp \in \mathcal{L}^d_{d-k}$ the totally orthogonal linear subspace. Moreover, for $F \in \mathcal{E}^d_k$, we use π_F to denote the orthogonal projection onto F.

The spaces $\mathcal{E}_k^d$ and $\mathcal{L}_k^d$ carry (up to normalization) unique Haar measures; we will explicitly use the invariant probability measure ν_k on $\mathcal{L}_k^d$. For $i, k \in \{0, \ldots, d\}$ and $F \in \mathcal{L}_i^d$, $\mathcal{L}_k^{(F)}$ consists of all $E \in \mathcal{L}_k^d$ which contain (or are contained in) F; $\nu_k^{(F)}$ is the corresponding invariant probability measure on $\mathcal{L}_k^{(F)}$. The Radon transform R_{ik} maps continuous functions f on $\mathcal{L}_i^d$ to continuous functions on $\mathcal{L}_k^d$ and is defined as

$$R_{ik} f(F) := \int_{\mathcal{L}_i^{(F)}} f(E) \nu_i^{(F)}(dE), \qquad F \in \mathcal{L}_k^d. \tag{2.1}$$

It is extended by duality to a transformation R_{ki} from the space of measures on $\mathcal{L}_k^d$ to the space of measures on $\mathcal{L}_i^d$,

$$\int_{\mathcal{L}_i^d} f(E) R_{ki} \mu(dE) := \int_{\mathcal{L}_k^d} R_{ik} f(F) \mu(dF),$$

where f is a continuous function on $\mathcal{L}_i^d$. For $1 \leq i < k \leq d-1$, R_{ik} is injective, if and only if $i + k \leq d$, whereas for $d - 1 \geq i > k \geq 1$ injectivity holds, if and only if $i + k \geq d$. We shall mostly consider the case where $k = 1$ (or $k = d-1$), in which injectivity holds if and only if $i = d-1$ (resp. $i = 1$). The (injective) transforms $R_{d-1\,1}$ and $R_{1\,d-1}$ can be identified with the spherical Radon transform R on even functions (and even measures) on S^{d-1}, if lines and hyperplanes are identified with (pairs of) antipodal points on S^{d-1}. Information on Radon transforms (in this particular geometric setting) can be found in [8], [9]. We have slightly extended the usual definition of the Radon transform to cover the cases $i \in \{0, d\}$ and $k \in \{0, d\}$, since we want to include ordinary point processes in our later considerations. Because of $\mathcal{L}_0^{(F)} = \{0\}$ and $\mathcal{L}_d^{(F)} = \{\mathbb{R}^d\}$, the integrals in (2.1) then are trivial and the Radon transforms $R_{0k} f$ and $R_{dk} f$, respectively $R_{i0} f$ and $R_{id} f$, are constant functions; moreover $R_{kk} f = f$, $k \in \{0, \ldots, d\}$.

We further need the set

$$\mathcal{F}_k^d := \{F \in \mathcal{S}^d \,:\, F = \bigcup_{i=1}^{\infty} E_i, E_i \in \mathcal{E}_k^d\},$$

for $k = 0, \ldots, d-1$. As in [12], we fix a gauge body $B \in \mathcal{K}^d$ which contains the origin 0 and define, for $L \in \mathcal{S}^d$ and $F \in \mathcal{F}_k^d$, the distance from L to F with respect to B by

$$d_B(L, F) := \inf\{t \geq 0 : (L + tB) \cap F \neq \emptyset\}.$$

As usual, we write $d_B(x, y)$ instead of $d_B(\{x\}, \{y\})$ and $d_B(x, F)$ instead of $d_B(\{x\}, F)$. We also omit the subscript B here and in the subsequent notions in case $B = B^d$. If $0 < d_B(L, F) < \infty$, we put

$$\Pi_B(L, F) := \{(x, y) \in \partial L \times F : d_B(x, y) = d_B(L, F)\},$$

and define the skeleton set

$$\mathcal{S}_B^d(F) := \{L \in \mathcal{S}^d : 0 < d_B(L, F) < \infty, \operatorname{card} \Pi_B(L, F) \geq 2\}.$$

If $0 < d_B(L,F) < \infty$ and $L \notin \mathcal{S}^d_B(F)$, then there are uniquely determined points $x \in \partial L$ and $y \in F$ such that $d_B(x,y) = d_B(L,F)$ and we put $p_B(L,F) := x$ and

$$u_B(L,F) := (y-x)/d_B(L,F) \in \partial B.$$

By definition, $p_B(L,F), d_B(L,F)$ and $u_B(L,F)$ then determine y through

$$y = p_B(L,F) + d_B(L,F)u_B(L,F).$$

In case $d_B(L,F) \in \{0,\infty\}$ or $L \in \mathcal{S}^d_B(F)$, we give $(p_B(L,F), u_B(L,F))$ some fixed value in $\mathbb{R}^d \times \partial B$, unless stated otherwise. As above, we omit the curly brackets in $u_B(\{x\},F)$, and in similar expressions, and write $u_B(x,F)$.

The basic geometric result underlying the later investigations on contact distributions of flat processes is a translative integral formula for a moving flat F with respect to a given convex body K. It can be seen as a counterpart to Theorem 2.1 in [12] and as a generalization of distance formulae, as they were obtained in [28] and [29] (see also [23, section 4]). For its formulation, we make use of mixed relative support measures $\Theta_{i,k;j+1}(K,F;B;\cdot)$ and define $\Theta^{(F)}_{i;j+1}(K;B;\cdot)$ by

$$\Theta^{(F)}_{i;j+1}(K;B;A\times A') := \binom{d-1}{i,k,j}\Theta_{i,k;j+1}(K,F;B;A\times A_F\times A'), \tag{2.2}$$

where $A, A' \subset \mathbb{R}^d$ are arbitrary Borel sets and $A_F \subset F$ is a Borel set with $\mathcal{H}^k(A_F) = 1$. Here we assume that K, B and F are in general relative position and that $i+k+j = d-1$. In this particular situation, general relative position means that there are no parallel line segments in ∂K and ∂B or in $\partial(K+B)$ and F. Also, in (2.2), we used the fact that mixed relative support measures are locally defined, and therefore allow an extension to unbounded closed convex sets (as Radon measures). For details on these measures and related notions, we refer to [15], [11] and [12].

Theorem 2.1. *Let K, B be convex bodies with $0 \in B$, and let $F \in \mathcal{E}^d_k$ be a k-flat such that K, B and F are in general relative position, $k \in \{0,\dots,d-1\}$. If $g : [0,\infty]\times\partial B\times\partial K \to [0,\infty)$ is a measurable function, then*

$$\int_{F^\perp} \mathbf{1}\{0 < d_B(K,F+z) < \infty\} g(d_B(K,F+z), u_B(K,F+z), p_B(K,F+z))\mathcal{H}^{d-k}(dz)$$

$$= \sum_{j=0}^{d-k-1} \int_0^\infty \int t^{d-k-1-j} g(t,b,x)\Theta^{(F)}_{j;d-k-j}(K;B;d(x,b))dt.$$

Proof. During this proof, we can assume that $F \in \mathcal{L}^d_k$, and we use the abbreviations $B_{F^\perp} := \pi_{F^\perp}(B)$ and $K_{F^\perp} := \pi_{F^\perp}(K)$. For each point x in the relative boundary $\partial' K_{F^\perp}$ of $K_{F^\perp}$ there is a unique point $x^{(K)} \in \partial K$ with $\pi_{F^\perp}(x^{(K)}) = x$ and, similarly, for each point b in the relative boundary $\partial' B_{F^\perp}$ of $B_{F^\perp}$ there is a unique point $b^{(B)} \in \partial B$ with $\pi_{F^\perp}(b^{(B)}) = b$. Consequently, we obtain

$$\int_{F^\perp} \mathbf{1}\{0 < d_B(K,F+z) < \infty\} g(d_B(K,F+z), u_B(K,F+z), p_B(K,F+z))\mathcal{H}^{d-k}(dz)$$

$$= \int_{F^\perp} \mathbf{1}\{0 < d_{B_{F^\perp}}(K_{F^\perp},z) < \infty\}$$

$$\times\, g(d_{B_{F^\perp}}(K_{F^\perp},z), (u_{B_{F^\perp}}(K_{F^\perp},z))^{(B)}, (p_{B_{F^\perp}}(K_{F^\perp},z))^{(K)})\,\mathcal{H}^{d-k}(dz)\,.$$

We apply the defining equation of the relative support measures in $F^\perp$ (see [15], [11]) to the function $g_{F^\perp} : [0,\infty]\times\partial' B_{F^\perp}\times\partial' K_{F^\perp} \to [0,\infty)$ given by $g_{F^\perp}(t,b,x) := g(t, b^{(B)}, x^{(K)})$. Then

$$\begin{aligned}\int_{F^\perp} &\mathbf{1}\{0 < d_{B_{F^\perp}}(K_{F^\perp}, z) < \infty\} \\ &\qquad \times g(d_{B_{F^\perp}}(K_{F^\perp}, z), (u_{B_{F^\perp}}(K_{F^\perp}, z))^{(B)}, (p_{B_{F^\perp}}(K_{F^\perp}, z))^{(K)})\, \mathcal{H}^{d-k}(dz) \\ &= \sum_{j=0}^{d-k-1} \binom{d-k-1}{j} \int_0^\infty \int t^{d-k-1-j} g_{F^\perp}(t,b,x) \Theta'_{j;d-k-j}(K_{F^\perp}; B_{F^\perp}; d(x,b))dt\,.\end{aligned}$$

Here, the prime indicates again that the quantity is taken with respect to the underlying subspace $F^\perp$. Since

$$\binom{d-k-1}{j}\Theta'_{j;d-k-j}(K_{F^\perp}; B_{F^\perp}; \cdot)$$

is the image of $\Theta^{(F)}_{j;d-k-j}(K; B; \cdot)$ under $\pi_{F^\perp}$, the assertion follows. □

We later consider, in Corollary 3.5, the special case $K = \{0\}$. Then, $\Theta^{(F)}_{j;d-k-j}(\{0\}; B; \cdot) = 0$, for $j = 1, \ldots, d-k$, while

$$\Theta^{(F)}_{d-k}(B; \cdot) := \Theta^{(F)}_{0;d-k}(\{0\}; B; \cdot)$$

equals, up to the factor $\binom{d-1}{k}$, the relative support measure $\Theta_{k;d-k}(F; B; A_F \times \cdot)$ (compare (2.2)).

3 Contact distributions

In the sequel, we consider the space $\mathcal{F}^d$ of all closed subsets of $\mathbb{R}^d$ and equip $\mathcal{F}^d$ (and its subsets $\mathcal{S}^d$, $\mathcal{K}^d$, $\mathcal{F}^d_k$ and $\mathcal{E}^d_k$) with the σ-field generated by the Fell-Matheron "hit-or-miss" topology (see [16] or [22]). The subject of this paper are processes of k-flats, i.e. point processes on $\mathcal{E}^d_k$. Any such point process X can be considered as a random variable defined on an abstract probability space with probability measure $\mathbb{P}$ taking values in the space $\mathbf{N}$ of locally finite counting measures on $\mathcal{F}^d$, and such that the *intensity measure* $\Theta := \mathbb{E}X$ of X is concentrated on $\mathcal{E}^d_k$. We equip $\mathbf{N}$ with the Borel σ-field generated by the vague topology. Since we will later pose conditions on X which imply that X has to be simple, we concentrate on simple point processes right from the beginning. Therefore we may, alternatively, consider X as a locally finite subset of $\mathcal{E}^d_k$; this allows us to write $E \in X$ instead of $X(\{E\}) > 0$. For more details on processes of flats, we refer to [16] or [22].

Throughout the following, we assume that Θ is locally finite in the sense that

$$\Theta(\{E \in \mathcal{E}^d_k : E \cap C \neq \emptyset\}) < \infty \tag{3.1}$$

for all compact sets $C \subset \mathbb{R}^d$. Condition (3.1) implies that

$$X(\{E \in \mathcal{E}^d_k : E \cap C \neq \emptyset\}) < \infty \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

If X is a *Poisson process*, then the random variable in (3.2) has a Poisson distribution. Hence, in this case (3.2) is equivalent to (3.1).

In this work, we study processes of k-flats X with *translation regular* intensity measures (cf. [21]). To be more precise, we assume that the intensity measure Θ of X can be represented in the form

$$\Theta = \int_{\mathcal{L}_k^d} \int_{F^\perp} \mathbf{1}\{F + x \in \cdot\} \eta(F + x) \mathcal{H}^{d-k}(dx) \mathbb{Q}(dF), \tag{3.3}$$

where $\mathbb{Q}$ is a probability measure on $\mathcal{L}_k^d$ and η is a real-valued, nonnegative measurable function on $\mathcal{E}_k^d$. If X is stationary, then (3.3) is fulfilled with a constant function $\eta \equiv \gamma$. In that case, γ and $\mathbb{Q}$ are uniquely determined, γ is called the *intensity* and $\mathbb{Q}$ the *directional distribution* of X. If X is stationary and isotropic, then the directional distribution equals the invariant probability measure ν_k. In general, even for a Poisson process X with translation regular intensity measure, the function η and the probability measure $\mathbb{Q}$ in (3.3) are not uniquely determined. For example, if $\mathbb{Q}$ is absolutely continuous with respect to ν_k, equation (3.3) can be re-written as

$$\Theta = \int_{\mathcal{L}_k^d} \int_{F^\perp} \mathbf{1}\{F + x \in \cdot\} \tilde{\eta}(F + x) \mathcal{H}^{d-k}(dx) \nu_k(dF), \tag{3.4}$$

with another real-valued, nonnegative measurable function $\tilde{\eta}$ on $\mathcal{E}_k^d$.

For a Poisson process X, (3.3) with a constant function $\eta \equiv \gamma$ is equivalent to stationarity of X and (3.4) with a constant function $\tilde{\eta} \equiv \gamma$ is equivalent to stationarity and isotropy of X. We therefore call a Poisson process X of k-flats *translation regular*, if its intensity measure Θ is translation regular.

Following Schneider [21], we introduce the *direction measure* $\varphi(z, \cdot)$ of X at $z \in \mathbb{R}^d$ by

$$\varphi(z, \mathcal{A}) := \int_{\mathcal{A}} \eta(F + z) \mathbb{Q}(dF), \tag{3.5}$$

for Borel sets $\mathcal{A} \subset \mathcal{L}_k^d$, and the *intensity function* γ of X by

$$\gamma(z) := \varphi(z, \mathcal{L}_k^d) = \int_{\mathcal{L}_k^d} \eta(F + z) \mathbb{Q}(dF).$$

For fixed $\mathcal{A}$, $\varphi(\cdot, \mathcal{A})$ is a measurable function, hence φ is a (measurable) kernel. Equations (3.3) and (3.5) yield that

$$\int \mathbf{1}\{E^\circ \in \mathcal{A}\} \lambda_E(A \cap E) \Theta(dE) = \int_A \varphi(z, \mathcal{A}) \mathcal{H}^d(dz), \tag{3.6}$$

where again E° denotes the linear subspace parallel to $E \in \mathcal{E}_k^d$, λ_E is the k-dimensional Lebesgue measure on E, and $A \subset \mathbb{R}^d$ is a Borel set (see [21], for details). Equation (3.6) shows that the direction measure $\varphi(z, \cdot)$ (and thus also $\gamma(z)$) is uniquely determined for $\mathcal{H}^d$-almost all z by Θ and does not depend on the particular representation (3.3). Conversely, it is apparent from (3.5) that Θ is determined by φ, if either $\mathbb{Q}$ or η is known. In particular, if Θ allows a representation (3.4), then (3.5) transforms into

$$\varphi(z, \mathcal{A}) := \int_{\mathcal{A}} \tilde{\eta}(F + z) \nu_k(dF),$$

and thus φ determines $\tilde{\eta}$, and consequently Θ.

Note that the case $k = 0$ is included in these considerations. In this case, X is an ordinary point process in $\mathbb{R}^d$ and the measure $\mathbb{Q} = \delta_{\{0\}}$ is trivial (and unique). Thus, $\varphi(\cdot, \{0\}) = \eta = \gamma$, the intensity function of X.

Given a k-flat process X fulfilling (3.1), we form the union set

$$Z := \bigcup_{E \in X} E. \tag{3.7}$$

This is a random closed set (with values in $\mathcal{F}_k^d$) which is our main object of interest during this and the following section. For $K \in \mathcal{K}^d$, we define the *contact distribution function* $H_B(K, \cdot)$ of Z by

$$H_B(K, t) := \mathbb{P}(d_B(K, Z) \leq t \mid Z \cap K = \emptyset), \quad t \geq 0,$$

provided that $\mathbb{P}(Z \cap K = \emptyset) > 0$. In case $K = \{0\}$ (and for stationary Z), this coincides with the classical notion (see e.g. [27]). It is easy to check that

$$\mathbb{P}(d_B(K, Z) > t) = \mathbb{P}(Z \cap K = \emptyset)(1 - H_B(K, t)).$$

As we already indicated, the corresponding spherical contact distribution function (in which case $B = B^d$) is denoted by H.

If X is a Poisson process, then we also call Z a *Poisson network* (stationary or nonstationary according to whether X is stationary or nonstationary). For a Poisson network Z,

$$\mathbb{P}(Z \cap M = \emptyset) = \exp\left[-\Theta(\{E : E \cap M \neq \emptyset\})\right]$$

which is positive if M is a compact set and (3.1) is satisfied.

In our main theorem in this section, we consider a measurable function $g : [0, \infty) \times \partial B \times \partial K \times \mathcal{F}_k^d \to [0, \infty)$ which is K-admissible. A definition of this notion is obtained by an obvious adaptation of the one given in [12], and we refer to [12] for details. We also concentrate on Poisson networks Z, although a more general version of Theorem 3.1, which avoids the Poisson assumption, can be stated in terms of Palm measures; see [12] for corresponding results on particle processes.

Theorem 3.1. *Let Z be a Poisson network* (3.7) *defined by a translation regular Poisson process X of k-flats satisfying* (3.2) *and* (3.3), *$k \in \{0, \ldots, d-1\}$. Let $K, B \in \mathcal{K}^d$ be such that $0 \in B$ and K, B and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$. Then*

$$\mathbb{P}(0 < d_B(K, Z) < \infty, K \in \mathcal{S}_B^d(Z)) = 0. \tag{3.8}$$

If $g : [0, \infty] \times \partial B \times \partial K \times \mathcal{F}_k^d \to [0, \infty)$ is a K-admissible measurable function, then

$$\mathbb{E}\left[\mathbf{1}\{d_B(K, Z) < \infty\} g(d_B(K, Z), u_B(K, Z), p_B(K, Z), Z) \mid Z \cap K = \emptyset\right]$$

$$= \sum_{j=0}^{d-k-1} \int_0^\infty t^{d-1-k-j}(1 - H_B(K, t)) \int \int g(t, b, x, F + x + tb)$$

$$\times\, \eta(F + x + tb) \Theta_{j;d-k-j}^{(F)}(K; B; d(x, b)) \mathbb{Q}(dF) dt.$$

The proof of Theorem 3.1 requires the following auxiliary result. We skip the corresponding proof, since it is totally analogous to the one of Lemma 3.2 and Lemma 3.3 in [12]. The Poisson assumption can be replaced by an assumption on the second factorial moment measure of X, as in [12]. For the statement of Lemma 3.9 and the proof of Theorem 3.1, it is convenient to represent the k-flat process X in the form

$$X = \sum_{n=1}^{\tau} \delta_{\xi_n}, \tag{3.9}$$

where ξ_n, $n \in \mathbb{N}$, is a random variable in $\mathcal{E}_k^d$ and τ is a random variable taking values in $\mathbb{N}_0 \cup \{\infty\}$.

Lemma 3.2. *Let X be a Poisson process of k-flats fulfilling* (3.2), (3.3), *and represented as in* (3.9). *Then X is simple and*

(a)
$$\mathbb{P}(0 < d_B(K, \xi_n) = d_B(K, \xi_m) < \infty) = 0, \quad m \neq n;$$

(b) *K, B and ξ_n are in general relative position $\mathbb{P}$-almost surely for all $n \in \mathbb{N}$, if $K \in \mathcal{K}^d$ is such that K, B and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$.*

Proof of Theorem 3.1. Let T denote the mapping on $\mathbf{N}$ which is implicitly defined by (3.7), i.e., which satisfies $T(X) = Z$. Lemma 3.2 implies that

$$\{0 < d_B(K, Z) < \infty\} = \bigcup_{n=1}^{\infty} (A_n \cap B_n \cap C_n) \quad \mathbb{P}\text{-a.s.},$$

where

$$\begin{aligned} A_n &:= \{0 < d_B(K, \xi_n) < \infty\}, \\ B_n &:= \{d_B(K, T(X - \delta_{\xi_n})) > d_B(K, \xi_n)\}, \\ C_n &:= \{K, \xi_n, B \text{ are in general relative position}\}. \end{aligned}$$

This implies (3.8). Moreover, for all $n \in \mathbb{N}$ we have

$$p_B(K, Z) = p_B(K, \xi_n) \quad \text{on } A_n \cap B_n \cap C_n,$$

and similar relationships hold for $d_B(K, Z)$ and $u_B(K, Z)$. Since g is K-admissible by assumption, we deduce that

$$\begin{aligned} &g(d_B(K, Z), u_B(K, Z), p_B(K, Z), Z) = \\ &\qquad\qquad g(d_B(K, \xi_n), u_B(K, \xi_n), p_B(K, \xi_n), \xi_n) \quad \text{on } A_n \cap B_n \cap C_n. \end{aligned}$$

Using these relationships together with well-known properties of the Poisson process, we obtain, as in the proof of [12, Theorem 3.1],

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}\{d_B(K, Z) < \infty\} g(d_B(K, Z), u_B(K, Z), p_B(K, Z), Z) \mid K \cap Z = \emptyset\right] \\ &= \int \int_{F^\perp} g(d_B(K, F + z), u_B(K, F + z), p_B(K, F + z), F + z) \\ &\quad \times (1 - H_B(K, d_B(K, F + z)))\mathbf{1}\{0 < d_B(K, F + z) < \infty\} \eta(F + z) \mathcal{H}^{d-k}(dz) \mathbb{Q}(dF). \end{aligned}$$

For each $F \in \mathcal{L}_k^d$ (such that K, B and F are in general relative position), we define a function $g_F : [0,\infty) \times \partial B \times \partial K \to [0,\infty)$ by

$$g_F(t,b,x) := (1 - H_B(K,t))g(t,b,x,F+x+tb)\eta(F+x+tb)$$

and apply Theorem 2.1. Then

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}\{d_B(K,Z) < \infty\}g(d_B(K,Z), u_B(K,Z), p_B(K,Z), Z) \mid K \cap Z = \emptyset\right] \\
&= \int \int_{F^\perp} g_F(d_B(K,F+z), u_B(K,F+z), p_B(K,F+z)) \\
&\qquad \times \mathbf{1}\{0 < d_B(K,F+z) < \infty\}\mathcal{H}^{d-k}(dz)\mathbb{Q}(dF) \\
&= \int \sum_{j=0}^{d-k-1} \int_0^\infty \int t^{d-1-k-j} g_F(t,b,x)\Theta_{j;d-k-j}^{(F)}(K;B;d(x,b))dt\mathbb{Q}(dF) \\
&= \sum_{j=0}^{d-k-1} \int_0^\infty t^{d-1-k-j}(1 - H_B(K,t)) \iint g(t,b,x,F+x+tb) \\
&\qquad \times \eta(F+x+tb)\Theta_{j;d-k-j}^{(F)}(K;B;d(x,b))\mathbb{Q}(dK)dt\,.
\end{aligned}$$

This finally proves the theorem. □

In [12], various results were deduced from the corresponding theorem on Boolean models (Theorem 3.2 in [12]), the counterpart of the preceding theorem. The following corollaries can be deduced from our present Theorem 3.2 in a totally analogous manner and are therefore given without proof.

Corollary 3.3. *Let the assumptions of Theorem 3.1 be satisfied. Then*

$$H_B(K,t) = 1 - \exp\left[-\int_0^t \lambda_B(K,s)\,ds\right], \quad t \geq 0,$$

where

$$\lambda_B(K,s) = \sum_{j=0}^{d-k-1} s^{d-1-k-j} \iint \eta(F+x+sb)\,\Theta_{j;d-k-j}^{(F)}(K;B;d(x,b))\mathbb{Q}(dF).$$

Corollary 3.4. *Let the assumptions of Theorem 3.1 be satisfied and assume moreover that the Poisson process X is stationary with intensity γ. Then*

$$\begin{aligned}
&H_B(K,t) \\
&= 1 - \exp\left[-\sum_{m=0}^{d-k-1}\binom{d-k}{m} t^{d-k-m}\,\gamma \int V(\pi_{F^\perp}(K)[m], \pi_{F^\perp}(B)[d-k-m])\mathbb{Q}(dF)\right]
\end{aligned}$$

and

$$\mathbb{E}\left[\mathbf{1}\{d_B(K,Z)<\infty\}g(d_B(K,Z),u_B(K,Z),p_B(K,Z),Z)\mid Z\cap K=\emptyset\right]$$
$$=\sum_{j=0}^{d-k-1}\gamma\int_0^\infty t^{d-1-k-j}(1-H_B(K,t))\iint g(t,b,x,F+x+tb)$$
$$\times\,\Theta^{(F)}_{j;d-k-j}(K;B;d(x,b))\mathbb{Q}(dF)dt.$$

Corollary 3.5. *Let Z be a Poisson network* (3.7) *defined by a stationary Poisson process X of k-flats with intensity γ satisfying* (3.2)*, $k\in\{0,\dots,d-1\}$. Let $B\in\mathcal{K}^d$ with $0\in B$ be such that B and F are in general relative position, for $\mathbb{Q}$-almost all $F\in\mathcal{L}^d_k$. Further, let $A\subset\mathbb{R}^d$ be a Borel set and $r\geq 0$. Then*

$$\mathbb{E}\left[d_B(0,Z)\leq r,u_B(0,Z)\in A\right]$$
$$=\gamma\left(\int_0^r t^{d-1-k}(1-H_B(t))dt\right)\left(\int\Theta^{(F)}_{d-k}(B;A)\mathbb{Q}(dF)\right),$$

where

$$H_B(t):=H_B(0,t)=1-\exp\left[-t^{d-k}\gamma\int V_{d-k}(\pi_{F^\perp}(B))\mathbb{Q}(dF)\right].$$

For the definition of mixed volumes $V(\pi_{F^\perp}(K)[m],\pi_{F^\perp}(B)[d-k-m])$ and intrinsic volumes $V_{d-k}(\pi_{F^\perp}(B))$, as they occur in Corollaries 3.4 and 3.5, we refer to [19]; the connection to (mixed) relative support measures is explained in [15].

Corollary 3.5 shows, in particular, that the random variables $d_B(0,Z)$ and $u_B(0,Z)$ are stochastically independent.

4 Determination of the direction measure

In this section, we concentrate on a translation regular Poisson process X of k-flats and on the associated Poisson network Z. We assume that condition (3.2) is satisfied and that a representation (3.3) exists with a continuous density η.

The question which we pursue here concerns the degree to which the direction measure

$$\varphi(z,\cdot)=\int_{\mathcal{L}^d_k}\mathbf{1}\{F\in\cdot\}\eta(F+z)\mathbb{Q}(dF)$$

of X is determined by the distribution of the random vector $(d(z,Z),u(z,Z))$. The answer essentially depends on the injectivity properties of the Radon transforms R_{ij}. We formulate the following theorem only for $k\geq 1$, since in case $k=0$ a stronger result holds (and is well-known). Namely, for an ordinary Poisson process X in $\mathbb{R}^d$ with continuous intensity function η and for $z\in\mathbb{R}^d$, the distribution of $d(z,Z)$ determines $\eta(z)$.

Theorem 4.1. *Let X be a translation regular Poisson process of k-flats satisfying* (3.2)*, $k\in\{1,\dots,d-1\}$. Assume that the intensity measure of X has a representation* (3.3) *with a continuous density η. Let Z denote the associated Poisson network, and let $z\in\mathbb{R}^d$. Then the distribution of the random vector $(d(z,Z),u(z,Z))$ uniquely determines the Radon transform $R_{k\,d-1}\varphi(z,\cdot)$ of $\varphi(z,\cdot)$. In particular, $\varphi(z,\cdot)$ is completely determined if $k\in\{1,d-1\}$.*

Proof. Let the intensity measure of X be represented as in (3.3) with a continuous density η. The distribution of $(d(z, Z), u(z, Z))$ uniquely determines the expectation

$$\mathbb{E}\left[g(d(z, Z), u(z, Z))\right], \tag{4.1}$$

for any measurable function $g : [0, \infty) \times S^{d-1} \to [0, \infty)$. For $j \in \{0, \ldots, d-1-k\}$ and $F \in \mathcal{L}_k^d$,

$$\Theta_{j;d-k-j}^{(F)}(\{z\}; B^d; \cdot) = \begin{cases} \delta_z \otimes (\mathcal{H}^{d-1-k} \llcorner (F^\perp \cap S^{d-1})), & j = 0, \\ 0, & j \geq 1, \end{cases}$$

(here $\mu \llcorner A$ denotes the restriction of the measure μ to the set A). In fact, for any polytope B which is in general relative position with respect to F and contains 0, it follows from (2.2) and from the explicit representation of the mixed support measures for polytopes provided in [15] that

$$\Theta_{j;d-k-j}^{(F)}(\{z\}; B; \cdot) = \begin{cases} \delta_z \otimes \binom{d-1}{k} \Theta_{k;d-k}(F; B; A_F \times \cdot), & j = 0, \\ 0, & j \geq 1. \end{cases}$$

The general formula then follows by approximation of B^d with such polytopes B.

Therefore, Theorem 3.1 yields that

$$\begin{aligned} &\mathbb{E}\left[g(d(z, Z), u(z, Z))\right] \\ &\quad = \int_0^\infty t^{d-1-k}(1 - H(z, t)) \int_{\mathcal{L}_k^d} \int_{F^\perp \cap S^{d-1}} g(t, u) \eta(F + z + tu) \mathcal{H}^{d-1-k}(du) \mathbb{Q}(dF) dt, \end{aligned}$$

and

$$1 - H(z, t) = \exp\left[- \int_0^t \int_{\mathcal{L}_k^d} \int_{F^\perp \cap S^{d-1}} s^{d-1-k} \eta(F + z + su) \mathcal{H}^{d-1-k}(du) \mathbb{Q}(dF) ds \right].$$

Alternatively, these equations can be obtained directly using the Poisson property of the marked point process $\{(d(z, F), u(z, F)) : F \in X\}$ (see (6.1) and (6.2)). It follows that $H(z, \cdot)$ is continuous and $H(z, 0) = 0$. Since η is assumed to be continuous, a differentiation argument, similar to the one used in the proof of Theorem 4.1 in [12], implies that the expectations (4.1) also determine

$$\int_{\mathcal{L}_k^d} \int_{F^\perp \cap S^{d-1}} g(u) \eta(F + z) \mathcal{H}^{d-1-k}(du) \mathbb{Q}(dF)$$

for any continuous measurable function $g : S^{d-1} \to [0, \infty)$. But then also

$$\int_{\mathcal{L}_k^d} \int_{\mathcal{L}_{d-1}^{(F)}} f(U) \nu_{d-1}^{(F)}(dU) \varphi(z, dF) = \int_{\mathcal{L}_{d-1}^d} f(U) \left[R_{k\, d-1} \varphi(z, \cdot)\right](dU)$$

is determined for any continuous function f on $\mathcal{L}_{d-1}^d$.

The final assertion of the theorem now follows from the injectivity properties of the Radon transform. □

The proof of the preceding theorem leads to a more specific result if we additionally assume stationarity. So let X be a stationary Poisson process of k-flats. Then we obtain for any measurable function $g : S^{d-1} \to [0, \infty)$ that

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}\{d(0,Z) \le r\} g(u(0,Z))\right] \\ &\quad = \omega_{d-k}^{-1}\left(1 - \exp[-\gamma\kappa_{d-k} r^{d-k}]\right) \int_{\mathcal{L}_k^d} \int_{F^\perp \cap S^{d-1}} g(u) \mathcal{H}^{d-1-k}(du) \mathbb{Q}(dF), \end{aligned} \tag{4.2}$$

for $r \ge 0$. In the limit, as $r \to \infty$, we deduce from (4.2) that

$$\begin{aligned} \omega_{d-k} \mathbb{E}\left[g(u(0,Z))\right] &= \int_{\mathcal{L}_{d-k}^d} R_{1\,d-k} g\,(W) \mathbb{Q}^\perp(dW) \\ &= \int_{S^{d-1}} g(u) R_{d-k\,1} \mathbb{Q}^\perp(du), \end{aligned}$$

where g is an even continuous function on S^{d-1}. Here, for $k \in \{1, d-1\}$, we have identified functions and measures on $\mathcal{L}_k^d$ with even functions (resp. measures) on S^{d-1}. Correspondingly, we will replace $R_{1\,d-1}$ and $R_{d-1\,1}$ by the spherical Radon transform R. Then, we obtain, for a stationary Poisson line process X,

$$\omega_{d-1} \mathbb{E}\left[g(u(0,Z))\right] = \int_{S^{d-1}} Rg(u) \mathbb{Q}(du).$$

For smooth functions g, one can now use inversion formulae for the spherical Radon transform to determine integrals of given smooth test functions with respect to $\mathbb{Q}$; cf. [25] and the references cited there. Alternatively, one can employ spherical harmonics in the following way. Let $\mathbb{S}_n^d$ denote the space of spherical harmonics of degree n in $\mathbb{R}^d$ and set, for an even integer $n \in \mathbb{N}$,

$$a_{dn} := (-1)^{\frac{n}{2}} \frac{1 \cdot 3 \cdots (n-1)}{(d-1)(d+1) \cdots (d+n-3)}.$$

Then, using Lemma 3.4.7 in [10], we get the following consequence.

Corollary 4.2. *Let X be a stationary Poisson process of lines in $\mathbb{R}^d$, $d \ge 3$, with directional distribution $\mathbb{Q}$ and associated Poisson network Z. Let $n \in \mathbb{N}$ be even and $H_n \in \mathbb{S}_n^d$. Then*

$$\int_{S^{d-1}} H_n(u) \mathbb{Q}(du) = \mathbb{E}\left[a_{dn}^{-1} H_n(u(0,Z))\right];$$

in particular, $\mathbb{Q}$ is determined by the distribution of $u(0,Z)$.

A corresponding result for hyperplane processes could also be formulated, but this is rather trivial since then $\mathbb{Q}$ equals the distribution of $u(0,Z)$.

Continuing the discussion of line processes, we now assume that $\mathbb{Q}$ is absolutely continuous with respect to spherical Lebesgue measure and has a density q of differentiability class $\lfloor \frac{d-1}{2} \rfloor$. Hence, for $d = 3$ the density q is assumed to be continuously differentiable.

Since $\mathbb{Q}$ is an even measure, q is an even function, and we obtain the expansion of q into spherical harmonics

$$q(u) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}[q] H_{nj}(u), \qquad u \in S^{d-1},$$

where $c_{nj}[q] \in \mathbb{R}$ and H_{nj}, $j = 1, \ldots, N(d,n)$, is an orthonormal basis of the linear space of spherical harmonics of degree n (see [10]). Due to the smoothness assumption on q, the convergence is uniform in $u \in S^{d-1}$; see [14, Theorem 1]. Hence, using Corollary 4.2, we obtain

$$q(u) = \lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{n=0 \\ n \text{ even}}}^{N} \sum_{j=1}^{N(d,n)} a_{dn}^{-1} H_{nj}(u) H_{nj}(u(0,Z)) \right]$$

uniformly for $u \in S^{d-1}$. This shows that

$$\hat{q}_N(u) := \sum_{\substack{n=0 \\ n \text{ even}}}^{N} \sum_{j=1}^{N(d,n)} a_{dn}^{-1} H_{nj}(u) H_{nj}(u(0,Z)), \qquad u \in S^{d-1},$$

is a (uniformly) asymptotically unbiased estimator for $q(u)$, as $N \to \infty$.

5 Distance measurements from flats

In this section, we consider a translation regular process X of k-flats, $k \in \{0, \ldots, d-1\}$, for which condition (3.1) is satisfied and assume that a representation (3.3) of Θ is given. Furthermore, we replace the test set K from the last two sections by a j-flat $E \in \mathcal{E}_j^d$ for some $j \in \{0, \ldots, d-k\}$. In order to include the case $j = d-k$, we have to extend the notion of projection points by defining $p(\tilde{F}, E) = p(E, \tilde{F}) = p$, if $E \in \mathcal{E}_{d-k}^d$ and $\tilde{F} \in \mathcal{E}_k^d$ satisfy $E \cap \tilde{F} = \{p\}$, for some $p \in \mathbb{R}^d$.

We assume that E and $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$ are in general relative position (i.e. $E^\circ \cap F = \{0\}$). Equation (3.3) then implies that $\mathbb{P}$-almost surely the flats E and $\tilde{F}$ are in general relative position for each $\tilde{F} \in X$. In contrast to the previous situations, we are now studying the process of all distances $d(E, \tilde{F})$, $\tilde{F} \in X$, and the associated random vectors $u(E, \tilde{F})$ and $p(E, \tilde{F})$ (respectively $p(\tilde{F}, E) = p(E, \tilde{F}) + d(E, \tilde{F}) u(E, \tilde{F})$). In fact, the distance $d(E, Z)$ between E and the union set $Z = \bigcup_{\tilde{F} \in X} \tilde{F}$ might be zero. For a stationary process X, for instance, it follows from Corollary 5.2 below that the intensity measure of the set of all flats $\tilde{F} \in X$ whose distance from E is less than some fixed positive number is infinite as soon as the dimension j of E is strictly positive.

Let us define the point process

$$X^E := \sum_{\tilde{F} \in X} \delta_{p(\tilde{F}, E)}$$

of projection points. For $j + k = d$ this is the intersection process of X and E. Clearly, X^E is a Poisson process if X is Poisson. Since the intensity measure Θ of X is supposed to be locally finite, the same is true for the intensity measure Θ^E of X^E. Note however that even for a stationary process X of k-flats, the point process X^E is inhomogeneous, in general. If X is not Poisson, then the point process X^E might not be simple. However, this is not relevant in the following. The first main result of this section expresses Θ^E in terms of Θ (respectively, in terms of the characteristics $\eta, \mathbb{Q}$, which are related to Θ as in (3.3)). For its formulation we need to introduce generalized determinants between (affine) subspaces. For $E \in \mathcal{E}_i^d, F \in \mathcal{E}_j^d$, we choose an orthonormal basis in $E^\circ \cap F^\circ$ and extend it to an orthonormal basis B of E°, respectively an orthonormal basis B' of F°. Let $[E, F]$ be the $\min\{i + j, d\}$-dimensional volume of the parallelepiped spanned by the vectors in $B \cup B'$. This definition implies that $[E, F] = [E^\perp, F^\perp]$, where both sides are zero if E, F are not in general relative position. In case $i + j = d$, $[E, F]$ is the Jacobian of the orthogonal projection $\pi_{F^\perp}$ restricted to E.

Theorem 5.1. *Let X be a translation regular process of k-flats, $k \in \{0, \ldots, d-1\}$, satisfying* (3.1) *and let $\eta, \mathbb{Q}$ be related to Θ as in* (3.3)*. Let $E \in \mathcal{E}_j^d$ with $j \in \{0, \ldots, d-k\}$ be such that E and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$. Then*

$$\Theta^E = \int_{\mathcal{L}_k^d} \int_{E+(E+F)^\perp} \mathbf{1}\{y \in \cdot\}[E, F]\eta(F + y)\mathcal{H}^{d-k}(dy)\mathbb{Q}(dF).$$

Proof. It is sufficient to consider the special case $E \in \mathcal{L}_j^d$, hence $E = E^\circ$. In fact, if $E \in \mathcal{E}_j^d$, $E = E^\circ + x_E, x_E \in E^\perp$, we may consider the flat process

$$\tilde{X} := \sum_{\tilde{F} \in X} \delta_{\tilde{F} - x_E}$$

which has intensity measure $\tilde{\Theta}$ corresponding to $\mathbb{Q}$ and $\tilde{\eta}$, $\tilde{\eta}(\tilde{F}) := \eta(\tilde{F} + x_E)$. Then, $X^E = \tilde{X}^{E^\circ} + x_E$, hence Θ^E is the image of the intensity measure of $\tilde{X}^{E^\circ}$ under $x \mapsto x + x_E$. Since this translation transforms the measure

$$\int_{E^\circ+(E^\circ+F)^\perp} \mathbf{1}\{y \in \cdot\}[E, F]\tilde{\eta}(F + y)\mathcal{H}^{d-k}(dy)$$

into

$$\int_{E+(E+F)^\perp} \mathbf{1}\{y \in \cdot\}[E, F]\eta(F + y)\mathcal{H}^{d-k}(dy)\,,$$

we thus obtain the general case from the special one.

So let $E \in \mathcal{L}_j^d$, and let $g : \mathbb{R}^d \to [0, \infty)$ be a measurable function. We make use of the representation (3.3) and obtain

$$\begin{aligned} \int g(z)\Theta^E(dz) &= \int_{\mathcal{E}_k^d} g(p(\tilde{F}, E))\Theta(d\tilde{F}) \\ &= \int_{\mathcal{L}_k^d} \int_{F^\perp} g(p(F + z, E))\eta(F + z)\mathcal{H}^{d-k}(dz)\mathbb{Q}(dF). \end{aligned}$$

We now decompose the inner integral according to the orthogonal decomposition

$$F^\perp = (F^\perp \cap E^\perp) \oplus (F^\perp \cap (F + E)),$$

which holds since the orthogonal complement of $F^\perp \cap E^\perp$ with respect to $F^\perp$ is

$$F^\perp \cap (F^\perp \cap E^\perp)^\perp = F^\perp \cap (F + E).$$

Since we may assume that F and E are in general relative position, we have $\dim(F+E) = k+j$, and then $\dim(F^\perp \cap E^\perp) = d-k-j$ as well as $\dim(F^\perp \cap (F+E)) = j$. We obtain

$$\begin{aligned}&\int_{F^\perp} g(p(F+z,E))\eta(F+z)\mathcal{H}^{d-k}(dz)\\ &\qquad = \int_{F^\perp\cap E^\perp}\int_{F^\perp\cap(F+E)} g(p(F+x+y,E))\eta(F+x+y)\mathcal{H}^j(dy)\mathcal{H}^{d-k-j}(dx).\end{aligned}$$

Because of $\dim(F^\perp \cap (F+E)) = \dim E$, we may project E onto $F^\perp \cap (F+E)$ to replace the integral over $F^\perp \cap (F+E)$ by one over E. The Jacobian of this transformation is

$$[E, F + (F^\perp \cap E^\perp)] = [E, F].$$

For $w \in E$,

$$\pi_{F^\perp\cap(F+E)}(w) = w - \pi_F(w),$$

and hence

$$F + x + \pi_{F^\perp\cap(F+E)}(w) = F + x + w.$$

We obtain

$$\begin{aligned}&\int_{F^\perp} g(p(F+z,E))\eta(F+z)\mathcal{H}^{d-k}(dz)\\ &\qquad = [E,F]\int_{F^\perp\cap E^\perp}\int_{E} g(p(F+x+w,E))\eta(F+x+w)\mathcal{H}^j(dw)\mathcal{H}^{d-k-j}(dx).\end{aligned}$$

Generally, for $z \in \mathbb{R}^d$, we have

$$p(F+z,E) = p(E,F+z) + \pi_{(F+E)^\perp}(z). \tag{5.1}$$

For $w \in E$ and $x \in F^\perp \cap E^\perp$, this implies

$$p(F+x+w,E) = p(E,F+w) + x = w + x\,,$$

and we finally get

$$\begin{aligned}\int g(z)\Theta^E(dz) &= \int_{\mathcal{L}_k^d}[E,F]\int_{F^\perp\cap E^\perp}\int_E g(w+x)\\ &\qquad\qquad \times \eta(F+x+w)\mathcal{H}^j(dw)\mathcal{H}^{d-k-j}(dx)\mathbb{Q}(dF)\\ &= \int_{\mathcal{L}_k^d}[E,F]\left(\int_{E+(E+F)^\perp} g(y)\eta(F+y)\mathcal{H}^{d-k}(dy)\right)\mathbb{Q}(dF).\end{aligned}$$

This completes the proof. □

Now we assume that $j+k<d$. Then, in view of $p(\tilde F,E)=p(E,\tilde F)+d(E,\tilde F)u(E,\tilde F)$, which holds if E and $\tilde F$ are in general relative position, we may also consider the point process

$$\Phi^E := \sum_{\tilde F\in X} \delta_{(d(E,\tilde F),p(E,\tilde F),u(E,\tilde F))}$$

on $\mathbb{R}\times\mathbb{R}^d\times S^{d-1}$. The process Φ^E is a parametric representation of X^E and is concentrated on $[0,\infty)\times E\times(S^{d-1}\cap E^\perp)$. For the intensity measure Λ^E of Φ^E, we also obtain a representation in terms of η and $\mathbb{Q}$.

Corollary 5.2. *Let X be a translation regular process of k-flats, $k\in\{0,\dots,d-1\}$, satisfying* (3.1)*, and let $\eta,\mathbb{Q}$ be related to Θ as in* (3.3)*. Let $E\in\mathcal{E}_j^d$ with $j\in\{0,\dots,d-1-k\}$ be such that E and F are in general relative position, for $\mathbb{Q}$-almost all $F\in\mathcal{L}_k^d$. Then*

$$\Lambda^E = \int_0^\infty \int_{\mathcal{L}_k^d}\int_{F^\perp\cap E^\perp\cap S^{d-1}}\int_E \mathbf{1}\{(t,z,u)\in\cdot\}[E,F]t^{d-1-k-j} \times \eta(F+z+tu)\mathcal{H}^j(dz)\mathcal{H}^{d-1-k-j}(du)\mathbb{Q}(dF)dt.$$

Proof. We define the map $T:\mathbb{R}^d\setminus E\to[0,\infty)\times E\times(E^\perp\cap S^{d-1})$ by

$$T(y) := (d(E,y),p(E,y),(y-p(E,y))/d(E,y)).$$

Hence, $\mathbb{P}$-a.s. we have $\Phi^E=T(X^E)$, and thus $\Lambda^E=T(\Theta^E)$. Applying Theorem 5.1, using Fubini's theorem, and introducing polar coordinates in $F^\perp\cap E^\perp$, we obtain the result. □

It is useful to rewrite Corollary 5.2 as

$$\Lambda^E = \int_0^\infty \int_E\int_{S^{d-1}} \mathbf{1}\{(t,z,u)\in\cdot\}t^{d-1-k-j}\varphi_E(t,z,du)\mathcal{H}^j(dz)dt, \tag{5.2}$$

where φ_E is a finite kernel from $[0,\infty)\times E$ to S^{d-1} defined by

$$\varphi_E(t,z,\cdot) := \int_{\mathcal{L}_k^d}\int_{F^\perp\cap E^\perp\cap S^{d-1}} \mathbf{1}\{u\in\cdot\}\eta(F+z+tu)[E,F]\mathcal{H}^{d-1-k-j}(du)\mathbb{Q}(dF).$$

If $j\ge 1$, then the measure $\Lambda^E([0,\infty)\times\cdot\times S^{d-1})$ is in general not locally finite. In the stationary case, for instance,

$$\gamma_E := \varphi_E(t,z,S^{d-1}) = \omega_{d-k-j}\gamma\int_{\mathcal{L}_k^d}[E,F]\mathbb{Q}(dF) \tag{5.3}$$

is independent of $(t,z)\in[0,\infty)\times E$ and $\Lambda^E([0,\infty)\times A\times S^{d-1})=\infty$ for any Borel set $A\subset E$ of positive volume. Hence the points of $\Phi^E([0,\infty)\times\cdot\times S^{d-1})$ accumulate in each nonempty and open set. Assuming η to be continuous, we have for all Borel sets $C\subset S^{d-1}$ that $\varphi_E(t,z,C)\to\varphi_E(z,C)$, as $t\to 0$, where

$$\varphi_E(z,C) := \int_{\mathcal{L}_k^d}\int_{F^\perp\cap E^\perp\cap S^{d-1}} \mathbf{1}\{u\in C\}[E,F]\mathcal{H}^{d-1-k-j}(du)\varphi(z,dF), \tag{5.4}$$

in close analogy to formula (12) in [21].

Irrespectively of η being continuous or not (and for $j \in \{0, \dots, d-1-k\}$), we use (5.4) to define, for each $z \in E$, an even Borel measure $\varphi_E(z, \cdot)$ on S^{d-1}, concentrated on $S^{d-1} \cap E^\perp$. Further we put

$$\gamma_E(z) := \varphi_E(z, S^{d-1}) = \omega_{d-k-j} \int_{\mathcal{L}_k^d} [E, F] \varphi(z, dF), \tag{5.5}$$

which coincides with the constant γ_E from (5.3), if X is stationary. We call $\gamma_E(z)$ the *local intensity* and $\varphi_E(z, \cdot)$ the *local direction measure* of Φ^E at $z \in E$. This notation is justified by the following consequence of (5.2).

Theorem 5.3. *Let X be a translation regular process of k-flats in $\mathbb{R}^d$, $k \in \{0, \dots, d-1\}$, and assume that Θ has a representation* (3.3) *with continuous η. Then, for any $j \in \{0, \dots, d-1-k\}$, any $E \in \mathcal{E}_j^d$ and any $z \in E$, the local direction measure $\varphi_E(z, \cdot)$ fulfills*

$$\varphi_E(z, \cdot) = (d-k-j) \lim_{r \downarrow 0} \lim_{A \downarrow z} \frac{\Lambda^E([0, r] \times A \times \cdot)}{r^{d-k-j} \mathcal{H}^j(A)}.$$

As a preparation for our next result, we associate with each $u \in S^{d-1}$ unit vectors $u_1, \dots, u_{d-1}$ from an orthonormal basis of $u^\perp$. Then, for $I \subset \{1, \dots, d-1\}$ with $|I| = j$, we define $E_I(u) \in \mathcal{L}_j^{(u^\perp)}$ as the linear subspace spanned by $\{u_i : i \in I\}$ (with $E_I(u) := \{0\}$, if $j = 0$). Any system $\{E_I(u) : |I| = j, u \in S^{d-1}\} \subset \mathcal{L}_j^d$ arising in this way, is called an *orthogonal system of j-spaces.*

Theorem 5.4. *Let X be a translation regular process of k-flats, $k \in \{0, \dots, d-1\}$, whose intensity measure has a representation* (3.4) *with continuous density $\tilde{\eta}$. Let $z \in \mathbb{R}^d$ and $j \in \{0, \dots, d-1-k\}$. Then, the local direction measures $\varphi_{E+z}(z, \cdot)$, where E runs through a fixed orthogonal system of j-spaces, uniquely determine the Radon transform $R_{k\, d-1} \varphi(z, \cdot)$ of $\varphi(z, \cdot)$.*

In particular, for a process X of lines, the direction measure $\varphi(z, \cdot)$ is determined.

Proof. Let g be a an even continuous function on S^{d-1} and let $E \in \mathcal{L}_j^d$ be from the given orthogonal system. By (5.4) and applying a very special case of Theorem 1 in [1], we obtain

$$\begin{aligned} \int_{S^{d-1}} g(v) \varphi_{E+z}(z, dv) &= \int_{\mathcal{L}_{d-k}^d} \int_{W \cap E^\perp \cap S^{d-1}} g(v) \tilde{\eta}(W^\perp + z) [E, W^\perp] \mathcal{H}^{d-1-k-j}(dv) \nu_{d-k}(dW) \\ &= c_{dkj} \int_{E^\perp \cap S^{d-1}} \int_{\mathcal{L}_{d-k}^{(v)}} g(v) \tilde{\eta}(W^\perp + z) [E, W^\perp]^2 \nu_{d-k}^{(v)}(dW) \mathcal{H}^{d-1-j}(dv), \end{aligned} \tag{5.6}$$

where $\mathcal{L}_{d-k}^{(v)}$ is the set of all $W \in \mathcal{L}_{d-k}^d$ with $v \in W$, $\nu_{d-k}^{(v)}$ is the corresponding Haar probability measure on $\mathcal{L}_{d-k}^{(v)}$, and $c_{dkj} \neq 0$ is a constant. Above we have used the fact that, for $\nu_{d-k}^{(v)}$-almost all $W \in \mathcal{L}_{d-k}^{(v)}$ and for $v \in E^\perp$, the Jacobian $J(T_v(E^\perp \cap S^{d-1}), W)$, appearing in [1], can be simplified in the following way:

$$\begin{aligned} J(T_v(E^\perp \cap S^{d-1}), W) =& [(v^\perp \cap E^\perp \cap W)^\perp \cap (v^\perp \cap E^\perp), W] \\ =& [(E_v^\perp \cap W) + E_v, W^\perp] \\ =& [E_v, W^\perp] = [E, W^\perp], \end{aligned}$$

where $E_v := E + \operatorname{lin}(v)$.

Since this is true for all even continuous functions g, the local direction measure $\varphi_{E+z}(z, \cdot)$ determines the integrals

$$\int_{\mathcal{L}_k^{(v^\perp)}} [E, V]^2 \tilde{\eta}(V + z) \nu_k^{(v^\perp)}(dV) \tag{5.7}$$

for all $v \in S^{d-1} \cap E^\perp$.

Now we consider a fixed $u \in S^{d-1}$. Then, the subspaces $E_I(u) \in \mathcal{L}_j^{(u^\perp)}$ with $I \subset \{1, \ldots, d-1\}$ and $|I| = j$ from the given orthogonal system satisfy

$$\sum_{|I|=j} [E_I(u), V]^2 = \binom{d-1-k}{j}.$$

Hence, replacing v and E in (5.7) by u and $E_I(u)$ and summing over I, we obtain the integral

$$\int_{\mathcal{L}_k^{(u^\perp)}} \tilde{\eta}(V + z) \nu_k^{(u^\perp)}(dV).$$

Since this holds for all $u \in S^{d-1}$, the first assertion is shown.

The second assertion follows from the injectivity of the Radon transform $R_{1\,d-1}$. □

Remark. It is possible to obtain the same result without the assumption of absolute continuity of $\mathbb{Q}$, if, instead of working with a fixed orthogonal system of j-spaces, we assume that the local direction measures $\varphi_{E+z}(z, \cdot)$ are given for all $E \in \mathcal{L}_j^d$. The proof is similar, but uses an additional approximation argument as an initial step (cf. [7]).

The arguments used at the end of the above proof do not allow us to deduce that, for $k \in \{2, \ldots, d-2\}$, the direction measure $\varphi(z, \cdot)$ is **not** determined by the given collection of local direction measures $\varphi_{E+z}(z, \cdot)$. For $j = d-1-k$, we obtain such a more precise result, if we combine (5.6) with a theorem of Goodey and Howard [6]. For $z \in \mathbb{R}^d$ and $j \in \{0, \ldots, d-1\}$, we denote by $\mathcal{E}_j^{(z)}$ the set of all j-flats through z.

Theorem 5.5. *Let X be a translation regular process of k-flats, $k \in \{0, \ldots, d-1\}$, and let $z \in \mathbb{R}^d$. Then, the family $\{\varphi_E(z, \cdot) : E \in \mathcal{E}_{d-1-k}^{(z)}\}$ of local direction measures at z uniquely determines the direction measure $\varphi(z, \cdot)$ of X at z, if and only if $k \in \{0, 1, d-1\}$.*

Proof. By definition (5.4), the uniqueness holds trivially if $k = 0$. Hence, we assume now $k \geq 1$.

As we mentioned already in the remark above, the convolution arguments given in [7] allow us to assume further that $\mathbb{Q}$ has a density f with respect to ν_k, hence (3.4) holds with $\tilde{\eta}(F + z) := \eta(F + z) f(F)$.

Then, (5.6) shows that the family $\{\varphi_E(z,\cdot) : E \in \mathcal{E}_{d-1-k}^{(z)}\}$ determines $\varphi(z,\cdot)$, if and only if the function $f_z : F \mapsto \tilde{\eta}(F+z), F \in \mathcal{L}_k^d$, is determined by Sf_z. Here, for a continuous function h on $\mathcal{L}_k^d$, the function Sh is defined by

$$Sh(E,u) := \int_{\mathcal{L}_{d-k}^{(u)}} [E, W^\perp]^2 h(W^\perp) \nu_{d-k}^{(u)}(dW),$$

for $E \in \mathcal{L}_{d-1-k}^d$ and $u \in S^{d-1} \cap E^\perp$. Using Theorem 3.1 and p. 115, l. 2 in [6], we obtain that the operator S has a nontrivial kernel, if $k \in \{2,\dots,d-2\}$, whereas, for $k \in \{1, d-1\}$, [6, p. 114, l. -1] implies injectivity of S. □

We remark that the injectivity results in Theorems 5.4 and 5.5, for $k=1$, are not best possible. In fact, for a line process X, equation (5.5) and the injectivity of the generalized cosine transform (cf. Lemma 5.2 in [25]) yield the stronger result that, for given $j \in \{1,\dots,d-2\}$, $\varphi(z,\cdot)$ is even determined by the collection of local intensities $\gamma_E(z)$, $E \in \mathcal{E}_j^{(z)}$. In the special case $d \geq 3$, $j=k=1$ and $E = \text{lin}(e) + z \in \mathcal{E}_1^{(z)}$, for some $e \in S^{d-1}$, we consider $\varphi(z,\cdot)$ as an even measure on S^{d-1}. Then

$$\gamma_E(z) = \omega_{d-2} \int_{S^{d-1}} \sqrt{1 - \langle e, v\rangle^2} \varphi(z, dv)$$

and the uniqueness result just mentioned reduces to the well-known injectivity of the sine transform (on even measures). For a stationary process of lines in $\mathbb{R}^3$, this example is also discussed in [26].

If the line process X is stationary, then $\gamma_E = \gamma_E(z)$ is independent of z, and Theorem 5.4 (or the above remark) immediately implies the following result.

Corollary 5.6. *Let X be a stationary process of lines in $\mathbb{R}^d$ with intensity γ and directional distribution $\mathbb{Q}$. Then, for any $j \in \{1,\dots,d-2\}$, γ and $\mathbb{Q}$ are determined by $\{\gamma_E : E \in \mathcal{L}_j^d\}$.*

If X is a stationary process of k-flats, an unbiased estimator for $(d-k-j)^{-1}\gamma_E$ is given by

$$\text{card}\{\tilde{F} \in X : \tilde{F} \cap (B^d + E) \neq \emptyset, p(E, \tilde{F}) \in A_E\},$$

where $A_E \subset E$ is a Borel set with $\mathcal{H}^j(A_E) = 1$. In [26], for $d=3$ and $k=j=1$, this estimator is compared to another estimator. We extend this analysis to the case $d \geq 3$ and $k \in \{1,\dots,d-2\}$, but we also assume that $j=1$.

Let $E \in \mathcal{L}_1^d$ be fixed, and let $e \in S^{d-1}$ be such that $E = \text{lin}(e)$. Then we set $A_1 := B^d \cap E^\perp$, $A_2 := [-e, e]$ and define

$$\hat{\Phi}_{a,b}^E := \text{card}\{\tilde{F} \in X : \tilde{F} \cap (aA_1 + bA_2) \neq \emptyset\},$$

for $a, b > 0$. For the expectation of $\hat{\Phi}^E_{a,b}$, we deduce

$$\begin{aligned}
\mathbb{E}\left[\hat{\Phi}^E_{a,b}\right] &= \mathbb{E}\left[\operatorname{card}\{\tilde{F} \in X : d(E,\tilde{F}) \le a, p(E,\tilde{F}) \in bA_2\}\right] \\
&\qquad + \mathbb{E}\left[\operatorname{card}\{\tilde{F} \in X : p(E,\tilde{F}) \notin bA_2, \tilde{F} \cap (aA_1 + bA_2) \neq \emptyset\}\right] \\
&= a^{d-1-k} 2b(d-1-k)^{-1}\gamma_E \\
&\qquad + \mathbb{E}\left[\operatorname{card}\{\tilde{F} \in X : p(E,\tilde{F}) \notin bA_2, \tilde{F} \cap (aA_1 + bA_2) \neq \emptyset\}\right] \\
&= a^{d-1-k} 2b(d-1-k)^{-1}\gamma_E \\
&\qquad + \gamma \int_{\mathcal{L}^d_k} \int_0^a t^{d-2-k} \int_{F^\perp \cap E^\perp \cap S^{d-1}} [E,F] \int_E \mathbf{1}\{z \notin [-be, be]\} \\
&\qquad\qquad \times \mathbf{1}\{(F + z + tu) \cap (aA_1 + bA_2) \neq \emptyset\} \mathcal{H}^1(dz) \mathcal{H}^{d-2-k}(du)\, dt\, \mathbb{Q}(dF).
\end{aligned}$$

For $\|z\| > b$, the second indicator function is one if and only if

$$0 \le t \le a \qquad \text{and} \qquad \|z\| - b \le \sqrt{a^2 - t^2}/\tan\alpha,$$

where $\alpha \in [0, \pi/2]$ is defined by $[E,F] = \sin\alpha$. Hence, we deduce that

$$\mathbb{E}\left[\hat{\Phi}^E_{a,b}\right] - a^{d-1-k} 2b(d-1-k)^{-1}\gamma_E = \kappa_{d-k} a^{d-k} \gamma \int_{\mathcal{L}^d_k} \|\pi_F e\| \mathbb{Q}(dF). \tag{5.8}$$

For $d = 3$ and $k = 1$ we thus recover Theorem 4.4 in [26]. Further, from (5.8) one can deduce similar conclusions as in [26, Corollary 4.5].

6 Distance measurements from flats: Poisson case

We now consider a translation regular Poisson process X of k-flats satisfying (3.3), and the corresponding union set Z. Generalizing the situation discussed in Section 4, we fix $j \in \{0, \dots, d-1-k\}$ and consider a test flat $E \in \mathcal{E}^d_j$, such that E and F are in general relative position for $\mathbb{Q}$-almost all $F \in \mathcal{L}^d_k$. Further, let $C \subset E$ be a compact set. If $\{\tilde{F} \in X : p(E,\tilde{F}) \in C\} \neq \emptyset$, we define $\xi_C \in X$ as the ($\mathbb{P}$-a.s. uniquely determined) flat in X for which

$$d(E, \xi_C) = \min\{d(E,\tilde{F}) : \tilde{F} \in X, p(E,\tilde{F}) \in C\},$$

and we put $T^{C,E} := d(E, \xi_C)$ and $U^{C,E} := u(E, \xi_C)$. If $\{\tilde{F} \in X : p(E,\tilde{F}) \in C\} = \emptyset$, we put $T^{C,E} := \infty$ and give $U^{C,E}$ some fixed value. The random variable $T^{C,E}$ can be interpreted as a localized distance from E to X. We now discuss the question in how far the direction measure $\varphi(z, \cdot)$ of X at some point $z \in \mathbb{R}^d$ is determined by the distributions of the random vectors $(T^{C,E}, U^{C,E})$, where $E \in \mathcal{E}^{(z)}_j$ and $C \subset E$ vary. Since $\Phi^E(\cdot \times C \times \cdot)$ is a Poisson process and $T^{C,E}$ is the smallest point (on $(0,\infty)$) of $\Phi^E(\cdot \times C \times S^{d-1})$ we obtain from (5.2) and a well-known point process result that

$$\mathbb{P}\left(T^{C,E} > r\right) = \exp\left[-\int_0^r \int_C t^{d-1-k-j} \varphi_E(t, z, S^{d-1}) \mathcal{H}^j(dz)\, dt\right], \tag{6.1}$$

and

$$\mathbb{P}\left((T^{C,E},U^{C,E})\in d(t,u)\right)=\mathbb{P}\left(T^{C,E}>t\right)\int_C t^{d-1-k-j}\varphi_E(t,z,du)\mathcal{H}^j(dz)dt. \qquad (6.2)$$

In case $j=0$ we have $E=C=\{z\}$ for some $z\in\mathbb{R}^d$, and

$$\varphi_E(t,z,\cdot)=\int_{\mathcal{L}_k^d}\int_{F^\perp\cap S^{d-1}}\mathbf{1}\{u\in\cdot\}\eta(F+z+tu)\mathcal{H}^{d-1-k}(du)\mathbb{Q}(dF).$$

Therefore (6.2) boils down to a formula from Section 4.

Assume that the intensity measure of X has a representation (3.4) with continuous density $\tilde\eta$, and let $z\in\mathbb{R}^d$ and $E\in\mathcal{E}_j^{(z)}$. Then, (6.2) and a differentiation argument show that the distributions of $(T^{C,E},U^{C,E})$, where $C\subset E$ varies among the compact subsets, determine the measure $\varphi_E(z,\cdot)$. From Theorem 5.4, we therefore get directly the following result.

Theorem 6.1. *Let X be a translation regular Poisson process of k-flats, $k\in\{0,\dots,d-1\}$, whose intensity measure has a representation* (3.4) *with continuous density $\tilde\eta$, let $z\in\mathbb{R}^d$ and $j\in\{0,\dots,d-1-k\}$.*

Then, the distributions of the random vectors $(T^{C,E+z},U^{C,E+z})$, where E runs through a fixed orthogonal system of j-spaces and C varies among all compact subsets of $E+z$, uniquely determine the Radon transform $R_{k\,d-1}\varphi(z,\cdot)$ of $\varphi(z,\cdot)$.

In particular, for $k\in\{0,1,d-1\}$, $\varphi(z,\cdot)$ is determined.

Comparing Theorems 4.1 and 6.1, we see that both make similar statements about the determination of the direction measure $\varphi(z,\cdot)$ of the Poisson process X (at z). As was already mentioned before Theorem 4.1, there is a well-known stronger result for $k=0$. For $j=0$, Theorem 6.1 is weaker than Theorem 4.1, since it requires the absolute continuity of $\mathbb{Q}$ (with respect to ν_k). This assumption is needed because we work with a fixed orthogonal system of j-spaces. However, if the orthogonal system of j-spaces is replaced by $\mathcal{L}_j^d$ (compare the remark after Theorem 5.4), then it is possible to obtain a common generalization of Theorems 4.1 and 6.1 which holds for $j\geq 0$ and without the assumption of absolute continuity.

We also remark that, for $E\in\mathcal{E}_j^{(z)}$, $j\in\{1,\dots,d-1-k\}$, it follows from (6.1) and again by a differentiation argument, that the distributions of the random distances $T^{C,E}$, where C varies among all compact subsets of E, determine $\gamma_E(z)$.

In the stationary case, we obtain the following special case of Theorem 6.1.

Corollary 6.2. *Let X be a stationary Poisson process of lines in $\mathbb{R}^d$. Then, for any $j\in\{1,\dots,d-2\}$, the distribution of X is determined by the distribution of the random vectors $(T^{B^d\cap E,E},U^{B^d\cap E,E})$, where E varies among a fixed orthogonal system of j-spaces.*

In Corollary 6.2, for a given $E\in\mathcal{L}_j^d$ we can choose any Borel set $A\subset E$ with $\mathcal{H}^j(A)\in(0,\infty)$ instead of $B^d\cap E$. A particularly simple case of Corollary 6.2 is obtained for $d=3$ and $j=1$. Here we associate with any unit vector $u\in S^2$ two further unit vectors $E_1(u),E_2(u)$ such that $u,E_1(u),E_2(u)$ is an orthonormal basis of $\mathbb{R}^3$ in order to obtain the required orthogonal system of lines in $\mathbb{R}^3$.

7 Distance measurements from flags

In this final section, we generalize some of the previous results by replacing the fixed flat E by a chain of flats with increasing dimension. Again, we consider a translation regular process X of k-flats, $k \in \{0, \dots, d-1\}$, for which condition (3.1) is satisfied and assume that a representation (3.3) of Θ is given. For $i \in \{0, \dots, d-1-k\}$ we consider flags

$$\mathcal{E} = (E_i, E_{i+1}, \dots, E_{d-1-k})$$

of flats $E_j \in \mathcal{E}_j^d$ with $E_i \subset E_{i+1} \subset \dots \subset E_{d-1-k}$. We assume that F and E_{d-1-k} are in general relative position for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$, which implies that F and E_j are in general relative position for $j = i, \dots, d-1-k$.

We now consider the point process

$$X^{\mathcal{E}} := \sum_{F \in X} \delta_{(p(F,E_i), \dots, p(F,E_{d-1-k}))}$$

of $(d-k-i)$-tuples of projection points, which is a point process on $(\mathbb{R}^d)^{d-k-i}$. As in the previous section, $X^{\mathcal{E}}$ has a locally finite intensity measure $\Theta^{\mathcal{E}}$, and it is a Poisson process if X is Poisson.

The following result is an immediate extension of Theorem 5.1.

Theorem 7.1. *Let X be a translation regular process of k-flats, $k \in \{0, \dots, d-1\}$, satisfying* (3.1) *and let $\eta, \mathbb{Q}$ be related to Θ as in* (3.3)*. Let $\mathcal{E} = (E_i, \dots, E_{d-1-k})$, $i \in \{0, \dots, d-1-k\}$, be a flag of flats such that E_{d-1-k} and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$. Then, for a measurable function $g : (\mathbb{R}^d)^{d-k-i} \to [0, \infty)$,*

$$\mathbb{E}\left[\sum_{F \in X} g(p(F, E_i), \dots, p(F, E_{d-1-k}))\right] = \int_{\mathcal{L}_k^d} [F, E_i] \int_{E_i} \int_{F^\perp \cap E_i^\perp}$$
$$g(x+y, p(E_{i+1}, F+y) + x + \pi_{F^\perp \cap E_{i+1}^\perp}(y), \dots, p(E_{d-1-k}, F+y) + x + \pi_{F^\perp \cap E_{d-1-k}^\perp}(y))$$
$$\times\ \eta(F+x+y)\mathcal{H}^{d-k-i}(dy)\mathcal{H}^i(dx)\mathbb{Q}(dF).$$

Proof. We can proceed as in the proof of Theorem 5.1. First, it is sufficient to consider the case $0 \in E_i$, hence $E_j \in \mathcal{L}_j^d$ for $j = i, \dots, d-1-k$. Then we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sum_{F \in X} g(p(F, E_i), \dots, p(F, E_{d-1-k}))\right] \\
&\quad = \int_{\mathcal{L}_k^d} [E_i, F] \int_{F^\perp \cap E_i^\perp} \int_{E_i} g(p(F+x+y, E_i), \dots, p(F+x+y, E_{d-1-k})) \\
&\qquad\quad \times\ \eta(F+x+y)\mathcal{H}^i(dx)\mathcal{H}^{d-k-i}(dy)\mathbb{Q}(dF).
\end{aligned}$$

For $j \in \{i, \dots, d-1-k\}$, $x \in E_i$ and $y \in F^\perp \cap E_i^\perp$, we get from (5.1)

$$\begin{aligned}
p(F+x+y, E_j) &= p(E_j, F+x+y) + \pi_{F^\perp \cap E_j^\perp}(x+y) \\
&= p(E_j, F+y) + x + \pi_{F^\perp \cap E_j^\perp}(y),
\end{aligned}$$

since $x \in E_i \subset E_j$. In particular, for $j = i$, we obtain $p(E_i, F+y) = 0$ and $\pi_{F^\perp \cap E_i^\perp}(y) = y$. From this the result follows. □

We now assume again that X is a stationary process of k-flats. Since uniqueness theorems for line and hyperplane processes (as well as ordinary point processes) have been stated previously in this paper, we now concentrate on the case $k \in \{2, \dots, d-2\}$. In view of the assumed stationarity, it is sufficient to consider test flags consisting of linear subspaces. Let $\mathcal{F}_i$ denote the set of all flags $\mathcal{E} = (E_i, \dots, E_{d-1-k})$, where $E_j \in \mathcal{L}_j^d$, $j = i, \dots, d-1-k$. We also define, for $E_i \in \mathcal{L}_i^d$,

$$Z(E_i) := (B^d \cap E_i^\perp) + A_{E_i},$$

where $A_{E_i} \subset E_i$ is an arbitrary Borel set with $\mathcal{H}^i(A_{E_i}) = 1$.

Our final theorem shows that, for certain pairs (k, i), the intensity and the directional distribution of a stationary point process X of k-flats are uniquely determined by the intensity measures of the point processes $X^{\mathcal{E}}$, $\mathcal{E} \in \mathcal{F}_i$. The condition on (k, i) is that

$$i = 1, d-k \text{ odd}, \quad \text{or} \quad i = d-2k+1, k < d-k, k \text{ odd}, \quad \text{or} \quad 0 < i \leq d-2k\,. \tag{7.1}$$

Theorem 7.2. *Let X be a stationary process of k-flats, $k \in \{2, \dots, d-2\}$, fulfilling (3.1), and assume that (k, i) satisfies condition (7.1). Then γ and $\mathbb{Q}$ are uniquely determined by the collection of measures*

$$\mathbb{E}\left[\sum_{F \in X} \mathbf{1}\{\mathrm{lin}(u(F, E_i), \dots, u(F, E_{d-1-k})) \in \cdot\}\mathbf{1}\{p(F, E_i) \in Z(E_i)\}\right]$$

on $\mathcal{L}_{d-k-i}^d$, where $\mathcal{E} = (E_i, \dots, E_{d-1-k})$ varies in $\mathcal{F}_i$.

Proof. We first show that E and F are in general relative position, for ν_{d-1-k}-almost all $E \in \mathcal{L}_{d-1-k}^d$ and $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$.

In fact, let R denote the set of all $(E, F) \in \mathcal{L}_{d-1-k}^d \times \mathcal{L}_k^d$ for which E and F are in general relative position. By Lemma 4.5.1 in [19], we get for all $F \in \mathcal{L}_k^d$ that

$$\nu_{d-1-k}(\{E \in \mathcal{L}_{d-1-k}^d : (E, F) \in R\}) = 1.$$

Using Fubini's theorem, we deduce that

$$\begin{aligned} 1 &= \int_{\mathcal{L}_k^d} \int_{\mathcal{L}_{d-1-k}^d} \mathbf{1}\{(E, F) \in R\} \nu_{d-1-k}(dE) \mathbb{Q}(dF) \\ &= \int_{\mathcal{L}_{d-1-k}^d} \int_{\mathcal{L}_k^d} \mathbf{1}\{(E, F) \in R\} \mathbb{Q}(dF) \nu_{d-1-k}(dE). \end{aligned}$$

Hence, for ν_{d-1-k}-almost every $E \in \mathcal{L}_{d-1-k}^d$, we find that

$$\mathbb{Q}(\{F \in \mathcal{L}_k^d : (E, F) \in R\}) = 1.$$

It follows that, for ν_i-almost all $E_i \in \mathcal{L}_i^d$, there is some $E_{d-1-k} \in \mathcal{L}_{d-1-k}^d$ such that E_{d-1-k} and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$. Therefore, we may concentrate on flags $\mathcal{E} = (E_i, \dots, E_{d-1-k}) \in \mathcal{F}_i$, for which E_{d-1-k} and F are in general relative position, for $\mathbb{Q}$-almost all F.

For such flags, we apply Theorem 7.1 with the function g on $\mathbb{R}^{d-k-i}$ defined by

$$g(x_i, \dots, x_{d-1-k}) := \mathbf{1}\{\mathrm{lin}(x_i - \pi_{E_i}(x_i), \dots, x_{d-1-k} - \pi_{E_{d-1-k}}(x_{d-1-k})) \in \mathcal{A}\}\mathbf{1}\{x_i \in Z(E_i)\},$$

where $\mathcal{A} \subset \mathcal{L}^d_{d-k-i}$ is a Borel set. Let $F \in \mathcal{L}^d_k$ be such that E_{d-1-k} and F are in general relative position. Let $x \in E_i$, $y \in F^\perp \cap E_i^\perp$ and $j \in \{i, \dots, d-1-k\}$. Then

$$p(E_j, F + y) + x + \pi_{F^\perp \cap E_j^\perp}(y) - \pi_{E_j}(p(E_j, F + y) + x + \pi_{F^\perp \cap E_j^\perp}(y)) = \pi_{F^\perp \cap E_j^\perp}(y),$$

since $x \in E_i \subset E_j$ and $E_j^\perp \subset E_i^\perp$. Obviously, we have

$$\begin{aligned}&\mathrm{lin}(u(F, E_i), \dots, u(F, E_{d-1-k})) \\ &\quad = \mathrm{lin}(p(F, E_i) - \pi_{E_i}(p(F, E_i)), \dots, p(F, E_{d-1-k}) - \pi_{E_{d-1-k}}(p(F, E_{d-1-k}))).\end{aligned}$$

Hence, Theorem 7.1 implies that

$$\begin{aligned}&\mathbb{E}\left[\sum_{F \in X} \mathbf{1}\{\mathrm{lin}(u(F, E_i), \dots, u(F, E_{d-1-k})) \in \mathcal{A}\}\mathbf{1}\{p(F, E_i) \in Z(E_i)\}\right] \\ &\quad = \gamma \int_{\mathcal{L}^d_k} [F, E_i] \int_{E_i} \int_{F^\perp \cap E_i^\perp} \mathbf{1}\{\mathrm{lin}(\pi_{F^\perp \cap E_i^\perp}(y), \dots, \pi_{F^\perp \cap E_{d-1-k}^\perp}(y)) \in \mathcal{A}\} \\ &\qquad\quad \times \mathbf{1}\{x + y \in Z(E_i)\}\, \mathcal{H}^{d-k-i}(dy)\mathcal{H}^i(dx)\mathbb{Q}(dF) \\ &\quad = \kappa_{d-k-i}\gamma \int_{\mathcal{L}^d_k} [F, E_i]\mathbf{1}\{(F + E_i)^\perp \in \mathcal{A}\}\mathbb{Q}(dF).\end{aligned}$$

In the last step, we have used that $\pi_{F^\perp \cap E_j^\perp}(y) \in (F + E_i)^\perp$, for $j = i, \dots, d-1-k$, and that these vectors are linearly independent, for $\mathcal{H}^{d-k-i}$-almost all $y \in F^\perp \cap E_i^\perp$, whenever F and E_{d-1-k} are in general relative position.

An approximation argument shows that the measures

$$\int_{\mathcal{L}^d_k} [F^\perp, E^\perp]\mathbf{1}\{F^\perp \cap E^\perp \in \cdot\}\gamma\mathbb{Q}(dF)$$

are determined, for all $E \in \mathcal{L}^d_i$. Now we can use the main theorem from [7] to infer that $\gamma\mathbb{Q}$ is determined, which gives the required result. □

Similarly to some of the previous results, we could formulate a corresponding theorem with the set $\mathcal{F}_i$ of all flags replaced by the smaller set $\mathcal{F}_i^* := \{\mathcal{E}(E_i) : E_i \in \mathcal{L}^d_i\}$, where for each $E_i \in \mathcal{L}^d_i$ we have selected an arbitrary fixed flag $\mathcal{E}(E_i) = (E_i, \dots, E_{d-1-k}) \in \mathcal{F}_i$. In general, however, the corresponding set of $(d-1-k)$-flats occurring in these flags might be rather small then. Therefore, we would need a regularity assumption on $\mathbb{Q}$ or a suitable choice of the flags (in fact, almost all choices would be fine) in order to guarantee that the general position assumption of Theorem 7.1 is fulfilled.

We also remark that the injectivity result in [7], which we used, shows that (7.1) describes precisely the conditions for injectivity. Hence, in all other cases a uniqueness result corresponding to Theorem 7.2 fails.

In the proof of Theorem 7.2 we have established the equation

$$\mathbb{E}\left[\sum_{F\in X}\mathbf{1}\{\mathrm{lin}(u(F,E_i),\dots,u(F,E_{d-1-k}))\in\cdot\}\mathbf{1}\{p(F,E_i)\in Z(E_i)\}\right]$$
$$=\kappa_{d-k-i}\gamma\int_{\mathcal{L}_k^d}[F,E_i]\mathbf{1}\{F^\perp\cap E_i^\perp\in\cdot\}\mathbb{Q}(dF),$$

for all flags $(E_i,\dots,E_{d-1-k})$ for which E_{d-1-k} and F are in general relative position, for $\mathbb{Q}$-almost all $F\in\mathcal{L}_k^d$. For $r>0$ and a Borel set $A\subset E_i$, the following slight generalization is obtained in the same manner,

$$\mathbb{E}\left[\sum_{F\in X}\mathbf{1}\{\mathrm{lin}(u(F,E_i),\dots,u(F,E_{d-1-k}))\in\cdot\}\mathbf{1}\{p(E_i,F)\in A,d(E_i,F)\le r\}\right]$$
$$=\kappa_{d-k-i}r^{d-k-i}\mathcal{H}^i(A)\gamma\int_{\mathcal{L}_k^d}[F,E_i]\mathbf{1}\{F^\perp\cap E_i^\perp\in\cdot\}\mathbb{Q}(dF),$$

where the flag $(E_i,\dots,E_{d-1-k})$ is chosen as above. For $r>0$ and a Borel set $A\subset E_i$, the following slight generalization is obtained in the same manner,

$$\mathbb{E}\left[\sum_{F\in X}\mathbf{1}\{\mathrm{lin}(u(F,E_i),\dots,u(F,E_{d-1-k}))\in\cdot\}\mathbf{1}\{p(E_i,F)\in A,d(E_i,F)\le r\}\right]$$
$$=\kappa_{d-k-i}r^{d-k-i}\mathcal{H}^i(A)\gamma\int_{\mathcal{L}_k^d}[F,E_i]\mathbf{1}\{F^\perp\cap E_i^\perp\in\cdot\}\mathbb{Q}(dF).$$

We now concentrate on one-dimensional subspaces E_i, i.e., on the case $i=1$. In view of the established injectivity results, this restriction is justified since condition (7.1) is satisfied for $(k,1)$ whenever it is satisfied for (k,i) and a suitable choice of i. As at the end of Section 5, we now introduce the estimator

$$\tilde{\Phi}_{a,b}^{E_1}:=\mathrm{card}\big\{F\in X:\mathrm{lin}(u(F,E_1),\dots,u(F,E_{d-1-k}))\in\cdot\},$$
$$F\cap\big(a(B^d\cap E_1^\perp)+b(B^d\cap E_1)\big)\neq\emptyset\big\},$$

for $E_1\in\mathcal{L}_1^d$, $B^d\cap E_1=[-e,e]$, and $a,b>0$. Then we deduce that

$$\mathbb{E}\left[\tilde{\Phi}_{a,b}^{E_1}\right]=\kappa_{d-1-k}a^{d-1-k}2b\gamma\int_{\mathcal{L}_k^d}[F^\perp,E_1^\perp]\mathbf{1}\{F^\perp\cap E_1^\perp\in\cdot\}\mathbb{Q}(dF)$$
$$+a^{d-k}\kappa_{d-k}\gamma\int_{\mathcal{L}_k^d}\|\pi_F e\|\mathbf{1}\{F^\perp\cap E_1^\perp\in\cdot\}\mathbb{Q}(dF).$$

This shows that $(\kappa_{d-1-k}a^{d-1-k}2b)^{-1}\tilde{\Phi}_{a,b}^{E_1}$ is an asymptotically unbiased estimator for

$$\gamma\int_{\mathcal{L}_k^d}[F^\perp,E_1^\perp]\mathbf{1}\{F^\perp\cap E_1^\perp\in\cdot\}\mathbb{Q}(dF)$$

as $a/b \to 0$.

Finally, we mention that a result corresponding to Theorem 7.2 can be easily shown for nonstationary processes as well. Namely, if we assume that Θ has a representation (3.3) with continuous η, then we can extend our notion of local direction measure and define $\varphi_{E_i}(z,\cdot)$, for $z \in \mathbb{R}^d$ and $E_i \in \mathcal{E}_i^{(z)}$, as a measure on $\mathcal{L}_{d-k-i}^d$ by

$$\varphi_{E_i}(z,\cdot) := (d-k-i) \lim_{r\downarrow 0} \lim_{A\downarrow z} \frac{1}{r^{d-k-i}\mathcal{H}^i(A)}$$
$$\times \mathbb{E}\left[\sum_{F\in X} \mathbf{1}\{\mathrm{lin}(u(F,E_i),\dots,u(F,E_{d-1-k})) \in \cdot\}\mathbf{1}\{d(E_i,F) \le r, p(E_i,F) \in A\}\right]$$

whenever $(E_i,\dots,E_{d-1-k})$ is a flag of flats such that E_{d-1-k} and F are in general relative position, for $\mathbb{Q}$-almost all $F \in \mathcal{L}_k^d$. As in the proof of Theorem 7.2 it can be shown that, for ν_i-almost all $E_i \in \mathcal{L}_i^d$, $\varphi_{E_i+z}(z,\cdot)$ is well-defined (and independent of the choice of a suitable flag), since then

$$\varphi_{E_i+z}(z,\cdot) = (d-k-i) \lim_{r\downarrow 0} \lim_{A\downarrow z} \frac{1}{r^{d-k-i}\mathcal{H}^i(A)} \int_{\mathcal{L}_k^d}\int_A\int_0^r\int_{F^\perp\cap E^\perp\cap S^{d-1}} t^{d-1-k-i}[F,E_i]$$
$$\times\, \mathbf{1}\{F^\perp \cap E_i^\perp \in \cdot\}\, \eta(F+z+tu)\mathcal{H}^{d-1-k-i}(du)dt\mathcal{H}^i(dz)\mathbb{Q}(dF)$$
$$= \omega_{d-k-i}\int_{\mathcal{L}_k^d} [F,E_i]\mathbf{1}\{F^\perp \cap E_i^\perp \in \cdot\}\eta(F+z)\mathbb{Q}(dF)$$
$$= \omega_{d-k-i}\int_{\mathcal{L}_k^d} [F,E_i]\mathbf{1}\{F^\perp \cap E_i^\perp \in \cdot\}\varphi(z,dF).$$

Hence, if the collection of local direction measures $\{\varphi_{E_i}(z,\cdot) : E_i \in \mathcal{E}_i^{(z)}\}$ is known and (k,i) satisfies condition (7.1), then the direction measure $\varphi(z,\cdot)$ is determined.

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